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Gain Margins of Multivariable MRAC Systems

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Abstract— **This paper studies the gain margins (GM) of multivariable model reference adaptive control (MRAC) systems: the parameter range of a control gain matrix in a designed MRAC system for maintaining the desired closed-loop signal boundedness and asymptotic output tracking. Analytical GM results are obtained for both continuous-time and discrete-time direct MRAC schemes applied to multi-input multi-output (MIMO) LTI systems. The GM problem is also studied for a class of indirect multivariable MRAC systems.**

Keywords: Gain margin, high frequency gain matrix, LDS decomposition, model reference adaptive control.

I. Introduction

The design of control systems for aircraft to make its controlled outputs track desired trajectories despite of parametric, structural, or environmental uncertainties, is of both theoretical and practical interests. Their ability to automatically adjust a controller by adaptive laws to deal with such uncertainties to achieve desired system performance makes adaptive control schemes attractive for aircraft control applications. Recently there has been considerably increased effort in research on adaptive control for aircraft flight systems in the presence of uncertainties and failures. As a main approach of adaptive control, model reference adaptive control (MRAC), in which a reference model is chosen to generate the desired output trajectories, is capable of making the outputs of the controlled system to track the outputs of the reference model system in addition to closed-loop stability [4], [10]. The study of multivariable MRAC systems expands to some important new issues including adaptive failure compensation and gain margin specification.

The gain margin (GM) concept, originally defined for linear time-invariant (LTI) control systems to specify a necessary and sufficient range of a control gain for stability [5], is also important for MRAC, of major interest for aircraft control applications. This paper derives the gain margins for MRAC schemes applied to MIMO LTI systems based on their stability properties [1], [3], [4], [6], [7], [10], as the generalization of our recent work in [8] for SISO systems. As pointed out in [8], for MRAC the gain margin also defines a range for a control gain variation, but this range provides only sufficient conditions ensuring stability (signal boundedness) and asymptotic output tracking under any initial conditions. It is shown that continuous-time multivariable direct MRAC systems have gain margin equal to infinity, while discrete-time multivariable direct MRAC systems have a finite gain margin. For indirect MRAC of MIMO systems, the problem of nonsingular estimation of a general system high frequency gain matrix using system input and output measurements is still a problem to be solved. In this paper we will consider a special case to demonstrate the typical feature of gain margins for indirect MRAC.

The paper is organized as follows. The formulation of the GM problem for multivariable MRAC systems is presented in Section II. The results of GM analysis of direct multivariable MRAC systems are presented in Section III for the continuous-time case, and in Section IV for the discrete-time case, respectively. The gain margin of an indirect multivariable MRAC system is studied in Section V.

II. Problem Statement

The gain margin (GM) problem of multivariable MRAC systems is formulated in this section, followed by a summary of the analytical GM results.

We consider the M-input M-output LTI plant in the transfer matrix representation

$$y(t) = G(D)[u](t), \tag{1}$$

where $u(t),\ y(t) \in \mathbb{R}^M$ are the plant input and output, $G(D) = Z(D)P^{-1}(D)$ is strictly proper and full rank, and $Z(D),\ P(D) \in \mathbb{R}^{M\times M}$ are right coprime polynomial matrices with $P(D)$ being column proper. [1]

The GM problem for an adaptive control system is formulated as the problem of specifying the stability ranges of the elements of a positive definite and diagonal matrix

$$K = \mathrm{diag}\{k_1, k_2, \ldots, k_M\},\ k_i > 0,\ i = 1, 2, \ldots, M \tag{2}$$

which is present in the forward loop between the controlled plant $G(D)$ and the adaptive controller (denoted as the blocks $C_1(D)$ and $C_2(D)$) that has been designed to be able to ensure desired closed-loop system properties for $K = I_M$, the $M \times M$ identity matrix, as shown in Fig. 1.

The adaptive controller is designed such that for $K = I_M$ all signals in the closed-loop system are bounded for any bounded initial conditions, and asymptotic tracking is achieved. That is, the plant output $y(t)$ asymptotically tracks a reference signal $y_m(t) \in \mathbb{R}^M$ generated from a reference model system $y_m(t) = W_m(D)[r](t)$, where $W_m(D) \in \mathbb{R}^{M\times M}$ is a rational transfer matrix.

[1] The symbol D is used, in the continuous-time case, as the Laplace transform variable or the time-differentiation operator: $D[x](t) = \dot{x}(t)$, $t \in [0, +\infty)$; or in the discrete-time case, as the z-transform variable or the time-advance operator: $D[x](t) = x(t+1)$, $t \in \{0, 1, 2, 3, \ldots\}$.

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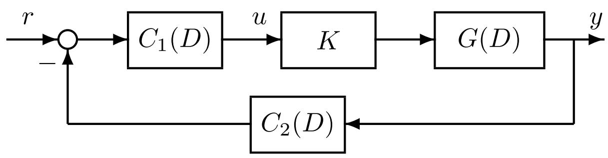

Fig. 1. Adaptive control system with a gain matrix K.

