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Robust stability analysis of a class of neutral type time delay equations

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Abstract— **In this contribution the robust stability of a class of linear neutral type time delay equations with bounded uncertain coefficients is analyzed via Lyapunov-Krasovskii functionals of complete type. For any exponentially stable nominal systems, we are able to present bounds on the uncertainties that guarantee the stability of the perturbed system. The quadratic Lyapunov-Krasovskii functionals of complete type for scalar neutral type time delay equations depend on the so-called Lyapunov function for neutral time delay systems. Thus, the sufficient robust stability conditions depend on the Lyapunov function.**

I. INTRODUCTION

Neutral type time delay systems form an important class of time delay systems where the system dynamics depend both on the delayed state and its derivative. Many engineering systems can be modeled using functional differential equations of neutral type: processes including steam or water pipes, heat exchangers (see [5] and the references therein), distributed networks containing lossless transmission lines [1] in electrical engineering, and the control of constrained manipulators with delay measurements [7] in mechanical engineering. In practice, some model parameters are not precisely known, leading to the study of the robustness of the stability with respect to uncertainties; for example, lossless transmission line models may have uncertain parameters. It is then of interest to consider neutral systems represented by uncertain models.

In the present paper, based on Lyapunov-Krasovskii functionals of complete type [4], robust stability conditions are given in the case of a scalar neutral type time delay equation with multiple delays. In contrast with the known robust stability conditions which are based on the LMI approach, [3], [6], our conditions provide positive bounds on the uncertainties for any exponentially stable nominal system.

The note is organized as follows: the problem statement is presented in Section II. Because of their key role in obtaining the sufficient robust stability conditions, some basic results concerning the quadratic Lyapunov Krasovskii functionals of complete type are recalled in Section III. In Section IV we show how the functionals of complete type are used for the robust stability analysis of the neutral type time delay equations. The main result of the contribution is illustrated with two examples in Section V. Some concluding remarks end the contribution.

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II. PROBLEM STATEMENT

Consider the nominal neutral type time delay equation with multiple delays

$$\sum_{j=0}^{m} a_j \dot{x}(t-jh) = \sum_{j=0}^{m} b_j x(t-jh), \quad t \geq 0 \tag{1}$$

where $a_j \in \mathbb{R}$ and $b_j \in \mathbb{R}$, $j = 0,\dots,m$, are real coefficients, $a_0 = 1$, and $h > 0$ is the basic delay. The initial condition is

$$x(\theta,\varphi) = \varphi(\theta), \quad \theta \in [-mh, 0]$$

where $\varphi \in C([-mh,0] \rightarrow \mathbb{R})$. $x_t(\varphi)$ stands for the restriction of $x(t,\varphi)$ to the interval $[t-mh,t]$. The state x_t is defined by $x_t(\theta) = x(t+\theta)$ for $\theta \in [-mh,0]$.

Consider now the neutral type time delay equation with uncertain coefficients

$$\sum_{j=0}^{m} (a_j + \Delta_{a_j}) \dot{y}(t-jh) = \sum_{j=0}^{m} (b_j + \Delta_{b_j}) y(t-jh) \tag{2}$$

where the uncertain values $\Delta_{a_j} \in \mathbb{R}$ and $\Delta_{b_j} \in \mathbb{R}$, $j = 0,\dots,m$, are such that

$$\begin{array}{c} \Delta_{a_0} = 0, \quad |\Delta_{a_j}| \leq \delta_{a_j}, \quad j = 1,2,\dots,m, \\ |\Delta_{b_j}| \leq \delta_{b_j}, \quad j = 0,1,\dots,m, \end{array} \tag{3}$$

where δ_{a_j} and δ_{b_j} are positive numbers.

Problem 1: Determine the values of δ_{a_j} and δ_{b_j} for which the neutral equation with uncertain coefficients (2) remains exponentially stable provided that the nominal system (1) is exponentially stable.

III. PREVIOUS RESULTS

We summarize here the main useful results exposed in [8] on Lyapunov Krasovskii functionals of complete type for exponentially stable neutral type time delay equations of the form (1). These functionals are such that their derivative along the solutions of the system is more substantial than a quadratic form of the state, allowing to prove that they admit a quadratic lower bound. It appears that they depend on the Lyapunov function denoted $u(\cdot)$ associated to the system.

