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Alternative Optimal Filter for Linear Systems with Multiple State and Observation Delays

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Abstract— **In this paper, the optimal filtering problem for linear systems with multiple state and observation delays is treated using the optimal estimate of the state transition matrix. As a result, the alternative optimal filter is derived in the form similar to the traditional Kalman-Bucy one, i.e., consists of only two equations, for the optimal estimate and the estimation error variance. This presents a significant advantage in comparison to the previously obtained optimal filter [1], which includes infinite or variable number of covariance equations, unboundedly growing as the filtering horizon tends to infinity. Performances of the two optimal filters are compared in example; the obtained results are discussed.**

I. INTRODUCTION

The optimal filtering problem for linear system states and observations without delays was solved in 1960s [2], and this closed form solution is known as the Kalman-Bucy filter. However, the related optimal filtering problem for linear states with delay has not been solved in a closed form, regarding as a closed form solution a closed system of a finite number of ordinary differential equations for any finite filtering horizon. The optimal filtering problem for time delay systems itself did not receive so much attention as its control counterpart, since most of the research was concentrated on the filtering problems with observation delays (the papers [3], [4], [5] could be mentioned to make a reference). A few particular cases, the optimal filtering problems for linear systems with state delay and/or multiple observation delays, have recently been solved in [6], [7], [8], [9], [1]. A Kalman-like estimator for linear systems with observation delay has recently been designed in [10]. The optimal filter for linear systems with multiple state and observation delays, derived in [1], has solved the same filtering problem as the present paper. However, that solution is not free of computational disadvantages: it includes a variable number of covariance equations, which unboundedly grows as the filtering horizon tends to infinity, and the structure of the covariance equations also varies with the number. There also exists a considerable bibliography related to the robust control and filtering problems for time-delay systems (such as [11]–[22]). A number of papers, published in 1970s, were dedicated to some particular optimal filtering problems for time-delay systems (see [23]). Comprehensive reviews of theory and algorithms for time delay systems are given in [24]–[30].

In this paper, the optimal filtering problem for linear systems with multiple state and observation delays is treated using the optimal estimate of the state transition matrix from the current time moment to the delayed ones. In doing so, the employed method closely resembles the well-known Smith predictor approach [31] (see [32] for more research on this resemblance). As a result, the optimal filter is derived in the form similar to the traditional Kalman-Bucy one, i.e., consists of only two equations, for the optimal estimate and the estimation error variance. This presents a significant advantage in comparison to the previously obtained optimal filter [1] consisting of the infinite number of covariance equations in the general case of non-commensurable delays, or a variable number of covariance equations, which unboundedly grows as the filtering horizon tends to infinity, in some particular cases. It should be further emphasized that, in contrast to the previously obtained results [6], [7], [8], [9], [1], this paper designs the optimal mean-square finite-dimensional filter for linear time-delay systems with arbitrary, even non-commensurable delays, in both state and observation equations.

Note that the approach based on the optimal estimation of the state transition matrix would be applicable to any system of state and observation equations with time delays, where the the optimal estimate of the state transition matrix is uncorrelated with the estimation error variance, including certain classes of nonlinear systems. The obtained results remain valid, if the time-delays in both state and observation equations are time-varying but non-random; neither design method, nor optimal filtering equations themselves are changed. However, if some system parameters are uncertain, application of joint optimal state filtering and parameter identification methods is needed.

Finally, performance of the designed alternative optimal filter for linear systems with multiple state and observation delays is compared in the illustrative example with the performance of the optimal filter obtained in [1]. The simulation results show an insignificant difference in values of the obtained estimates for both filters at the final simulation time.

The paper is organized as follows. Sections 2 and 3 present the filtering problem statement for a linear system with multiple state and observation delays and its solution, respectively. In Section 4, performance of the obtained alternative optimal filter for linear systems with with multiple state and observation delays is verified in the illustrative example

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against the optimal filter obtained in [1]. The simulation results are discussed in Conclusion.

