Proceedings of the
47th IEEE Conference on Decision and Control
Cancun, Mexico, Dec. 9-11, 2008

Probabilistic Analysis of the Rapid Convergence of a Class of Progressive Second Price Auctions

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Abstract— **A quantized progressive second price auction mechanism called ADQ-PSP is presented which improves upon the performance of the so-called Q-PSP mechanism presented in [1], [2]. The Q-PSP mechanism was developed for the fast convergence properties that can be proven for it subject to the assumption that all agents share similar demand (i.e. marginal valuation) functions. For the ADQ-PSP mechanism applied to agent populations with randomly distributed demand functions it is shown in this paper that the states (i.e. bid prices and quantities) of the corresponding dynamical systems rapidly converge with high probability to a quantized (Nash) equilibrium with a common price for all agents. This property holds for ensembles containing populations which have significantly different demand functions. Furthermore, the convergence rates are independent of the number of quantization levels. Finally, the ϵ-efficiency of the quantized equilibria is established and numerical examples are given.**

I. Introduction

This work is motivated by pricing problems for communication networks. Flat rate pricing is a scheme that charges a fixed fee for service regardless of usage and it is widely employed not only in traditional telecommunications, electricity and transportation areas, but also, for instance, for pricing Internet access. However, it has many well known drawbacks (see e.g. [1], [3], [4]); this has led to the study of game theoretic methods for the design of market pricing mechanisms because each agent (consumer) in such networks will typically have its own demand (marginal valuation) function and revenue function.

The classical Vickrey-Clarke-Groves (VCG) mechanism [5] has been applied to distributed large-scale systems due to the fact that for VCG (i) incentive compatibility (i.e. an agent's bids corresponds to true valuations) is a dominant strategy and (ii) knowledge of other agents' valuations (demand functions) cannot improve an agent's expected utility. This reduces the complexity of auction mechanism design and of the decision making itself. The modification and generalization of the VCG mechanism has been studied in [3], [6], and [7], among others. Recently, a non-VCG efficient mechanism design was proposed by R. Jain [8] and R. Jain and P. Varaiya [9] to achieve efficiency (i.e achievement of social optima), budget-balance and individual rationality by compromising on incentive compatibility in combinatorial (i.e. agents may bid on combinations of items) and double (i.e. sellers and buyers submit bids) auctions.

The convergence analysis of bidding processes usually focuses upon the existence and stability of those Nash equilibria which can in principle be obtained from the differential equations describing the time evolution of allocated resources and agents' payments (see e.g. [10], [11]). Furthermore, approximations have been used [12] to obtain computational tractability for the linear programming and integer programming problems of equilibrium computation which would otherwise be NP-hard. In general, convergence rates of auction processes have not been thoroughly studied, an exception being the PSP analysis [4] where the rate depends upon the reserve price, see below.

In [3], [4], a so-called Progressive Second Price auction mechanism (PSP) was proposed for dynamic market-pricing and the allocation of variable-size resources in communication networks. This mechanism efficiently and dynamically allocates a divisible resource to price-taking agents capable of exerting their market power to generate successive price and quantity bids. It was proved in [4] that this VCG-like PSP mechanism has the desirable properties of incentive compatibility (i.e truthful bidding) and efficiency (i.e achievement of social optima), in that an (ϵ-Nash) equilibrium is achieved if all agents use truthful ϵ-best responses to the strategy profile (i.e. bid set) of their opponents at each iteration. However, the rate of convergence is inversely proportional to ϵ which corresponds to a bid fee. P. Maille and B. Tuffin presented a one-step bid version of PSP in [13] to avoid the slow convergence of the PSP algorithm; but the accelerated convergence is achieved at a computational cost, furthermore it requires a large message space. This is the case since agents are required to use a high dimensional bid set in order to closely approximate their own demand functions and hence to guarantee the precision and efficiency of the mechanism.