TABLE I
GAIN MARGINS OF ADAPTIVE CONTROL SYSTEMS

Adaptive control scheme	**Gain margin**
CT MIMO direct MRAC	$k_i \in (0, +\infty),\ i = 1, 2, \ldots, M$
CT SISO direct MRAC	$k \in (0, +\infty)$
DT MIMO direct MRAC	$k_i \in (0, k_i^0]$ [a]; $i = 1, 2, \ldots, M$
DT SISO direct MRAC	$k \in \left(0, \frac{k_p^0}{\lvert k_p\rvert}\right],\ 0 < \lvert k_p\rvert \le k_p^0$
MIMO indirect MRAC	$k_i \in [k_{i0}, +\infty)$ [b]; $i = 1, 2, \ldots, M$
SISO indirect MRAC	$k \in \left[\frac{k_{p0}}{\lvert k_p\rvert}, +\infty\right),\ 0 < k_{p0} \le \lvert k_p\rvert$

[a] The stability ranges of the gains k_i, $i = 1, 2, \ldots, M$, have some finite upper bounds which depend on the knowledge (used in adaptive laws for control adaptation) of some upper bounds and coupling terms of the plant high frequency gain matrix K_p.
[b] The stability ranges of the gains k_i, $i = 1, 2, \ldots, M$, have some positive lower bounds which depend on the knowledge (used in adaptive laws for plant identification) of some lower bounds of the plant high frequency gain matrix K_p.

For $K = \mathrm{diag}\{k_1, k_2, \ldots, k_M\} \neq I_M$, we want to find the ranges of k_i such that under any initial conditions, the MRAC scheme still ensures the closed-loop signal boundedness and output tracking when K is in the range.

In this paper, the gain margins of different MRAC systems with a constant K in (2) are studied, and the GM results are summarized in Table I for continuous-time (CT) and discrete-time (DT) MRAC systems, where the results for MIMO direct cases are derived for the systems designed based on the LDS decomposition of the plant high frequency gain matrix K_p. The results for the single-input single-output (SISO) cases with $K = k$ are provided here for comparison, where k_p^0 and k_{p0} are the upper and lower bounds of the magnitude of the plant high frequency gain k_p, respectively, used in the design of the adaptive control schemes [8], [10].

III. Gain Margin Analysis of Continuous-Time MIMO Direct MRAC Systems

For a continuous-time MIMO MRAC system, its gain margin can be derived based on the adaptive control scheme applied. We present the result for the MRAC design based on the LDS decomposition of K_p.

A. *LDS Decomposition of* K_p

The high frequency gain matrix of the controlled plant is defined as $K_p = \lim_{D\to\infty} \xi_m(D)G(D)$, where $\xi_m(D)$, which is assumed to be known and has a stable inverse, is the modified interactor matrix of $G(D)$. In the MRAC design based on decompositions of K_p, the reference model transfer matrix $W_m(D)$ is chosen to be $W_m(D) = \xi_m^{-1}(D)$.

We assume that K_p is finite and nonsingular, and all its leading principle minors, denoted as $\Delta_i, i = 1, 2, \ldots, M$, are nonzero and their signs are known. Besides, in the discrete-time design, we assume that some upper bounds d_i^0 of $|d_i^*| = |\frac{\Delta_i}{\Delta_{i-1}}|$ with $\Delta_0 = 1$, such that $0 < |d_i^*| \le d_i^0, i = 1, 2, \ldots, M$, are known [10].

The nonunique LDS decomposition of K_p, $K_p = L_s D_s S$, follows from its unique LDU decomposition $K_p = LD^*U$, where $L_s = LD_sU^{-T}D_s^{-1}$, $S = U^TD_s^{-1}D^*U$ for some $M \times M$ unity lower triangular matrix L, unity upper triangular matrix U, diagonal matrix

$$D^* = \mathrm{diag}\{d_1^*, \ldots, d_M^*\} = \mathrm{diag}\left\{\Delta_1, \ldots, \frac{\Delta_M}{\Delta_{M-1}}\right\} \quad (3)$$

with $\Delta_i \neq 0$ being the leading principle minors of K_p, and

$$D_s = \mathrm{diag}\left\{\mathrm{sign}[\Delta_1]\gamma_1, \ldots, \mathrm{sign}\left[\frac{\Delta_M}{\Delta_{M-1}}\right]\gamma_M\right\} \quad (4)$$

with $\gamma_i > 0$, $i = 1, 2, \ldots, M$, which can be arbitrary.

B. *Design Based on the LDS Decomposition of* K_p

When the parameters of plant $G(D)$ in (1) are known, the model reference controller

$$u(t) = \Theta_1^{*T}\omega_1(t) + \Theta_2^{*T}\omega_2(t) + \Theta_{20}^*y(t) + \Theta_3^*r(t), \quad (5)$$

with controller parameters computed from the plant-model transfer matrix matching equation

$$\begin{aligned}&\Theta_1^{*T}A(D)P(D) + \left(\Theta_2^{*T}A(D) + \Lambda(D)\Theta_{20}^*\right)Z(D)\\ &= \Lambda(D)\left(P(D) - \Theta_3^*\xi_m(D)Z(D)\right),\end{aligned} \quad (6)$$

achieves the control objective of signal boundedness and asymptotic output tracking, where $\Theta_1^* = [\Theta_{11}^*, \ldots, \Theta_{1\nu-1}^*]^T$, $\Theta_2^* = [\Theta_{21}^*, \ldots, \Theta_{2\nu-1}^*]^T$ with ν being the known observability index of $G(D)$, $\Theta_{ij}^*, \Theta_{20}^*, \Theta_3^* \in \mathbb{R}^{M\times M}$, $i = 1, 2, j = 1, \ldots, \nu-1$, and $\omega_1(t) = F(D)[u](t),\ \omega_2(t) = F(D)[y](t)$ with $F(D) = \frac{A(D)}{\Lambda(D)},\ A(D) = [I_M, DI_M, \ldots, D^{\nu-2}I_M]^T$ for a stable monic polynomial $\Lambda(D)$ of degree $\nu - 1$.