II. Filtering Problem for Linear Systems with Multiple State and Observation Delays

Let (Ω, F, P) be a complete probability space with an increasing right-continuous family of σ-algebras $F_t, t \geq 0$, and let $(W_1(t), F_t, t \geq 0)$ and $(W_2(t), F_t, t \geq 0)$ be independent Wiener processes. The partially observed F_t-measurable random process $(x(t), y(t))$ is described by a delay differential equation for the system state

$$dx(t) = (\sum_{i=0}^{p} a_i(t)x(t-h_i))dt + b(t)dW_1(t), \; x(t_0) = x_0, \quad (1)$$

with the initial condition $x(s) = \phi(s)$, $s \in [t_0 - h, t_0]$, $h = \max(h_1, \ldots, h_p)$, and a delay differential equation for the observation process

$$dy(t) = (\sum_{j=0}^{q} A_j(t)x(t-\tau_j))dt + B(t)dW_2(t), \quad (2)$$

where $x(t) \in R^n$ is the state vector, $y(t) \in R^m$ is the observation process, $h_0 = \tau_0 = 0$, and $\phi(s)$ is a mean square piecewise-continuous Gaussian stochastic process (see [33] for definition) given in the interval $[t_0 - h, t_0]$ such that $\phi(s)$, $W_1(t)$, and $W_2(t)$ are independent. The system state $x(t)$ dynamics and the observations $y(t)$ depend on the current state $x(t)$, as well as on a bunch of delayed states $x(t-h_i)$ and $x(t-\tau_i)$, $h_i > 0$, $\tau_i > 0$, $i = 1, \ldots, p$, $j = 1, \ldots, q$, which actually make the system state space infinite-dimensional (see, for example, [26]). The vector-valued function $a_0(s)$ describes the effect of system inputs (controls and disturbances). It is assumed that at least one of the matrices $A_j(t)$ is not zero matrix, and $B(t)B^T(t)$ is a positive definite matrix. All coefficients in (1)–(2) are deterministic functions of appropriate dimensions.

The estimation problem is to find the best estimate of the system state $x(t)$ based on the observation process $Y(t) = \{y(s), 0 \leq s \leq t\}$, that minimizes the Euclidean 2-norm

$$J = E[(x(t) - \hat{x}(t))^T (x(t) - \hat{x}(t)) \mid F_t^Y]$$

at every time moment t. Here, $E[z(t) \mid F_t^Y]$ means the conditional expectation of a stochastic process $z(t) = (x(t) - \hat{x}(t))^T (x(t) - \hat{x}(t))$ with respect to the σ - algebra F_t^Y generated by the observation process $Y(t)$ in the interval $[t_0, t]$. As known [33], this optimal estimate is given by the conditional expectation

$$\hat{x}(t) = m(t) = E(x(t) \mid F_t^Y)$$

of the system state $x(t)$ with respect to the σ - algebra F_t^Y generated by the observation process $Y(t)$ in the interval $[t_0, t]$. The matrix functions

$$P(t) = E[(x(t) - m(t))(x(t) - m(t))^T \mid F_t^Y],$$

that is the estimation error variance, and

$$P(t, t - t_1) = E[(x(t) - m(t))(x(t - t_1) - m(t - t_1))^T \mid F_t^Y],$$

that is the covariance between the estimation error values at different time moments, $P(t,t) = P(t)$, are used to obtain a system of filtering equations.

The proposed solution to this optimal filtering problem is based on the formulas for the Ito differential of the conditional expectation $E(x(t) \mid F_t^Y)$ and its variance $P(t)$ (cited after [33]) and given in the following section.