A quantized version of PSP, called the Quantized Progressive Second Price auction mechanism (Q-PSP), was developed in [1] to deal with the slow convergence and signal overhead problems of PSP. Q-PSP uses the basic allocation rules and cost functions of the PSP mechanism in that at each stage each agent submits a two-dimensional bid consisting of the reserve price (the highest price an agent is willing to pay for the unit of a resource) and the corresponding quantity of this resource. However the Q-PSP mechanism operates as follows: at each step the best quantity response for each agent is achieved with respect to the previous strategy profile of its opponents, i.e. the quantity such that the marginal valuation of the agent exceeds its current reserve price; the unit bid price is then chosen based on this best quantity and on the agent's market price (inverse residual supply) function; finally the bid quantity at this stage is calculated based on the

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unit bid price and the agent's own demand function. In Q-PSP auctions all agents submit quantized bids synchronously until a (Nash) equilibrium in the quantized framework is reached or periodic behaviour (i.e. an oscillation) sets in. It was proved in [1] that if all agents have similar demand functions the nonlinear dynamical system corresponding to this mechanism is such that its state (equivalently, bid prices and bid quantities) process converges, or begins to oscillate, in at most five iterations; not only is this behaviour independent of the number of agents involved, but it is also independent of the number of quantization levels.

In this paper an extended version of Q-PSP, called the Aggressive-Defensive Quantized Progressive Second Price auction mechanism (ADQ-PSP) is presented in order to handle the more general case where the agents may have significantly different demand functions. We consider agents' behaviours in two different situations during the dynamical auction process: that where the agents' aggregate demand is less than, or, respectively, exceeds, the market supply (resource constraint). In these cases two different strategies are respectively employed: the defensive quantized strategy where the response is that used in the Q-PSP mechanism, and the aggressive response which requires that the price offer is set equal to the lowest price which exceeds each agent's marginal valuation and the quantity offer is then set equal to the value of inverse marginal valuation function corresponding to the calculated price response. (This scheme may be motivated by the fact that the agents' knowledge that the market supply is insufficient with respect to agents' current aggregate demand leads to aggressive bidding.)

By considering agent populations with randomly distributed demand functions, it is shown here that the nonlinear dynamics induced by the ADQ-PSP mechanism converges with high probability to a small set of quantized bid prices at the first step. Subsequently the available quantized bid price set shrinks linearly until the price component of the state process of the associated dynamical system converges (see Fig. 1). In addition, approximate efficiency and (Nash) equilibria are shown to be achieved within this quantized framework.

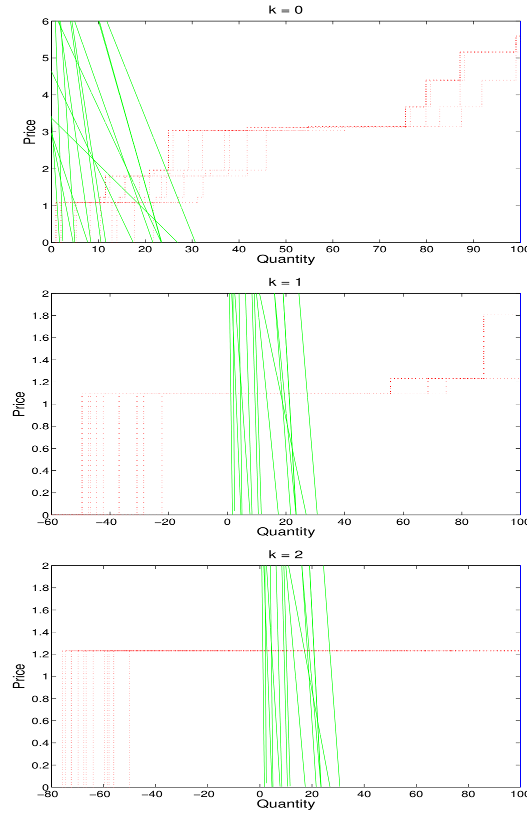

Fig. 1. Rapid convergence to a final bid price of an ADQ-PSP system with 15 agents (see Theorem III.4). *(Note that in this and all subsequent simulation figures neither the horizontal nor the vertical scales are necessarily the same.)*

II. AD Quantized Progressive Second Price Auctions

A. Quantized Progressive Second Price Auction

To begin we give a summary of the PSP and Q-PSP auctions introduced respectively in [4] and [1].