When the plant parameters are unknown, the model reference controller (5) cannot be implemented. Instead, the adaptive version of (5) is used, that is, the model reference adaptive controller

$$u(t) = \Theta_1^T(t)\omega_1(t) + \Theta_2^T(t)\omega_2(t) + \Theta_{20}(t)y(t) + \Theta_3(t)r(t) \quad (7)$$

is applied, where $\Theta_i(t)$ are the time-varying estimates of $\Theta_i^*,\ i = 1, 2, 20, 3$, and are updated from an adaptive law developed based on the error model derived as follows.

Error model. With $\Theta_3^* = K_p^{-1}$ and $K_p = L_sD_sS$, the matching equation (6) leads to

$$\begin{aligned}&D_sS(u(t) - \Theta_1^{*T}\omega_1(t) - \Theta_2^{*T}\omega_2(t) - \Theta_{20}^*y(t) - \Theta_3^*r(t))\\ &= L_s^{-1}\xi_m(D)[y - y_m](t).\end{aligned} \quad (8)$$

Letting $\Theta(t) = [\Theta_1^T(t), \Theta_2^T(t), \Theta_{20}(t), \Theta_3(t)]^T$ be the estimate of $\Theta^* = [\Theta_1^{*T}, \Theta_2^{*T}, \Theta_{20}^*, \Theta_3^*]^T$, and denoting $\tilde\Theta(t) = \Theta(t) - \Theta^*$, (7) and (8) yield

$$\xi_m(D)[y - y_m](t) + \Theta_0^*\xi_m(D)[y - y_m](t) = D_sS\tilde\Theta^T(t)\omega(t)$$

where $\omega(t) = [\omega_1^T(t), \omega_2^T(t), y(t), r(t)]^T$ and $\Theta_0^* = L_s^{-1} - I_M$ has a special lower triangular form with zero diagonal elements. We define the parameter vectors consisting of the nonzero parameters in each row of Θ_0^* to be $\theta_i^* = [\theta_{i1}^*, \dots, \theta_{ii-1}^*]^T \in \mathbb{R}^{i-1}$ and let their estimates to be $\theta_i(t) = [\theta_{i1}(t), \dots, \theta_{ii-1}(t)]^T \in \mathbb{R}^{i-1}$, $i = 2, 3, \dots, M$. By defining the estimation error to be

$$\epsilon(t) = \bar{e}(t) + [0, \theta_2^T(t)\eta_2(t), \dots, \theta_M^T(t)\eta_M(t)]^T + \Psi(t)\xi(t),$$

the following error model is obtained with D replaced by s:

$$\begin{aligned}\epsilon(t) = [0, \tilde{\theta}_2^T(t)\eta_2(t), \tilde{\theta}_3^T(t)\eta_3(t), \dots, \tilde{\theta}_M^T(t)\eta_M(t)]^T \\ + D_s S\tilde{\Theta}^T(t)\zeta(t) + \tilde{\Psi}(t)\xi(t),\end{aligned} \quad (9)$$

where $\tilde{\theta}_i(t) = \theta_i(t) - \theta_i^*$, $\eta_i(t) = [\bar{e}_1(t), \dots, \bar{e}_{i-1}(t)]^T \in \mathbb{R}^{i-1}$, $i = 2, 3, \dots, M$, with the filtered tracking error $\bar{e}(t) = \xi_m(s)h(s)[y - y_m](t) = [\bar{e}_1(t), \dots, \bar{e}_M(t)]^T$, $h(s) = \frac{1}{f(s)}$ with $f(s)$ being a chosen stable monic polynomial of the same degree as the maximum degree of $\xi_m(s)$, $\tilde{\Psi}(t) = \Psi(t) - \Psi^*$ with $\Psi(t)$ being the estimate of $\Psi^* = D_s S$, and $\xi(t) = \Theta^T(t)\zeta(t) - h(s)[\Theta^T\omega](t)$ with $\zeta(t) = h(s)[\omega](t)$.

Adaptive law. Based on the error model (9), we choose the following gradient adaptive laws:

$$\dot{\theta}_i(t) = -\frac{\Gamma_{\theta i}\epsilon_i(t)\eta_i(t)}{m^2(t)}, i = 2, 3, \dots, M \quad (10)$$

$$\dot{\Theta}^T(t) = -\frac{D_s\epsilon(t)\zeta^T(t)}{m^2(t)} \quad (11)$$

$$\dot{\Psi}(t) = -\frac{\Gamma\epsilon(t)\xi^T(t)}{m^2(t)} \quad (12)$$

where $\epsilon(t) = [\epsilon_1(t), \epsilon_2(t), \dots, \epsilon_M(t)]^T$, $\Gamma = \Gamma^T > 0$, $\Gamma_{\theta i} = \Gamma_{\theta i}^T > 0$, and $m^2(t) = 1 + \zeta^T(t)\zeta(t) + \xi^T(t)\xi(t) + \sum_{i=2}^M \eta_i^T(t)\eta_i(t)$.