III. Alternative Optimal Filter for Linear Systems with Multiple State and Observation Delays

The optimal filtering equations can be obtained using the formula for the Ito differential of the conditional expectation $m(t) = E(x(t) \mid F_t^Y)$ (see [33])

$$dm(t) = E(\varphi(x) \mid F_t^Y)dt + E(x[\varphi_1 - E(\varphi_1(x) \mid F_t^Y)]^T \mid F_t^Y) \times \big(B(t)B^T(t)\big)^{-1} (dy(t) - E(\varphi_1(x) \mid F_t^Y)dt), \quad (3)$$

where $\varphi(x)$ is the drift term in the state equation equal to $\varphi(x) = a_0(t) + \sum_{i=0}^{p} a_i(t)x(t-h_i)$ and $\varphi_1(x)$ is the drift term in the observation equation equal to $\varphi_1(x) = \sum_{j=0}^{q} A_j(t)x(t-\tau_j)$. Note that the conditional expectation equality $E(x(t-h) \mid F_t^Y) = E(x(t-h) \mid F_{t-h}^Y) = m(t-h)$ is valid for any $h > 0$, since, in view of a positive delay shift $h > 0$, the treated problem (1),(2) is a filtering problem, not a smoothing one, and, therefore, the formula (3) yields the optimal estimate $m(s)$ for any time s, $t_0 < s \leq t$, if the observations (2) are obtained until the current moment t (see [33], [8]). Upon performing substitution of the expressions for φ and φ_1 into (3) and taking into account the conditional expectation equality, the estimate equation takes the form

$$dm(t) = (\sum_{i=0}^{p} a_i(t)m(t-h_i))dt +$$
$$E(x(t)[\sum_{j=0}^{q} A_j(t)(x(t-\tau_j) - m(t-\tau_j))]^T \mid F_t^Y) \times$$
$$(B(t)B^T(t))^{-1}(dy(t) - (\sum_{j=0}^{q} A_j(t)m(t-\tau_j)dt)) =$$
$$(\sum_{i=0}^{p} a_i(t)m(t-h_i))dt +$$
$$(\sum_{j=0}^{q} E([x(t) - m(t)][x(t-\tau_j) - m(t-\tau_j)]^T \mid F_t^Y)A_j^T(t)) \times$$
$$(B(t)B^T(t))^{-1}(dy(t) - (\sum_{j=0}^{q} A_j(t)m(t-\tau_j)dt)) =$$
$$(\sum_{i=0}^{p} a_i(t)m(t-h_i))dt + \quad (4)$$
$$(\sum_{j=0}^{q} P(t, t-\tau_j)A_j^T(t))(B(t)B^T(t))^{-1} \times$$
$$(dy(t) - (\sum_{j=0}^{q} A_j(t)m(t-\tau_j))dt),$$

where $P(t,t-\tau_j) = E[(x(t)-m(t))(x(t-\tau_j)-m(t-\tau_j))^T \mid F_t^Y]$.

To compose a system of the filtering equations, the equations for the conditional expectations $E([x(t)-m(t)][(x(t-h_i)-m(t-h_i))]^T \mid F_t^Y)$, $i=0,\dots,p$, and $E([x(t)-m(t)][(x(t-\tau_j)-m(t-\tau_j))]^T \mid F_t^Y)$, $j=0,\dots,q$, should be obtained. This can be done using the equation (1) for the state $x(t)$, the equation (4) for the estimate $m(t)$, and the formula for the Ito differential of a product of two processes satisfying Ito differential equations (see [33]):

$$d(z_1 z_2^T) = z_1 dz_2^T + (z_2 dz_1^T)^T + (1/2)[y_1 \nu y_2^T + y_2 \nu y_1^T]dt. \quad (5)$$

Here, the stochastic process z_1 satisfies the equation

$$dz_1 = x_1 dt + y_1 dw_1,$$

the stochastic process z_2 satisfies the equation

$$dz_2 = x_2 dt + y_2 dw_2,$$

and ν is the covariance intensity matrix of the Wiener vector $[w_1\ w_2]^T$.