The functional with prescribed derivative

$$\frac{d}{dt} v(x_t)\Big|_{(1)} = -w(x_t), \quad t \geq 0,$$

where

$$w(x_t) = \sum_{j=0}^{m} \mu_j x^2(t-jh) + \sum_{j=1}^{m} \nu_j \int_{-jh}^{0} x^2(t+\theta) d\theta, \tag{4}$$

and μ_j, $j = 0, 1, \ldots, m$, and ν_j, $j = 1, 2, \ldots, m$, are given positive constants is

$$v(x_t) = \left[\sum_{j=0}^{m}(\mu_j + jh\nu_j)\right]v_0(x_t) + \sum_{j=1}^{m}\int_{-jh}^{0}\left[\mu_j + (jh+\theta)\,\nu_j\right]x^2(t+\theta)d\theta, \tag{5}$$

where $v_0(x_t)$ is defined as

$$\begin{aligned} v_0\left(x_t\right) = {} & u(0)\left[\sum_{j=0}^{m} a_j x(t-jh)\right]^2 + 2\left[\sum_{j=0}^{m} a_j x(t-jh)\right]\times \\ & \times\sum_{i=1}^{m}\int_{-ih}^{0}\left[u(ih+\theta)b_i + \frac{\partial u(ih+\theta)}{\partial\theta}a_i\right]x(t+\theta)d\theta + \\ & + \sum_{j=1}^{m}\sum_{i=1}^{m}\left[\int_{-jh}^{0}\int_{-ih}^{0}x(t+\theta_1)\Big[b_j u((j-i)h+\theta_1-\theta_2)b_i + \right. \\ & +2a_j\frac{\partial u((j-i)h+\theta_1-\theta_2)}{\partial\theta_1}b_i\Big]x(t+\theta_2)d\theta_2 d\theta_1 - \\ & - \sum_{k=1}^{j}\sum_{l=1}^{i}\int_{-kh}^{(-k+1)h}x(t+\theta_1)a_j\times \\ & \times\Big[\int_{\substack{-lh \\ \theta_2\neq\theta_1+(k-l)h}}^{(-l+1)h}\frac{\partial^2 u((i-j)h-\theta_1+\theta_2)}{\partial\theta_1\partial\theta_2}a_i x(t+\theta_2)d\theta_2 + \\ & \left. + \triangle\, u'((i-j+k-l)h)a_i x(t+\theta_1+(k-l)h)\Big]d\theta_1\right]. \end{aligned}$$

Under the assumption that system (1) is exponentially stable, the functional (5) has the following quadratic bounds, see [8]

$$\alpha_1\Big[\sum_{j=0}^{m} a_j\varphi(-jh)\Big]^2 \leq v(\varphi) \leq \alpha_2\left|\varphi\right|_{mh}^2$$

for some positive constants α_1 and α_2. Here $|\varphi|_{mh} = \sup_{\theta\in[-mh,0]}|\varphi(\theta)|$.

The definition and computation of the Lyapunov function $u(\cdot)$ associated to equation (1) is addressed in [4]. It is shown to be the solution of the dynamic equation

$$\sum_{j=0}^{m} a_j u'(t-jh) = \sum_{j=0}^{m} b_j u(t-jh), \quad t \geq 0,$$

where $u'(\tau) = \frac{d}{d\tau}u(\tau)$, satisfy the symmetry condition

$$u(-\tau) = u(\tau), \quad \tau \geq 0,$$

and the algebraic condition

$$\sum_{j=0}^{m}\sum_{i=0}^{m} a_j b_i u((j-i)h) = -1/2.$$

The problem of its computation is reduced to the computation of solutions of an auxiliary two-point boundary problem for a special delay free system equations.

The jump size of the first derivative of the Lyapunov function $u(\tau)$ can be computed as follows

$$\begin{aligned} \Delta u'(lh) &= -e_1^T P S^l e_1, \quad l = 0, 1, 2, \ldots, \\ \Delta u'(-lh) &= -e_1^T (S^T)^l P e_1, \quad l = 1, 2, \ldots, \end{aligned}$$

where $e_1^T = \left[\begin{array}{cccc} 1 & 0 & \cdots & 0 \end{array}\right]$, matrix S is

$$S = \begin{bmatrix} -a_1 & -a_2 & \cdots & -a_{m-1} & -a_m \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & & 0 & 0 \\ & & \ddots & & \vdots \\ 0 & 0 & & 1 & 0 \end{bmatrix},$$

and matrix P is the solution of the matrix equation $S^T PS - P = -W_0$ with $W_0 = e_1 e_1^T$, see [8].