For $K = I_M$, the adaptive controller (7) with the adaptive laws (10)–(12) ensures closed-loop signal boundedness and asymptotic output tracking, $\lim_{t\to\infty}(y(t) - y_m(t)) = 0$ [10].

C. LDU Decomposition of K_pK

In the presence of $K \neq I_M$, the controlled plant is $y(t) = G(s)K[u](t)$, and its high frequency gain matrix is K_pK. The following lemma establishes the link between the LDU decomposition of K_p and that of K_pK, which is crucial for the GM analysis of multivariable MRAC systems designed based on the LDS decomposition of K_p.

Lemma 1. *The gain matrix $K_pK \in \mathbb{R}^{M\times M}$, with $K = \mathrm{diag}\{k_1, k_2, \dots, k_M\} > 0$, has a unique LDU decomposition*

$$K_pK = \bar{L}\bar{D}^*\bar{U}, \ \bar{L} = L, \ \bar{D}^* = D^*K, \ \bar{U} = K^{-1}UK \quad (13)$$

where L, D^*, and U are from the unique LDU decomposition of the nonsingular matrix K_p with its leading principle minors being all nonzero, that is, $K_p = LD^*U$.

Proof: Suppose the matrix K_p is represented as

$$K_p = \begin{bmatrix} k_{p11} & k_{p12} & \cdots & k_{p1M} \\ k_{p21} & k_{p22} & \cdots & k_{p2M} \\ \cdots & \cdots & \cdots & \cdots \\ k_{pM1} & k_{pM2} & \cdots & k_{pMM} \end{bmatrix}, \quad (14)$$

and its nonzero leading principle minors are Δ_i, $i = 1, 2, \dots, M$. With a diagonal $K > 0$ as in (2), we have

$$K_pK = \begin{bmatrix} k_{p11}k_1 & k_{p12}k_2 & \cdots & k_{p1M}k_M \\ k_{p21}k_1 & k_{p22}k_2 & \cdots & k_{p2M}k_M \\ \cdots & \cdots & \cdots & \cdots \\ k_{pM1}k_1 & k_{pM2}k_2 & \cdots & k_{pMM}k_M \end{bmatrix}, \quad (15)$$

and its leading principle minors are

$$\bar{\Delta}_i = \Delta_i \prod_{j=1}^{i} k_j, \ i = 1, 2, \dots, M, \quad (16)$$

from which and D^* in (3), we can obtain

$$\bar{D}^* = \mathrm{diag}\left\{k_1\Delta_1, k_2\frac{\Delta_2}{\Delta_1}, \dots, k_M\frac{\Delta_M}{\Delta_{M-1}}\right\} = D^*K. \quad (17)$$

With $K_pK = LD^*UK = \bar{L}D^*K\bar{U}$, (17) implies $\bar{L} = L$ and $\bar{U} = K^{-1}UK$. This decomposition is unique from the uniqueness of matrix LDU decomposition. ∇

D. Gain Margin Analysis

The desired closed-loop properties of signal boundedness and asymptotic output tracking hold for $K = \mathrm{diag}\{k_1, k_2, \dots, k_M\} \neq I_M$ with $k_i > 0$ being constant, that is, we have the gain margin result:

Proposition 1. *A continuous-time multivariable direct MRAC system, designed based on the LDS decomposition of K_p, has gain margins $(0, +\infty)$ for k_i of the input control gain variation matrix $K = \mathrm{diag}\{k_1, k_2, \dots, k_M\}$.*

Proof: To prove that the closed-loop system with the adaptive controller designed for $K = I_M$ retains the desired performance for $K \neq I_M$, we need to prove that the assumptions under which the adaptive controller is designed for $K = I_M$ are satisfied.

Since $k_i > 0$, from (16), we have $\bar{\Delta}_i \neq 0$, and $\mathrm{sign}[\bar{\Delta}_i] = \mathrm{sign}[\Delta_i]$, that is, the presence of K does not violate the assumptions of nonzero leading principle minors and the knowledge of their signs of the high frequency gain matrix, based on which the MRAC scheme is designed for $K = I_M$. To be precise, the design parameter D_s in (11) is not affected by a gain matrix $K \neq I_M$. Therefore, for $k_i \in (0, +\infty)$, $i = 1, \dots, M$, the controller (7) with the adaptive laws (10)–(12) still ensures the desired system performance, and the MRAC system has gain margin $(0, +\infty)$. ∇

Remark 1: The conclusion in Proposition 1 reduces to the SISO case [8] when $M = 1$, that is, continuous-time SISO direct MRAC system has gain margin $(0, +\infty)$. □

IV. Gain Margin Analysis of Discrete-Time MIMO Direct MRAC Systems

While discrete-time multivariable MRAC shares similar controller structure and matching conditions as the continuous-time MRAC schemes, it has different stability characterization, which leads to different signal filters and adaptation gains, as well as stability and robust analysis. Besides, there are extra assumptions on plant models. We present the gain margin result for the MRAC design based on the LDS decomposition of K_p.

A. *Design Based on the LDS Decomposition of* K_p

As described in Section III-B, for $K = I_M$, when the plant parameters in $P(z)$ and $Z(z)$, the polynomial $\Lambda(z)$, and the modified interactor matrix $\xi_m(z)$ are specified, the controller (5) with the controller parameters $\Theta_i^*, i = 1, 2, 20, 3$ computed from the matching equation (6) with D replaced by z, is applied to the plant to achieve the control objective.