Let us obtain the formula for the Ito differential of the general expression $P(t,t-t_1) = E([x(t)-m(t)][x(t-t_1)-m(t-t_1)]^T \mid F_t^Y)$, where $t_1>0$ is an arbitrary delay, not necessarily equal to τ. Upon representing $P(t,t-t_1)$ as $P(t,t-t_1) = E([x(t)(x(t-t_1))^T] \mid F_t^Y) - m(t)m(t-t_1)$, using first $x(t)$ as z_1 and $x(t-t_1)$ as z_2 and then $m(t)$ as z_1 and $m(t-t_1)$ as z_2 in the formula (5), taking into account independence of the Wiener processes W_1 and W_2 in the equations (1) and (2), and finally subtracting the second derived equation from the first one, the following formula is obtained

$$dP(t,t-t_1)/dt = \sum_{i=0}^{p} a_i(t)P(t-h_i,t-t_1)+ \quad (6)$$
$$\sum_{i=0}^{p} P(t,t-t_1-h_i)a_i^T(t-t_1)+$$
$$(1/2)[b(t)b^T(t-t_1)+b(t-t_1)b^T(t)]-$$
$$(1/2)[(\sum_{j=0}^{q} P(t,t-\tau_j)A_j^T(t))(B(t)B^T(t))^{-1}\times$$
$$B(t)B^T(t-t_1)(B(t-t_1)B^T(t-t_1))^{-1}\times$$
$$(\sum_{j=0}^{q} A_j(t-t_1)P^T(t-t_1,t-t_1-\tau_j))-$$
$$(\sum_{j=0}^{p} P(t-t_1,t-t_1-\tau_j)A_j^T(t-t_1))(B(t-t_1)B^T(t-t_1))^{-1}\times$$
$$B(t-t_1)B^T(t)(B(t)B^T(t))^{-1}(\sum_{j=0}^{q} A_j(t)P^T(t,t-\tau_j))].$$

Equating $t_1=0$ in (6) yields the equation for the conditional error variance $P(t) = E[(x(t)-m(t))(x(t)-m(t))^T \mid F_t^Y]$

$$dP(t)/dt = \sum_{i=0}^{p} a_i(t)P(t-h_i,t) + \sum_{i=0}^{p} P(t,t-h_i)a_i^T(t)+ \quad (7)$$
$$b(t)b^T(t) - [(\sum_{j=0}^{q} P(t,t-\tau_j)A_j^T(t))\times$$
$$(B(t)B^T(t))^{-1}(\sum_{j=0}^{q} A_j(t)P^T(t,t-\tau_j))].$$

Consider alternative representations for the terms $P(t,t-t_1) = E((x(t)-m(t))(x(t-t_1)-m(t-t_1))^T \mid F_t^Y)$, $t_1=h_i$, $h=1,\dots,p$, and $t_1=\tau_j$, $j=1,\dots,q$, in the last equation. Denote as $x_1(t)$ the solution of the equation $\dot{x}_1(t) = a_0(t) + \sum_{i=0}^{p} a_i(t)x(t-h_i)$ with the initial condition $x_1(t_0)=x_0$. Then, the solution $x(t)$ of the equation (1) can be represented in the form

$$x(t) = x_1(t) + \int_{t_0}^{t} b(s)dW_1(s). \quad (8)$$

Let us now introduce the matrix $\Phi(s,t)$, which would serve as a nonlinear analog of the state transition matrix in the inverse time. Indeed, define $\Phi(s,t)$ as a such matrix that the equality $\Phi(s,t)x_1(t) = x_1(s)$, $s\le t$, holds for any $t,s\ge t_0$ and $s\le t$. Naturally, $\Phi(s,t)$ can be defined as the diagonal matrix with elements equal to $x_{1_i}(s)/x_{1_i}(t)$, where $x_{1_i}(t)$ are components of the vector $x_1(t)$, if $x_{1_i}(t)\neq 0$ almost surely. The definition of $\Phi(s,t)$ for the case of $x_{1_i}(t)=0$ will be separately considered below.