- In a non-cooperative game, N agents buy the fixed amount of bandwidth C from one seller.
- Each agent $A_i, 1 \le i \le N$, makes a two-dimensional *bid* $s_i = (p_i, q_i)$ to the seller, where q_i is the quantity the agent desires and p_i is the unit-price the agent would like to pay for q_i.
- The *bidding profile* is defined as $s \triangleq [s_i]_{1 \le i \le N}$ and
$$s_{-i} \triangleq [s_1, \cdots, s_{i-1}, s_{i+1}, \cdots, s_N]$$
is the profile of Agent A_i's opponents.
- The *market price function (MPF)* of Agent A_i is a left-continuous function defined as:
$$P_i(z, s_{-i}) = \inf \left\{ y \ge 0 : C - \sum_{p_k > y, k \ne i} q_k \ge z \right\}$$
which is interpreted as the minimum price an agent bids in order to obtain the bandwidth z given the opponents' profile s_{-i}. Clearly the function is only reasonable when $z > 0$. Its inverse function Q_i is defined as follows:
$$Q_i(y, s_{-i}) = \left[C - \sum_{p_k > y, k \ne i} q_k \right]^+,$$
which means the maximum available quantity at a bid price of y given s_{-i}.
- The PSP *allocation rule* ([14]) is defined as
$$a_i(s) = \min\{q_i, \frac{q_i}{\sum_{k:p_k = p_i} q_k} Q_i(p_i, s_{-i})\}, \quad \text{(II.1)}$$
$$c_i(s) = \sum_{j \ne i} p_j \left[a_j(0; s_{-i}) - a_j(s_i; s_{-i}) \right], \quad \text{(II.2)}$$
where a_i denotes the quantity Agent A_i obtains by a bid price p_i (when the opponents bid s_{-i}) and the charge to Agent A_i by the seller is denoted c_i.

- A real valued function $\theta(\cdot)$ is an *(elastic) valuation function* on $[0, C]$ if
  - $\theta(0) = 0$;
  - θ is differentiable;
  - $\theta^{'} \geq 0$, non-increasing and continuous;
  - There exists $\gamma, \gamma > 0$, such that for all $z, z \in [0, C], \theta^{'}(z) > 0$ implies that for all $\eta \in [0, z)$, $\theta^{'}(z) \leq \theta^{'}(\eta) - \gamma(z - \eta)$.

  Its derivative function $\theta^{'}(\cdot)$ on $[0, C]$ is called an *(elastic) demand function*.
- Agent A_i's *utility* is defined as

$$u_i(s) = \theta_i(a_i(s)) - c_i(s), \tag{II.3}$$

  which implies the agent's preferences.
- Given s_{-i}, Agent A_i's ϵ-*best response* $s_i = (w_i, v_i)$ ([4]) is given by:

$$v_i = \sup\left\{z \geq 0 : \theta_i^{'}(z) > P_i(z)\right\} - \frac{\epsilon}{\theta_i^{'}(0)}$$
$$\text{(best quantity reply)} \tag{II.4a}$$
$$w_i = \theta_i^{'}(v_i) \quad \text{(best unit-price reply)}, \tag{II.4b}$$

  where $\int_0^z P_i(\eta)d\eta \leq b_i$, $\epsilon > 0$ is the bid fee, b_i is Agent A_i's budget, and every agent has an elastic demand function. Further it is shown in [4] that the bidding iterations converge at a rate inversely proportional to ϵ to an ϵ-Nash equilibrium.

We now introduce the Quantized Progressive Second Price Auction and the associated dynamical system in [1].

Hypothesis I. In the context of the equations (II.4), we adopt the following hypotheses:

1) All bid quantities q are bounded by C, i.e. $0 \leq q \leq C$,
2) There is no bid fee, i.e. $\epsilon = 0$,
3) The budget b_i of each agent is sufficiently large that the condition $\int_0^z P_i(\eta)d\eta \leq b_i$ in (II.4) is always satisfied. □

Due to 3) of Hypothesis I above, the integral constraint in (II.4) is not referred to again.