Without knowledge of the plant parameters, by following a similar procedure of derivation as in Section III with the only difference to be replacing s (or D) by z, we can obtain the estimation error model (9), based on which we choose the adaptive laws:

$$\theta_i(t+1) - \theta_i(t) = -\frac{\Gamma_{\theta i}\epsilon_i(t)\eta_i(t)}{m^2(t)}, \ i = 1, \ldots, M \quad (18)$$

$$\Theta^T(t+1) - \Theta^T(t) = -\frac{D_s\epsilon(t)\zeta^T(t)}{m^2(t)} \quad (19)$$

$$\Psi(t+1) - \Psi(t) = -\frac{\Gamma\epsilon(t)\xi^T(t)}{m^2(t)} \quad (20)$$

where $0 < \Gamma_{\theta i} = \Gamma_{\theta i}^T < 2I_{i-1}$, $0 < \Gamma = \Gamma^T < 2I_M$, and $D_s = \mathrm{diag}\left\{\mathrm{sign}[\Delta_1]\gamma_1, \ldots, \mathrm{sign}\left[\frac{\Delta_M}{\Delta_{M-1}}\right]\gamma_M\right\}$ with $\gamma_i > 0$, as in (4), is chosen to satisfy

$$0 < D_s U^T D_s^{-1} D^* U D_s < 2I_M, \quad (21)$$

that is, $\gamma_i \in (0, \gamma_i^0)$ for some $\gamma_i^0 > 0$, $i = 1, 2, \ldots, M$.

For $K = I_M$, the controller (7) with the adaptive laws (18)–(20) ensures closed-loop signal boundedness and asymptotic output tracking, $\lim_{t\to\infty}(y(t) - y_m(t)) = 0$ [10].

B. *Gain Margin Analysis*

In the presence of $K \neq I_M$ with $k_i > 0$ being constant and within some ranges, the same desired closed-loop system performance also holds, that is, we have the GM result:

Proposition 2. *A discrete-time multivariable direct MRAC system, designed based on the LDS decomposition of* K_p, *has gain margins* $\left(0, k_i^{GM}\right]$ *for* k_i *of the input control gain variation matrix* $K = \mathrm{diag}\{k_1, k_2, \ldots, k_M\}$, *where* $K^{GM} = \mathrm{diag}\{k_1^{GM}, k_2^{GM}, \ldots, k_M^{GM}\}$ *satisfies*

$$0 < D_s K^{GM} U^T (K^{GM})^{-1} D_s^{-1} D^* U K^{GM} D_s < 2I_M \quad (22)$$

for D_s *chosen to meet the condition (21).*

Proof: From Lemma 1, the presence of a gain matrix $K \neq I_M$ leads to the new high frequency gain matrix K_pK with its LDU decomposition as $K_pK = \bar{L}\bar{D}^*\bar{U}$ for $\bar{D}^* = D^*K$ from the LDU decomposition of K_p, $K_p = LD^*U$, that is, the sign information of the leading principle minors of K_pK is the same as that of K_p so that the adaptive law with D_s chosen for $K = I_M$ can still be used for $K \neq I_M$.

However, to ensure the closed-loop signal boundedness and asymptotic output tracking, the new condition for $K \neq I_M$, similar to (21) for $K = I_M$, is $0 < D_s\bar{U}^T D_s^{-1}\bar{D}^*\bar{U}D_s < 2I_M$, which needs to be satisfied for the chosen D_s for (21). From Lemma 1, it is equivalent to

$$0 < D_s K U^T K^{-1} D_s^{-1} D^* U K D_s < 2I_M, \quad (23)$$

from which it can be seen that for k_i of K, there is an upper bound $k_i^{GM} > 0$, which depends on γ_j, k_j, $j = 1, 2, \ldots, i-1$, d_j^*, d_j^0, $j = 1, 2, \ldots, i$ and the nonzero elements of U, with $K^{GM} = \mathrm{diag}\{k_1^{GM}, k_2^{GM}, \ldots, k_M^{GM}\}$ satisfying (22) such that the closed-loop performance is still achieved with the control gain variation $K \neq I_M$. That is, the gain margins are $\left(0, k_i^{GM}\right]$. ∇

Remark 2: The gain margin result in Proposition 2 reduces to the SISO case when $M = 1$, that is, a discrete-time SISO direct MRAC system has gain margin $\left(0, \frac{k_p^0}{|k_p|}\right]$, where k_p^0 is the known upper bound of the magnitude of the plant high frequency gain k_p [8]. □

Remark 3: It is desirable to have an explicit expression for the design parameters γ_i, $i = 1, 2, \ldots, M$ in (21), and K^{GM} in (22). The case for $M = 1$ is studied in [8]. For the general case $M > 1$, an explicit solution becomes complicated and may not exist because of the coupling of γ_i with each other and the unknown form of U. The results for $M = 2$ is provided here as a demonstration of the GM result presented in Proposition 2.