Hence, using the representation (8) and the notion of the matrix $\Phi(s,t)$, the term $P(t,t-t_1) = E((x(t)-m(t))(x(t-t_1)-m(t-t_1))^T \mid F_t^Y)$ can be transformed as follows

$$E((x(t)-m(t))(x(t-t_1)-m(t-t_1))^T \mid F_t^Y) =$$
$$E((x(t)-m(t))(x(t-t_1))^T \mid F_t^Y) =$$
$$E((x(t)-m(t))(x_1(t-t_1)+\int_{t_0}^{t-t_1} b(s)dW_1(s))^T \mid F_t^Y) =$$
$$E((x(t)-m(t))(x_1(t-t_1))^T \mid F_t^Y) =$$
$$E((x(t)-m(t))(\Phi(t-t_1,t)x_1(t))^T \mid F_t^Y) =$$
$$E((x(t)-m(t))(x_1(t))^T \mid F_t^Y)(\Phi^*(t-t_1,t))^T =$$
$$E((x(t)-m(t))(x_1(t)+\int_{t_0}^{t-t_1} b(s)dW_1(s))^T \mid F_t^Y)\times$$
$$(\Phi^*(t-t_1,t))^T = E((x(t)-m(t))(x(t))^T \mid F_t^Y)\times$$
$$(\Phi^*(t-t_1,t))^T = P(t)(\Phi^*(t-t_1,t))^T. \quad (9)$$

Here, $P(t) = E((x(t)-m(t))(x(t)-m(t))^T \mid F_t^Y)$ is the conditional error variance and $\Phi^*(t-t_1,t)$ is the state transition matrix in the inverse time for the process $x_1^*(t)$, that is the solution of the equation $\dot{x}_1^*(t) = \sum_{i=0}^{p} a_i(t)x_1^*(t-h_i))$ with the initial condition $x_1^*(t_0) = m_0 = E(x(t_0) \mid F_{t_0}^Y)$. Note that the transition from the third to fourth line in (9) is valid in view of independence of the conditional error variance $P(t)$ from both, $x(t)$ and $m(t)$, as it follows from the filtering equations in [1]. This is the same situation that takes place in the Kalman-Bucy filter [2].

Let us now define the matrix $\Phi(t-t_1,t)$ in the case of $x_{1_i}(t)=0$ almost surely for one of the components of $x_1(t)$.

Then, the corresponding diagonal entry of $\Phi_{ii}(t-t_1,t)$ can be set to 0 for any $h>0$, because, for the component $x_i(t)$,

$$E(x_i(t)(x_j(t-t_1)-m_j(t-t_1)) \mid F_t^Y) =$$
$$E((x_{1_i}(t)+(\int_{t_0}^t b(s)dW_1(s))_i)\times$$
$$(x_j(t-t_1)-m_j(t-t_1)) \mid F_t^Y) =$$
$$E(x_{1_i}(t)(x_j(t-t_1)-m_j(t-t_1)) \mid F_t^Y) = 0,$$

almost surely for any $j=1,\ldots,m$. Hence, the definition $\Phi_{ii}(s,t)=0$ for any $s<t$, if $x_{1_i}(t)=0$, leads to the same result as in the equation (7) and can be employed. The diagonal element $\Phi^*_{ii}(s,t)$ of the matrix $\Phi^*(s,t)$ is defined accordingly and set to 0 for any $s<t$, if the corresponding component of the process $x_1^*(t)$ is equal to zero at the moment t, $x^*_{1_i}(t)=0$.

Thus, in view of the transformation (9), the equation (4) for the optimal estimate takes the form

$$dm(t) = (\sum_{i=0}^{p} a_i(t)m(t-h_i))dt+ \tag{10}$$
$$(\sum_{j=0}^{q} P(t)(\Phi^*(t-\tau_j,t))^T A_j^T(t))(B(t)B^T(t))^{-1}\times$$
$$(dy(t)-(\sum_{j=0}^{q} A_j(t)m(t-\tau_j)dt),$$

with the initial condition $m(t_0)=E(x(t_0)\mid F_{t_0}^Y)$.