IV. MAIN RESULT

In this section, functionals of complete type (5) are used for the robust stability analysis of perturbed neutral type time delay equations. The nominal system (1) is assumed to be stable. We derive the values δ_{a_j} and δ_{b_j} for which equation (2) remains exponentially stable for all perturbation values Δ_{a_j} and Δ_{b_j} satisfying (3).

To obtain such bounds, we use the following modification of the functional (5) computed for the nominal equation (1)

$$v(y_t) = \left[\sum_{j=0}^{m}(\mu_j + jh\nu_j)\right]v_0(y_t) + \sum_{j=1}^{m}\int_{-jh}^{0}\left[\mu_j + (jh+\theta)\,\nu_j\right]y^2(t+\theta)d\theta, \tag{6}$$

where

$$\begin{aligned} v_0\left(y_t\right) = {} & u(0)\left[\sum_{j=0}^{m}(a_j + \Delta_{a_j})y(t-jh)\right]^2 + \\ & +2\left[\sum_{j=0}^{m}(a_j + \Delta_{a_j})y(t-jh)\right]\times \end{aligned}$$

$$\times \sum_{i=1}^{m} \int_{-ih}^{0} \left[u(ih+\theta)b_i + \frac{\partial u(ih+\theta)}{\partial \theta} a_i \right] y(t+\theta)d\theta +$$

$$+ \sum_{j=1}^{m}\sum_{i=1}^{m} \Bigg[\int_{-jh}^{0}\int_{-ih}^{0} y(t+\theta_1)\Big[b_j u((j-i)h+\theta_1-\theta_2)b_i +$$

$$+2a_j \frac{\partial u((j-i)h+\theta_1-\theta_2)}{\partial \theta_1} b_i \Big] y(t+\theta_2)d\theta_2 d\theta_1 -$$

$$-\sum_{k=1}^{j}\sum_{l=1}^{i} \int_{-kh}^{(-k+1)h} y(t+\theta_1)a_j \times$$

$$\times \Big[\int_{\substack{-lh \\ \theta_2 \neq \theta_1+(k-l)h}}^{(-l+1)h} \frac{\partial^2 u((i-j)h-\theta_1+\theta_2)}{\partial \theta_1 \partial \theta_2} a_i y(t+\theta_2)d\theta_2 +$$

$$+ \triangle u'((i-j+k-l)h) a_i y(t+\theta_1+(k-l)h) \Big] d\theta_1 \Bigg].$$

Remark 1: Notice that this modified functional depends on the values Δ_{a_j}, $j = 1, 2, \ldots, m$, of the perturbed equation.

The time derivative of the functional (6) along the trajectories of equation (2) is

$$\frac{d}{dt}v(y_t) = -w(y_t) + 2\Big[\sum_{k=0}^{m}(\mu_k + kh\nu_k)\Big] \times$$

$$\times \Big[\sum_{j=0}^{m}\Delta_{b_j} y(t-jh)\Big]\Big[u(0)\sum_{i=0}^{m}(a_i+\Delta_{a_i})y(t-ih) +$$

$$+\sum_{i=1}^{m}\int_{-ih}^{0}[u(ih+\theta)b_i + u'(ih+\theta)a_i]y(t+\theta)d\theta\Big] +$$

$$+2\Big[\sum_{k=0}^{m}(\mu_k + kh\nu_k)\Big]\Big[\sum_{j=0}^{m}\Delta_{a_j} y(t-jh)\Big] \times$$

$$\times\Big[\frac{1}{2}\Delta u'(0)\sum_{i=0}^{m} a_i y(t-ih) -$$

$$-\sum_{i=1}^{m}\sum_{l=i}^{m} a_l \Delta u'((l-i)h)y(t-ih) -$$

$$-\sum_{i=1}^{m}\int_{-ih}^{0} u'(ih+\theta)b_i y(t+\theta)d\theta -$$

$$-\sum_{i=1}^{m} \int_{\substack{-ih \\ \theta\in(-ih,0),\ \theta\neq lh,\ l=-i,-i+1,\ldots,0}}^{0} u''(ih+\theta)a_i y(t+\theta)d\theta\Big],$$

where $w(\cdot)$ is the functional defined in (4).