Assume the nonzero off-diagonal element of the unity upper triangular matrix U is a. For $a = 0$, the design parameters γ_1 and γ_2 must be chosen to satisfy

$$0 < \gamma_1 < \frac{2}{d_1^0}, \quad 0 < \gamma_2 < \frac{2}{d_2^0}. \quad (24)$$

For $a \neq 0$, the same range for γ_1 in (24) holds, and

$$0 < \gamma_2 < \frac{\alpha(\gamma_1)d_2^0 + \sqrt{\alpha(\gamma_1)\beta(\gamma_1)}}{4a^2 d_1^0}, \quad (25)$$

where d_1^0 and d_2^0, assumed to be known, are the upper bounds of $|d_1^*|$ and $|d_2^*|$, and $\alpha(\gamma_1) = \gamma_1(d_1^0\gamma_1 - 2)$, $\beta(\gamma_1) = \alpha(\gamma_1)(d_2^0)^2 - 16a^2 d_1^0$.

The gain margin results for $a = 0$ are:

$$0 < k_1 \leq \frac{d_1^0}{|d_1^*|}, \quad 0 < k_2 \leq \frac{d_2^0}{|d_2^*|}. \quad (26)$$

For $a \neq 0$, the same range for k_1 in (26) holds, and

$$0 < k_2 \leq \frac{d_1^0}{|d_1^*|} \frac{\bar{\alpha}(k_1\gamma_1)|d_2^*| + \sqrt{\bar{\alpha}(k_1\gamma_1)\bar{\beta}(k_1\gamma_1)}}{\alpha(\gamma_1)d_2^0 + \sqrt{\alpha(\gamma_1)\beta(\gamma_1)}} \quad (27)$$

where $\bar{\alpha}(k_1\gamma_1) = k_1\gamma_1(|d_1^*|k_1\gamma_1 - 2)$, and $\bar{\beta}(k_1\gamma_1) = \bar{\alpha}(k_1\gamma_1)|d_2^*|^2 - 16a^2|d_1^*|$. [2] □

Remark 4: When K_p is lower triangular, from its LDU decomposition, $K_p = LD^*U$, we have $U = I_M$, and (21) is equivalent to $0 < D^*D_s < 2I_M$, that is, $0 < \mathrm{diag}\{|d_1^*|\gamma_1, |d_2^*|\gamma_2, \ldots, |d_M^*|\gamma_M\} < 2I_M$. It is satisfied if

$$\mathrm{diag}\{\gamma_1, \gamma_2, \ldots, \gamma_M\} < \mathrm{diag}\left\{\frac{2}{d_1^0}, \frac{2}{d_2^0}, \ldots, \frac{2}{d_M^0}\right\}. \quad (28)$$

For the case when $K \neq I_M$, the inequality in (23) yields $0 < \mathrm{diag}\{|d_1^*|k_1\gamma_1, |d_2^*|k_2\gamma_2, \ldots, |d_M^*|k_M\gamma_M\} < 2I_M$,

[2] The results in (24)–(27) can be verified by converting the matrix inequalities in (21) and (23) into scalar inequalities and solving them.

which, from (28), is satisfied if

$$0 < \mathrm{diag}\{k_1, k_2, \ldots, k_M\} \le \mathrm{diag}\left\{\frac{d_1^0}{|d_1^*|}, \frac{d_2^0}{|d_2^*|}, \ldots, \frac{d_M^0}{|d_M^*|}\right\}.$$

Therefore, we have the gain margin result

$$k_i \in \left(0,\ \frac{d_i^0}{|d_i^*|}\right],\ i = 1, 2, \ldots, M, \tag{29}$$

where d_i^0, assumed to be known, is the upper bound of $|d_i^*|$ such that $0 < |d_i^*| \le d_i^0$. □

Remark 5: For a direct discrete-time MRAC system, the gain margin is $(0, k_i^0]$, where k_i^0 can be made large by reducing the adaptation gain. This can been seen from (4) and (23). As D^* and U are constant matrices, we can only increase k^0 by reducing the design parameters $\gamma_i > 0$, which reduces the adaptation speed for the controller parameters $\Theta(t)$ in (19). Therefore, the gain margin can be enlarged by assuming larger $d_i^0,\ i = 1, 2, \ldots, M$ in the system design process, while at the same time maintaining the inequality in (23). This is more clear in (28) and (29) for $U = I_M$. □

V. Gain Margins of Inidirect MRAC Systems

Indirect multivariable MRAC schemes are of interest because there are less parameters to be estimated than those of direct MRAC. Moreover, the plant parameters carry more physical meanings than controller parameters, and it is more natural and practical to expect a priori knowledge about plant parameters than that of controller parameters, which further reduces computational burden. In this section, we present the design of continuous-time and discrete-time multivariable indirect MRAC schemes followed by the GM analysis in a unified framework.

A. *Elliott and Wolovich's Algorithm*

In [2], Elliott and Wolovich used the left coprime polynomial matrix decomposition of the plant transfer matrix in developing indirect adaptive control strategies, since this representation of the multivariable plant can be estimated by input and output data. That is, let $G(D) = P_l^{-1}(D)Z_l(D)$, where $Z_l(D)$ and $P_l(D)$ are left coprime with $P_l(D)$ being row reduced.

Assume the observability indices of $G(D)$, denoted as ν_i, $i = 1, 2, \ldots, M$, are known, and let $\nu = \max_{1\le i\le M} \nu_i$, which is the observability index of $G(D)$. Without loss of generality, assume the row degrees of $P_l^T(D)$ are $\partial_{ci}(P_l^T(D)) = \nu_i$, and the matrix $P_\nu \in \mathbb{R}^{M\times M}$, containing the coefficients of the D^{ν_i} term in each column of $P_l(D)$, is unity lower triangular. By filtering the input-output equation of the plant model, a parametrization linear in the unknown plant parameters contained in $Z_l(D)$ and $P_l(D)$ can be obtained, based on which standard adaptive estimation techniques can be used for estimation of the plant parameters.