Subject to Hypothesis I, *the quantized PSP (Q-PSP) dynamical (state space) system* (with state (v, p, q)) equations are defined as follows:

$$P_i^{k+1}(z, s_{-i}^k) = \inf\left\{y \geq 0 : C - \sum_{p_j^k > y, j \neq i} q_j^k \geq z\right\}$$
$$v_i^{k+1} = \sup\left\{z \geq 0 : \theta_i^{'}(z) > P_i^{k+1}(z, s_{-i}^k)\right\} \tag{II.5a}$$
$$p_i^{k+1} = P_i^{k+1}(v_i^{k+1}, s_{-i}^k) \tag{II.5b}$$
$$q_i^{k+1} = \theta_i^{'-1}(p_i^{k+1}), \tag{II.5c}$$

with the initial conditions $p_i^0 \in B_p^0$, $q_i^0 = \theta_i^{'-1}(p_i^0)$, $0 \leq k < \infty$, $1 \leq i \leq N$. Here B_p^0 is defined as the initial quantized bid price set. One may verify that $\{(p_i^k, q_i^k); 1 \leq i \leq N, k \geq 0\}$ constitutes a minimum dimension state process for the dynamical system (II.5) and for all k, $B_p^k \equiv \{p_i^k; 1 \leq i \leq N\} \subseteq B_p^0$ and hence $\{q_i^k, 1 \leq i \leq N\} \subseteq \bigcup_i \theta_i^{'-1}(B_p^0)$.

It was proved in [1] that the nonlinear dynamic system (II.5) converges or begins to oscillate in at most five iterations if all agents share similar demand functions.

B. AD Quantized Progressive Second Price Auctions

The Aggressive-Defensive Quantized Progressive Second Price auction (ADQ-PSP) is formulated as follows:

$$P_i^{k+1}(z, s_{-i}^k) = \inf\left\{y \geq 0 : C \geq z + \sum_{p_k > y, k \neq i} q_k\right\}$$
$$v_i^{k+1} = \sup\left\{z \geq 0 : \theta_i^{'}(z) > P_i^{k+1}(z, s_{-i}^k)\right\} \tag{II.6a}$$

$$p_i^{k+1} = \begin{cases} P_i^{k+1}(v_i^{k+1}, s_{-i}^k) \\ \qquad (\text{if } \sum_{1 \leq j \leq N} q_j^k < C) \\ \min\left\{p_j^k \in B_p^k : p_j^k > P_i^{k+1}(v_i^{k+1}, s_{-i}^k)\right\} \\ \qquad (\text{if } \sum_{1 \leq j \leq N} q_j^k \geq C) \end{cases} \tag{II.6b}$$

$$q_i^{k+1} = D_i(p_i^{k+1}) \equiv \theta_i^{'-1}(p_i^{k+1}), \tag{II.6c}$$

where $P_i^{k+1}(\cdot, s_{-i}^k)$ is seen to be defined as in the original Q-PSP system and may be verified to be continuous from the left. v_i^{k+1} is also defined as in the original system.

The aggressive and defensive strategies of ADQ-PSP are illustrated by Fig. 2.

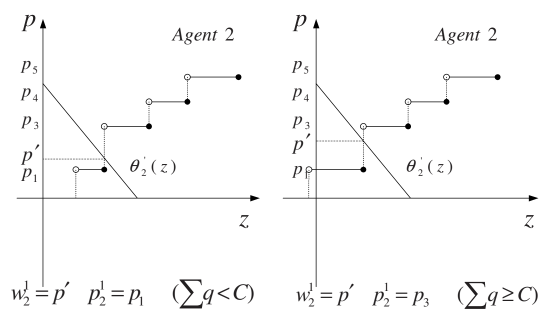

Fig. 2. Switching strategies in (II.6)

Compared with the Q-PSP mechanism, each agent in ADQ-PSP has a switching strategy which works in the following way. The quantized best strategy is one of the adjacent two prices on either side of the intersection of the agent's demand function and its market price function (the two prices corresponding to p_1 and p_3 in Fig. 2). The original Q-PSP strategy, where agents bid the lower one of these two prices (i.e. p_1 in Fig. 2), can be regarded as a defensive strategy. Bidding the higher price (p_3 in Fig. 2) can be viewed as an aggressive action. When the total quantity C is violated, i.e. the sum of bid quantities is greater than C, agents choose to act aggressively with

$$p_i^{k+1} = \min\left\{p_j^k \in B_p^k : p_j^k > P_i^{k+1}(v_i^{k+1}, s_{-i}^k)\right\}.$$