The functional (5) admits upper and lower quadratic bounds. According to the Lyapunov-Krasovskii Theorem for neutral type time delay systems [2] to claim stability, the time derivative must not only be negative, but must be a quadratic form with negative sign. Therefore it is necessary to find an upper estimate for the first derivative of the modified functional in such a manner the derivative of the modified functional remains negative definite in order to determine under which conditions the uncertain equation (2) remains exponentially stable. To do so, we introduce the following quantities

$$\begin{cases} a = \max\limits_{k=1,2,\ldots,m}|a_k|, \quad b = \max\limits_{k=0,1,\ldots,m}|b_k|, \\ u_1 = \max\limits_{\tau\in[-mh,mh]}|u(\tau)|, u_2 = \max\limits_{\tau\in[-mh,mh]}|u'(\tau)|, \\ u_3 = \sup\limits_{\tau\in(-mh,mh),\tau\neq kh}|u''(\tau)|, \quad \text{and} \\ u_4 = \max\limits_{k=-m,\ldots,0,\ldots,m}|\triangle u'(kh)|. \end{cases} \tag{7}$$

Now, by standard majorizations, we make an upper estimation of each term:

$$\Big|2\sum_{j=0}^{m}\Delta_{b_j} y(t-jh)\Big(u(0)\sum_{i=0}^{m}(a_i+\Delta_{a_i})y(t-ih)\Big)\Big|$$
$$\le u_1\sum_{j=0}^{m}\sum_{i=0}^{m}\delta_{b_j}(|a_i|+\delta_{a_i})\left[y^2(t-jh)+y^2(t-ih)\right]$$
$$= u_1\Big(\sum_{i=0}^{m}(|a_i|+\delta_{a_i})\Big)\sum_{j=0}^{m}\delta_{b_j}y^2(t-jh) +$$
$$+u_1\Big(\sum_{i=0}^{m}\delta_{b_i}\Big)\sum_{j=0}^{m}(|a_j|+\delta_{a_j})y^2(t-jh);$$

$$\Big|2\sum_{j=0}^{m}\Delta_{b_j} y(t-jh)\Big(\sum_{i=1}^{m}\int_{-ih}^{0}[u(ih+\theta)b_i +$$
$$+u'(ih+\theta)a_i]y(t+\theta)d\theta\Big)\Big|$$
$$\le (bu_1+au_2)\sum_{j=0}^{m}\sum_{i=1}^{m}\delta_{b_j}\int_{-ih}^{0}\left[y^2(t-jh)+y^2(t+\theta)\right]d\theta$$
$$= (bu_1+au_2)\Big[\tfrac{m(m+1)}{2}h\sum_{j=0}^{m}\delta_{b_j}y^2(t-jh) +$$
$$+\Big(\sum_{i=0}^{m}\delta_{b_i}\Big)\sum_{j=1}^{m}\int_{-jh}^{0}y^2(t+\theta)d\theta\Big];$$

$$\Big|2\sum_{j=0}^{m}\Delta_{a_j} y(t-jh)\left(\tfrac{1}{2}\Delta u'(0)\sum_{i=0}^{m}a_i y(t-ih)\right)\Big|$$
$$\le \tfrac{1}{2}|\Delta u'(0)|\sum_{j=0}^{m}\sum_{i=0}^{m}\delta_{a_j}|a_i|\left[y^2(t-jh)+y^2(t-ih)\right]$$
$$= \tfrac{1}{2}|\Delta u'(0)|\Big[\Big(\sum_{i=0}^{m}|a_i|\Big)\sum_{j=1}^{m}\delta_{a_j}y^2(t-jh) +$$
$$+\Big(\sum_{i=1}^{m}\delta_{a_i}\Big)\sum_{j=0}^{m}|a_j|y^2(t-jh)\Big];$$

$$\Big|2\sum_{j=0}^{m}\Delta_{a_j} y(t-jh)\Big(\sum_{i=1}^{m}\sum_{l=i}^{m}a_l\Delta u'((l-i)h)y(t-ih)\Big)\Big|$$
$$\le u_4\sum_{j=0}^{m}\sum_{i=1}^{m}\delta_{a_j}\Big(\sum_{l=i}^{m}|a_l|\Big)\left[y^2(t-jh)+y^2(t-ih)\right] =$$
$$= u_4\sum_{i=1}^{m}\Big(\sum_{l=i}^{m}|a_l|\Big)\sum_{j=1}^{m}\delta_{a_j}y^2(t-jh) +$$
$$+u_4\Big(\sum_{i=1}^{m}\delta_{a_i}\Big)\sum_{j=1}^{m}\Big(\sum_{l=j}^{m}|a_j|\Big)y^2(t-jh);$$