The transfer matrix $W_m(D)$ of the reference model is chosen to be $W_m(D) = \xi_m^{-1}(D)$, and the controller structure is in the form of (5) with the parameters obtained from the plant-model matching equation:

$$\begin{aligned} &I_M - \Theta_1^{*T} F(D) - (\Theta_2^{*T} F(D) + \Theta_{20}^*)\hat{G}(D) \\ &= \Theta_3^* \xi_m(D)\hat{G}(D), \end{aligned} \tag{30}$$

where $\hat{G}(D) = \hat{P}_l^{-1}(D)\hat{Z}_l(D)$ with $\hat{P}_l(D)$ and $\hat{Z}_l(D)$ being the estimates of $P_l(D)$ and $Z_l(D)$, and $F(D)$ is defined as in Section III-B.

To solve the plant-model matching equation (30) online, we need to assume that $\hat{G}(D)$ at each time instant t has the same interactor matrix $\xi_m(D)$, and the corresponding estimated high frequency gain matrix $\hat{K}_p$ is nonsingular. To ensure this, we can use parameter projection techniques in the adaptive estimation of $P_l(D)$ and $Z_l(D)$. However, since it is a row by row estimation, the parameter projection ensuring nonsingular $\hat{K}_p$ is still a problem to be solved. Thus here we consider a special case, which is a direct expansion from the SISO indirect MRAC algorithms [8], [10].

B. *Design for a Special Class of Systems*

We make the assumptions that the degrees of the polynomial matrices $P_l(D)$ and $Z_l(D)$ are ν and m, $\nu > m$, which are known. Moreover, we assume the highest order coefficient matrix of $P_l(D)$ is the identity matrix I_M, and that of $Z_l(D)$ is diagonal and nonsingular, that is, the plant to be controlled has the following representation:

$$\begin{aligned} &(I_M D^\nu + P_{\nu-1}D^{\nu-1} + \cdots + P_0)[y](t) \\ &= (Z_m D^m + Z_{m-1}D^{m-1} + \cdots + Z_0)[u](t) \end{aligned} \tag{31}$$

where $P_i,\ Z_j \in \mathbb{R}^{M\times M},\ i = 0, 1, \ldots, \nu-1,\ j = 0, 1, \ldots, m-1$ are constant coefficient matrices with $Z_m = \mathrm{diag}\{z_{m1}, \ldots, z_{mM}\}$ for some $z_{mi} \ne 0,\ i = 1, 2, \ldots, M$. Therefore, the plant transfer matrix $G(D)$ has an interactor matrix $\xi_m(D) = d(D)I_M$ with $d(D)$ being a monic stable polynomial of degree $n^* = \nu - m$, and the plant high frequency gain matrix is $K_p = Z_m$ for $K = I_M$.

Plant model parametrization. To obtain a parametrization of the plant model (31), we filter both sides with $\frac{1}{\Lambda_e(D)}$ for a chosen monic stable polynomial $\Lambda_e(D)$ of degree ν, after ignoring exponentially decaying terms, we can obtain

$$\begin{aligned} \bar{y}(t) &\triangleq \frac{D^\nu}{\Lambda_e(D)}[y](t) \\ &= Z_0\frac{1}{\Lambda_e(D)}[u](t) + \cdots + Z_m\frac{D^m}{\Lambda_e(D)}[u](t) \\ &\quad - P_0\frac{1}{\Lambda_e(D)}[y](t) - \cdots - P_{\nu-1}\frac{D^{\nu-1}}{\Lambda_e(D)}[y](t) \\ &= [\theta_1^{*T}\zeta_1(t), \ldots, \theta_M^{*T}\zeta_M(t)]^T, \end{aligned} \tag{32}$$

where $\theta_i^*,\ \zeta_i(t) \in \mathbb{R}^{M(\nu+m)+1},\ i = 1, 2, \ldots, M$ with

$$\begin{aligned} \theta_i^* &= [Z_{0i}^r, \ldots, Z_{m-1i}^r, z_{mi}, -P_{0i}^r, \ldots, -P_{\nu-1i}^r]^T \\ \zeta_i(t) &= \left[\frac{1}{\Lambda_e(D)}[u]^T(t), \ldots, \frac{D^{m-1}}{\Lambda_e(D)}[u]^T(t), \frac{D^m}{\Lambda_e(D)}[u_i](t),\right. \\ &\quad \left.\frac{1}{\Lambda_e(D)}[y]^T(t), \ldots, \frac{D^{\nu-1}}{\Lambda_e(D)}[y]^T(t)\right]^T, \end{aligned} \tag{33}$$

and the superscript r denotes the ith row of the corresponding coefficient matrix, and $u_i(t)$ is the ith input to the plant.

To ensure a nonsingular estimation of Z_m, we need to make this assumption: the sign of z_{mi}, sign$[z_{mi}]$, is known, and so is the lower bound z_{mi0} of $|z_{mi}|$ such that $0 < z_{mi0} \leq |z_{mi}|, \ \ i = 1, 2, \ldots, M$.