As stated above, this may be motivated by the interpretation that the MPF is a common knowledge and agents obtain the information that the total quantity is limited (see the right part of Fig. 2). On the other hand, they bid defensively with

$$p_i^{k+1} = P_i^{k+1}(v_i^{k+1}, s_{-i}^k),$$

when there is any unallocated quantity left (see the left part of Fig. 2).

Remark II.1. The best reply bid analysis in Section 2.3 of [1] still holds for (II.6). It has been noted above that the best reply bid for each agent A_i at the kth step, i.e. (w_i^k, v_i^k) with $w_i^k = \theta_i^{'}(v_i^k)$, is derived from the intersection of the agent's own demand function and its market price function. And the quantized best strategy is one of the adjacent two quantization levels on either side of this intersection, which correspond to the aggressive and defensive strategies. Hence, the aggressive-defensive quantized strategy is optimal up to a quantization level, that is to say, the strategy is a γ-best reply with

$$\gamma = u_i(v_i^{k+1}, s_{-i}^k) - u_i(a^{k+1}(p_i^{k+1}, s_{-i}^k), s_{-i}^k),$$

where u_i is the utility function of Agent A_i and defined in (II.3).

Moreover, if the states of an ADQ-PSP dynamical system converge to a quantized price p^*, then s^* is a δ-Nash equilibrium with $s_i^* = (p^*, \theta_i^{'-1}(p^*))$ in the sense that:

$$u_i(s_i^*, s_{-i}^*) \geq \sup_{p_i \in B_p^0} u_i(s_i, s_{-i}^*) - \delta, \qquad \text{(II.7)}$$

where δ is such that

$$\begin{aligned} \delta &> \max_i |u_i(s^*) - u_i((p^{**}, \theta_i^{'-1}(p^{**})), s_{-i}^*)| \quad \text{(II.8)} \\ p^{**} &= \min\{p : p > p^*, p \in B_p^0\}. \end{aligned}$$

□

III. RAPID CONVERGENCE OF ADQ-PSP

In order to study the convergence of (II.6) we will first create a new auction dynamical system by a possible left shift on both agents' demand functions and the corresponding market price functions. It is then shown that the bid prices generated by the new dynamical system for all agents are the same as those generated by the original ADQ-PSP dynamical system (II.6). The new dynamical system possesses a single market price function (which we term the Enveloping Market Price Function) for all agents at each step. This significantly simplifies an otherwise very complex analysis; moreover, under a set of reasonable probability assumptions on the initial bid prices and agents' demand functions, we will show that such a system converges with high probability to a small (compared with the number of agents) set of available bid prices at the first step. Subsequently, the bid price set decreases linearly until the system converges.

A. Enveloping Market Price Function (EMPF)

We begin by defining the *enveloping market price function* as

$$P^E(z, s) = \inf\left\{ y \geq 0 : C \geq z + \sum_{p_k > y} q_k \right\}, \quad 0 \leq z \leq C \qquad \text{(III.9)}$$

which denotes the market price function (MPF) without an agent being omitted; this gives an upper contour to the set of market price curves. Comparing $P^E(\cdot)$ with any market price function

$$P_i(z, s_{-i}) = \inf\left\{ y \geq 0 : C \geq z + \sum_{p_k > y, k \neq i} q_k \right\}, \quad 1 \leq i \leq N,$$

we obtain the following results (see Appendix of [15]):

(a) If $C \geq z > C - \sum_{p_k \geq p_i, k \neq i} q_k$,

$$P_i(z, s_{-i}) = P^E(z, s) \geq p_i, \qquad \text{(III.10)}$$

(b) If $0 < z \leq C - \sum_{p_k \geq p_i, k \neq i} q_k$,

$$P_i(z, s_{-i}) = P^E(z - q_i, s) \leq P^E(z, s) < p_i. \qquad \text{(III.11)}$$

Fig. 3 illustrates the results above.