To compose a system of the filtering equations, the equation (10) should be complemented with the equation for the error variance $P(t)$. In view of the transformation (9), the equation (7) for the error variance takes the form

$$dP(t)/dt = \sum_{i=0}^{p} a_i(t)\Phi^*(t-h_i,t)P(t)+ \tag{11}$$
$$\sum_{i=0}^{p} P(t)(\Phi^*(t-h_i,t))^T a_i^T(t)+$$
$$b(t)b^T(t)-[(\sum_{j=0}^{q} P(t)(\Phi^*(t-\tau_j,t))^T A_j^T(t))\times$$
$$(B(t)B^T(t))^{-1}(\sum_{j=0}^{q} A_j(t)(\Phi^*(t-\tau_j,t))P(t))].$$

The equation (11) should be complemented with the initial condition
$P(t_0)=E[(x(t_0)-m(t_0)(x(t_0)-m(t_0)^T \mid F_{t_0}^Y]$.

By means of the preceding derivation, the following result is proved.

Theorem 1. The optimal finite-dimensional filter for the linear state with multiple delays (1) over the linear observations with multiple delays (2) is given by the equation (10) for the optimal estimate $m(t)=E(x(t)\mid F_t^Y)$ and the equation (11) for the estimation error variance $P(t)=E[(x(t)-m(t))(x(t)-m(t))^T \mid F_t^Y]$.

In the next section, performance of the designed optimal filter is verified against the optimal filter for linear systems with multiple state and observation delays, that has recently been obtained in [1] in the form of a set of equations for the optimal state estimate and error covariances, whose number is infinite in the general case of non-commensurable delays, or grows as the current filtering horizon tends to infinity, in some particular cases.

IV. EXAMPLE

This section presents an example of designing the alternative optimal filter for linear systems with multiple state and observation delays and comparing it to the optimal filter for linear multiple time-delay systems, that has recently been obtained in [1].

Let the unobserved state $x(t)$ with delay be given by

$$\dot{x}(t)=x(t-20), \quad x(s)=\phi(s),\ s\in[-20,0], \tag{12}$$

where $\phi(s)=N(0,1)$ for $s\le 0$, and $N(0,1)$ is a Gaussian random variable with zero mean and unit variance. The observation process is given by

$$y(t)=x(t)+x(t-20)+\psi(t), \tag{13}$$

where $\psi(t)$ is a white Gaussian noise, which is the weak mean square derivative of a standard Wiener process (see [33]). The equations (12) and (13) present the conventional form for the equations (1) and (2), which is actually used in practice [34].

The filtering problem is to find the optimal estimate for the linear state with delay (12), using the linear observations with multiple delays (13) confused with independent and identically distributed disturbances modeled as white Gaussian noises. The filtering horizon is set to $T=40$.

The filtering equations (10),(11) take the following particular form for the system (12),(13)

$$\dot{m}(t)=m(t-20)+P(t)\times \tag{14}$$
$$(\Phi^*(t-20,t)+1)[y(t)-m(t)-m(t-20)],$$

with the initial condition $m(s)=E(\phi(s))=0$, $s\in[-20,0)$, and $m(0)=E(\phi(0)\mid y(0))=m_0$, $s=0$;

$$\dot{P}(t)=2P(t)(\Phi^*(t-20,t))-P^2(t)(\Phi^*(t-20,t)+1)^2, \tag{15}$$

with the initial condition $P(0)=E((x(0)-m(0))^2\mid y(0))=R_0$. The auxiliary variable $\Phi^*(t-20,t)$ is equal to $\Phi^*(t-20,t)=x^*(t-20)/x^*(t)$, where $x_1^*(t)$ is the solution of the equation

$$\dot{x}_1^*(t)=x_1^*(t-20),$$

with the initial condition $x_1^*(s)=E(\phi(s))=0$, $s\in[-20,0)$, and $x_1^*(0)=m_0$, $s=0$.