Error model and adaptive law. Based on the error equations $\epsilon_i(t) = \tilde{\theta}_i^T(t)\zeta_i(t)$ with $\tilde{\theta}_i(t) = \theta_i(t) - \theta_i^*$, we choose the gradient adaptive laws with parameter projection for $\theta_i(t)$:

$$\left.\begin{array}{r} \dot{\theta}_i(t) \\ \theta_i(t+1) - \theta_i(t) \end{array}\right\} = -\frac{\Gamma_i \zeta_i(t) \epsilon_i(t)}{m^2(t)} + f_i(t), \qquad (34)$$

$i = 1, 2, \ldots, M$, where $m^2(t) = 1 + \sum_{i=1}^{M} \zeta_i^T(t)\zeta_i(t)$, and the adaptive gain matrix $\Gamma_i = \text{diag}\{\Gamma_{i1}, \gamma_i, \Gamma_{i2}\}$ with $\Gamma_{i1} \in \mathbb{R}^{m\times m}$, $\gamma_i \in \mathbb{R}$, $\Gamma_{i2} \in \mathbb{R}^{\nu\times\nu}$, and for continuous-time case, $\Gamma_{i1} = \Gamma_{i1}^T > 0$, $\Gamma_{i2} = \Gamma_{i2}^T > 0$, $\gamma_i > 0$, while for discrete-time case, $0 < \Gamma_{i1} = \Gamma_{i1}^T < 2I_m$, $0 < \Gamma_{i2} = \Gamma_{i2}^T < 2I_\nu$, and $0 < \gamma_i < 2$. The parameter projection term $f_i(t)$ has the form $f_i(t) = [0_{1\times m}, f_{im+1}(t), 0_{1\times\nu}]^T$ with $f_{im+1}(t)$ designed to ensure the estimate of the $(m+1)$th component of $\theta_i(t)$, $\hat{z}_{mi}$, to be away from zero, that is, $|\theta_{im+1}(t)| = |\hat{z}_{mi}| \in [z_{mi0}, +\infty)$ (using the knowledge of $z_{mi0} \leq |z_{mi}|$, $i = 1, 2, \ldots, M$).

With the adaptive laws (34), the design equation (30), the control (5) applied to the plant (31) can achieve closed-loop signal boundedness and asymptotic output tracking.

C. Gain Margin Analysis

The same desired closed-loop performance holds for the case when a positive definite gain matrix $K > 0$ as in (2) is present at the control input, as long as the gains k_i are within some ranges, that is, we have the GM result:

Proposition 3. *The closed-loop system, consisting of the plant (31) and the controller (5), with the adaptive laws (34) and design equation (30), has gain margins* $\left[\frac{z_{mi0}}{|z_{mi}|}, +\infty\right)$ *for* k_i *of* $K = \text{diag}\{k_1, k_2, \ldots, k_M\}$*, where* z_{mi0}*, used in the adaptive law (34), are the known lower bounds of* $|z_{mi}|$ *such that* $0 < z_{mi0} \leq |z_{mi}|$*,* $i = 1, 2, \ldots, M$.

Proof: In the presence of K, the controlled plant is $y(t) = G(D)K[u](t)$, and its high frequency gain matrix is

$$K_pK = Z_MK = \text{diag}\{z_{m1}k_1, z_{m2}k_2, \ldots, z_{mM}k_M\}, \quad (35)$$

which needs to satisfy the assumptions under which the adaptive laws (34) are designed for $K = I_M$, in order for closed-loop stability and asymptotic tracking to be achieved. That is, for $K \neq I_M$, the entries of (35) must be greater than the assumed lower bounds z_{mi0}, i.e., $|z_{mi}k_i| \geq z_{mi0}$, from which we can obtain $k_i \in \left[\frac{z_{mi0}}{|z_{mi}|}, +\infty\right)$, $i = 1, 2, \ldots, M$. Therefore, we have the stated gain margin result. ∇

Remark 6: When $M = 1$, the MIMO plant (31) reduces to the SISO plant $P(D)[y](t) = k_pZ(D)[u](t)$. The GM result in Proposition 3 is a direct extension from the SISO result presented in [8]. □

Remark 7: For both continuous-time and discrete-time indirect MRAC designs, the gain margin is $[k_{i0}, \infty)$, where $k_{i0} > 0$ can be made small by reducing the parameter lower bound used in parameter projection of the adaptive laws (for plant identification) for avoiding control singularity. □

VI. Conclusions

In this paper, the gain margin problem has been studied for several multivariable model reference adaptive control (MRAC) systems: those with direct or indirect, continuous-time or discrete-time designs. For a direct continuous-time MRAC design, the gain margin is $(0, +\infty)$, while for a direct discrete-time design, the gain margin is finite with an upper bound that can be made large by reducing the adaptation gain. For indirect MRAC designs, in either continuous time or discrete time, the gain margin is infinity with a lower bound that can be made small by reducing the parameter lower bound used in parameter projection of the adaptive laws (for plant identification) for avoiding control singularity. In other words, the gain margins of MRAC systems can be enlarged by choosing proper design parameters, while ensuring both signal boundedness and asymptotic tracking. This indicates that the use of an MRAC scheme has a unique and significant advantage over a non-adaptive control scheme whose gain margin for signal boundedness is fixed (and is 1 for asymptotic tracking, that is, $K \neq I_M$ would lead to a non-zero tracking error). This advantage is important for aircraft flight control applications for which asymptotic tracking is a critical performance measure.

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