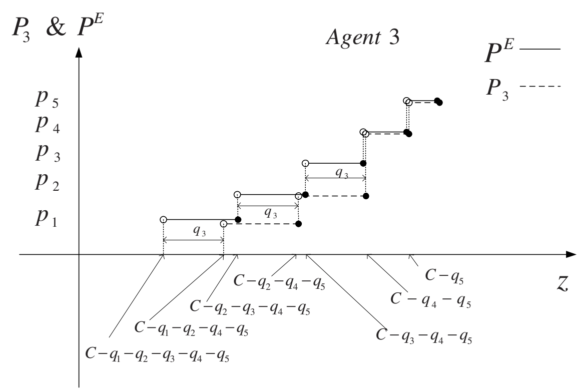

Fig. 3. Relationship between P_i and P^E

B. P^E for the Dynamical ADQ-PSP System

Define $p_{max}^k \in B_p^k$ as the *maximum bid price* at the kth step, and $p_T^k \in B_p^k \cup \{p_0 = 0\}$ as the *threshold price* which satisfies

$$P^{E^k}(0, s^k) = p_T^k. \qquad \text{(III.12)}$$

Define

$$\hat{\theta}_i^{'k}(z) \triangleq \begin{cases} \theta_i^{'}(z + q_i^k) & \text{if } p_i^k > p_T^k; \\ \theta_i^{'}(z) & \text{otherwise,} \end{cases} \qquad \text{(III.13)}$$

as the *shifted demand function*, which is adopted to compensate for the difference between between P_i and P^E in the sense that the recursion using P^E and $\hat{\theta}_i^{'}$ generates the same bid $\{p_i^k\}$ as the original ADQ-PSP system.

For convenience we define

$$D_i(p) \triangleq \theta_i^{'-1}(p), \quad \text{and} \quad \hat{D}_i^k(z) \triangleq \hat{\theta}_i^{'k-1}(z).$$

Then the P^E *dynamical ADQ-PSP system* is specified by:

$$P^{E^{k+1}}(z, \hat{s}^k) = \inf \left\{ y \geq 0 : C \geq z + \sum_{\hat{p}_j^k > y} \hat{q}_j^k \right\}$$

$$\hat{v}_i^{k+1} = \sup \left\{ z : \hat{\theta}_i^{'k}(z) > P^{E^{k+1}}(z, \hat{s}_{-i}^k) \right\} \quad \text{(III.14a)}$$

$$\hat{p}_i^{k+1} = \begin{cases} P^{E^{k+1}}(\hat{v}_i^{k+1}, \hat{s}_{-i}^k) & \text{if } \sum_{1 \leq j \leq N} \hat{q}_j^k < C \\ \hat{p}_i^{'k+1} & \text{if } \sum_{1 \leq j \leq N} \hat{q}_j^k \geq C \end{cases} \quad \text{(III.14b)}$$

$$\hat{q}_i^{k+1} = D_i(\hat{p}_i^{k+1}), \quad \text{(III.14c)}$$

with

$$\begin{aligned} \hat{p}_i^{'k+1} &= \min \left\{ \hat{p}_j^k \in \hat{B}_p^k : \hat{p}_j^k > P^{E^{k+1}}(\hat{v}_i^{k+1}, \hat{s}_{-i}^k) \right\} \\ \hat{w}^k &= \hat{\theta}_i^{'k}(\hat{v}_i^{k+1}) \\ \hat{s}^k &= \left\{ (\hat{p}_i^k, \hat{q}_i^k); 1 \leq i \leq N \right\} \\ \hat{B}_p^k &= \left\{ \hat{p}_i^k; 1 \leq i \leq N \right\}. \end{aligned}$$

Compared to (II.6), we now apply an EMPF to take the place of each agent's individual MPF in such a way that only one MPF needs to be considered at each iteration for all agents in (III.14). In order to guarantee for each agent the same system behavior as in (II.6), we employ the shifted demand functions $\hat{\theta}_i^{'k}$ defined in (III.13) instead of $\theta_i^{'}$ in (III.14). We claim that with the same initial states, both systems (II.6) and (III.14) possess the same system trajectories. Hence, if the sequence of prices (i.e. the price components of the states) generated by one system converges then so it must in the other system.