The estimates obtained upon solving the equations (14),(15) are compared to the estimates satisfying the optimal filtering equations for linear systems with multiple state and observation delays, that have recently been obtained in [1], which take the following particular form for the system (12),(13)

$$\dot{m}_1(t) = m_1(t-20) + (P_{0,0} + P_{1,0})[y(t) - m_1(t) - m_1(t-20)], \quad (16)$$

with the initial condition $m(s) = E(\phi(s)) = 0$, $s \in [-5, 0)$ and $m(0) = E(\phi(0) \mid y(0)) = m_0$, $s = 0$;

$$\dot{P}_{0,0}(t) = P_{1,0}^T(t) + P_{1,0}(t) - (P_{0,0}(t) + P_{1,0})^2, \quad (17)$$

$$\dot{P}_{1,0}(t) = P_{0,0}(t-20) + P_{1,1}(t) - \quad (18)$$

$$(P_{0,0}(t) + P_{1,0}(t))(P_{0,0}(t-20) + P_{1,0}(t-20)),$$

with the initial condition $P_{0,0}(0) = E((x(s) - m_1(s))^2 \mid y(0)) = R_0$. The initial conditions for the covariances $P_{1k}(s)$, $k = 0, 1$, in the intervals $s \in [20k, 20(k+1)]$, are assigned as $P_{10}(s) = R_0/2$ and $P_{11}(s) = 0$.

Numerical simulation results are obtained solving the systems of filtering equations (14)–(15) and (16)–(18). The obtained values of the estimates $m(t)$ and $m_1(t)$ satisfying the equations (14) and (16), respectively, are compared to the real values of the state variable $x(t)$ in (12).

For each of the two filters (14)–(15) and (16)–(18) and the reference system (19) involved in simulation, the following initial values are assigned: $x_0 = 1$, $m_0 = 10$, $R_0 = 100$. Simulation results are obtained on the basis of a stochastic run using realizations of the Gaussian disturbances $\psi(t)$ in (13) generated by the built-in MatLab white noise function.

The following graphs are obtained: graphs of the reference state $x(t)$ (12), and the estimate $m(t)$, satisfying the equations (14), and the estimation error $x(t) - m(t)$ are shown in Fig. 1; graphs of the reference state $x(t)$ (12), and the estimate $m_1(t)$, satisfying the equations (16), and the estimation error $x(t) - m_1(t)$ are shown in Fig. 2. The graphs are shown on the simulation interval from $t_0 = 0$ to $T = 40$.

Thus, it can be concluded that the obtained alternative optimal filter (14)–(15) yields virtually indistinguishable values of the estimate $m(t)$ at the final simulation time $T = 40$, in comparison to the optimal filter (16)–(18) obtained in [1]. A larger divergence of values of the estimate (14), observed near $T = 20$, appears due to MatLab discretization scheme, which poorly handles the division by numbers close to zero employed for calculating the matrix $\Phi(t-20, t)$ in (14). However, the significant advantage of the alternative optimal filter is that it consists of the only two equations, whose number and structure do not change as the filtering horizon tends to infinity. In contrast, the optimal filter (16)–(18), obtained in [1], would include an unboundedly growing number of covariance equations, as tends time to infinity.

The conducted simulation provides only numerical comparison between two different forms of the optimal filter for the system (12),(13), whereas the comparison of the optimal filter to some approximate filters, such as extended Kalman filters (EKF), and the corresponding graphic representation, revealing a better performance of the optimal filter, can be found in [1].