C. Main Results

We now adopt the following general hypothesis concerning the initial states of the ADQ-PSP systems:

Hypothesis IC. The initial conditions

$$s^0 \equiv (p, q)^0 = \left[(p_1^0, q_1^0), 0_N, q_N^0) \right], \qquad N > 2$$

for the ADQ-PSP system are such that $0 < p_i^0$, $0 < q_i^0$, $1 \leq i \leq N$, where $p_i^0 < p_{i+1}^0$, $1 \leq i \leq N-1$. The initial set of prices shall be denoted by $B_p^0 = \{p_i^0; 1 \leq i \leq N\}$.

Moreover, if $p_T^0 = 0$, $p_{T+i}^0 \equiv p_i^0$ for all $1 \leq i \leq N$; and if $p_T^0 = p_j^0 > 0$, $p_{T+i}^0 \equiv p_{j+i}^0$ for all $1 \leq i, j, i+j \leq N$. □

Then we have

Lemma III.1. *([15]) Given the same initial states ($s^0 = \hat{s}^0$), the systems (II.6) and (III.14) generate the same trajectories (states). That is to say, $s^k = \hat{s}^k, k \geq 0$.*

From Lemma III.1 and the definition of the modified demand functions, we obtain in [15] the following facts:

Corollary III.2. *([15]) Subject to Hypothesis IC, all agents in an ADQ-PSP system (III.14) are such that $0 = p_T^k \leq p_i^k$ if $\sum_{1 \leq j \leq N} q_j^k < C$, and such that bid prices p_i^{k+1} at the next step satisfy*

$$0 = p_T^k \leq p_i^{k+1} < p_i^k \leq p_{\max}^k.$$

Otherwise, if $\sum_{1 \leq j \leq N} q_j^k \geq C$ at the k-th step,

$$p_T^k < p_i^{k+1} \leq p_{\max}^k$$

holds for all agents.

In all cases, $p_{max}^k \geq p_{\max}^{k+1}$ for all $k \geq 0$.

In the following, we first establish a general estimation of the number of convergence steps for an ADQ-PSP system (II.6) or (III.14).

Let us define N^k as the cardinality of B_p^k.

Lemma III.3. *([15]) For (III.14), if $N^k = m > 1$ at the kth step ($k \geq 0$), then the system converges within $2(m-1)$ additional steps.*

Outline of proof: The proof argument has three phases: First we show under the condition $\sum_{1 \leq j \leq N} q_j^k < C$ for $k = l$ and $l+1$ with $l \geq 0$, at least one bid price is removed from B_p^k at each two steps. Then we demonstrate under the condition $\sum_{1 \leq j \leq N} q_j^k \geq C$, at least one bid price leaves the system at each step until $\sum_{1 \leq j \leq N} q_j^k < C$. Finally, we prove that at least one price disappears at each switching between the two strategies. See [15] for the complete proof. □

From Lemma III.3, it is clear that a dynamical system (III.14) with N agents and N initial bid prices converges to a unique price within at most $2(N-1)$ steps. Furthermore, the following theorem proves that, under some assumptions of the initial bid price set and the probability measure of the agents' demand functions, the system (III.14) will converge rapidly in a limited number of steps with a high calculable probability.

We finally adopt the following hypothesis:

Hypothesis DF. The set of agents' demand functions $\{\theta_i^{'}, 1 \leq i \leq N\}$ are linear and determined independently by the intercepts $(x_i, 0)$ and $(0, y_i)$, where $\{x_i; 1 \leq i \leq N\}$ are independent random variables with uniform distribution on $[0, B]$ and $\{y_i; 1 \leq i \leq N\}$ are also independent random variables with uniform distribution on $[0, A], A > p_0^N$. □

Then we may establish the main result.

Theorem III.4. *([15]) Subject to Hypotheses I, IC and DF, the ADQ-PSP system (II.6) with N agents converges to one final bid price within at most $2m-1$ steps, $m \geq 3$, with probability greater than*

$$1 - (N - m + 1) \exp \frac{-2(m-2)^2 \alpha^2}{m};$$

here α is strictly greater than 0 and is determined by p_N^0, A, B.

Outline of proof: The proof argument is in two parts. First it is shown that, under the given hypotheses, the probability for the system to keep more than m, $m \geq 3$, bid prices after the first iteration decreases exponentially with respect to m. Then by use of Lemma III.3 it is shown that a system which has m bid prices initially will converge to a limit price in less than $(2m-2)$ steps. See [15] for the detailed proof. □

Remark III.5. (*Efficiency of ADQ-PSP*)

When the ADQ-PSP system converges to a quantized price p^* and the quantity allocation is a^*, the steady state is a δ-Nash equilibrium in the quantized framework as described in (II.7) and (II.8). Applying Proposition 3 in [4], we obtain

$$\max_{a \in A} \sum_i \theta_i(a_i) - \sum_i \theta_i(a_i^*) = O(\sqrt{\delta\kappa})$$

where A is the set of quantity allocations under the quantization assumption, and it is assumed that for all $i, 0 < i \leq N$, the elastic demand functions $\theta_i^{'}$ satisfy

$$\theta_i^{'}(z) - \theta_i^{'}(z') > -\kappa(z - z'),$$

whenever $z > z' \geq 0$ (see Assumption 2 in [4]). □

Two numerical simulations are presented below to demonstrate the convergence behaviors of dynamical ADQ-PSP systems. Fig. 1 displays such a system with 15 agents which converges in two steps. Fig. 4 illustrates convergence in two steps in the case where 40 agents compete for the quantity $C = 150$. These two cases illustrate the convergence behaviors of dynamical ADQ-PSP systems with significantly different demand curves, showing the rapid convergence property can still hold for dynamical ADQ-PSP systems in general cases.

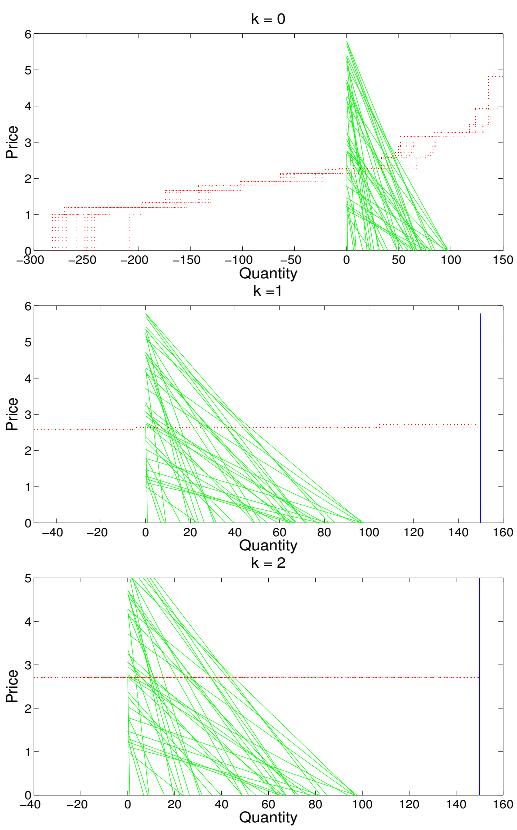

Fig. 4. Rapid convergence to a final bid price of an ADQ-PSP system with 40 agents.

IV. Conclusion and Future Work

Subject to probabilistic assumptions on the agents' demand functions, we provide a probabilistic analysis for the convergence of ADQ-PSP dynamical systems. Rapid convergence is achieved even if all agents have significantly different demand functions and efficiency is guaranteed up to a quantization level. Furthermore, the convergence results have been extend to the double auction case where sellers and buyers compete for one divisible commodity without budget constraints on the buyers' side in the recent work [15].

Finally we remark that is of interest to extend the analysis here to auctions where bids are made on both the supply and the demand sides for multiple commodities with and without budget constraints.

V. Acknowledgments

This work was supported by a grant from NSERC, Canada.

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