V. Conclusions

The simulation results show that the values of the estimate calculated by using the obtained alternative optimal filter for linear systems with multiple state and observation delays are only insignificantly different from the estimate values provided by the optimal filter previously obtained in [1]. Moreover, the estimates produced by both optimal filters asymptotically converge to the real values of the system state as time tends to infinity. The significant advantage of the alternative filter is that it consists of only two equations, for the optimal estimate and the estimation error variance, whose number and structure do not change as the filtering horizon tends to infinity. On the contrary, the previously obtained optimal filter of [1] includes the infinite number of covariance equations in the general case of non-commensurable delays, or a variable number of covariance equations, which unboundedly grows as the filtering horizon tends to infinity, in some particular cases. Moreover, the structure of the covariance equations in [1] also varies with the number. The obtained alternative filter is free from those complications and provides the equally good quality of the state estimation.

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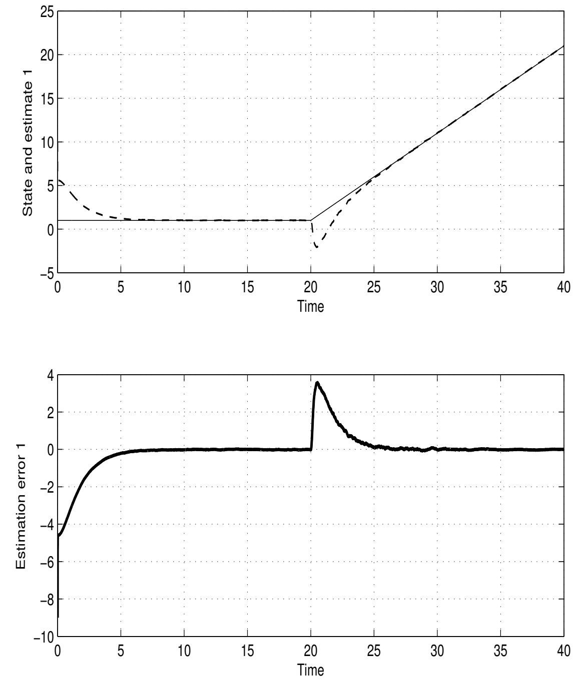

Fig. 1. Graphs of the reference state variable (12) $x(t)$ (solid line above), alternative optimal state estimate (14) $m(t)$ (dashed line above), estimation error $x(t)-m(t)$ (below) in the simulation interval $[0,40]$.

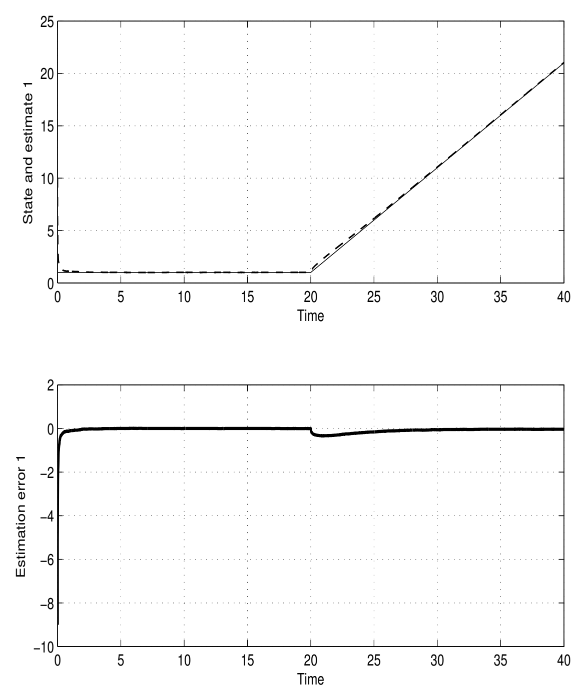

Fig. 2. Graphs of the reference state variable (12) $x(t)$ (solid line above), nominal optimal state estimate (16) $m(t)$ (dashed line above), estimation error $x(t)-m(t)$ (below) in the simulation interval $[0,40]$.