$$\left|2\sum_{j=0}^{m}\Delta_{a_j}y(t-jh)\left(\sum_{i=1}^{m}\int_{-ih}^{0}u'(ih+\theta)b_iy(t+\theta)d\theta\right)\right|$$
$$\leq bu_2\sum_{j=0}^{m}\sum_{i=1}^{m}\delta_{a_j}\int_{-ih}^{0}\left[y^2(t-jh)+y^2(t+\theta)\right]d\theta$$
$$=bu_2\left[\tfrac{m(m+1)}{2}h\sum_{j=1}^{m}\delta_{a_j}y^2(t-jh)+\right.$$
$$\left.+\left(\sum_{i=1}^{m}\delta_{a_i}\right)\sum_{j=1}^{m}\int_{-jh}^{0}y^2(t+\theta)d\theta\right];$$

$$\left|2\sum_{j=0}^{m}\Delta_{a_j}y(t-jh)\times\right.$$
$$\left.\times\left(\sum_{\substack{i=1\\ \theta\in(-ih,0),\ \theta\neq lh,\ l=-i,-i+1,\ldots,0}}^{m}\int_{-ih}^{0}u''(ih+\theta)a_iy(t+\theta)d\theta\right)\right|$$
$$\leq au_3\sum_{j=0}^{m}\sum_{i=1}^{m}\delta_{a_j}\int_{-ih}^{0}\left[y^2(t-jh)+y^2(t+\theta)\right]d\theta$$
$$=au_3\left[\tfrac{m(m+1)}{2}h\sum_{j=1}^{m}\delta_{a_j}y^2(t-jh)+\right.$$
$$\left.+\left(\sum_{i=1}^{m}\delta_{a_i}\right)\sum_{j=1}^{m}\int_{-jh}^{0}y^2(t+\theta)d\theta\right].$$

As a result, we obtain the following upper bound for the derivative

$$\frac{dv(y_t)}{dt}\bigg|_{(2)}\leq -M\left[\frac{\mu_0}{M}-u_1\left(\delta_{b_0}\sum_{i=0}^{m}(|a_i|+\delta_{a_i})+\sum_{i=0}^{m}\delta_{b_i}\right)-\right.$$
$$\left.-\frac{1}{2}\left(\delta_{b_0}(bu_1+au_2)m(m+1)h+|\Delta u'(0)|\sum_{i=1}^{m}\delta_{a_i}\right)\right]y^2(t)-$$
$$-\sum_{j=1}^{m}M\left[\frac{\mu_j}{M}-\delta_{b_j}\left(u_1\sum_{i=0}^{m}(|a_i|+\delta_{a_i})+\right.\right.$$
$$\left.+\frac{1}{2}(bu_1+au_2)m(m+1)h\right)-$$
$$-\delta_{a_j}\left(u_1\sum_{i=0}^{m}\delta_{b_i}+\frac{1}{2}|\Delta u'(0)|\sum_{i=0}^{m}|a_i|+\right.$$
$$\left.+u_4\sum_{i=1}^{m}\left(\sum_{l=i}^{m}|a_l|\right)+\frac{1}{2}(bu_2+au_3)m(m+1)h\right)-$$
$$-|a_j|\left(u_1\sum_{i=0}^{m}\delta_{b_i}+\frac{1}{2}|\Delta u'(0)|\sum_{i=0}^{m}\delta_{a_i}\right)-$$
$$\left.-\sum_{l=j}^{m}|a_l|\left(u_4\sum_{i=1}^{m}\delta_{a_i}\right)\right]y^2(t-jh)-$$
$$-\sum_{j=1}^{m}\int_{-jh}^{0}M\left[\frac{\nu_j}{M}-(bu_1+au_2)\sum_{i=0}^{m}\delta_{b_i}-\right.$$
$$\left.-(bu_2+au_3)\sum_{i=1}^{m}\delta_{a_i}\right]y^2(t+\theta)d\theta,$$

where $M=\sum_{k=0}^{m}(\mu_k+kh\nu_k)$.

Our main result stated below follows immediately from the previous inequality.

Theorem 1: Let the nominal equation (1) be exponentially stable. Given $\mu_j,\ j=0,1,\ldots,m$ and $\nu_j,\ j=1,2,\ldots,m$, positive constants, the uncertain equation (2) remains exponentially stable for all perturbations satisfying (3) if the values $\Delta_{a_j},\ j=1,2,\ldots,m$ and $\Delta_{b_j},\ j=0,1,\ldots,m$, satisfy the following inequalities

- $\frac{\mu_0}{M}>u_1\left(\delta_{b_0}\sum_{i=0}^{m}(|a_i|+\delta_{a_i})+\sum_{i=0}^{m}\delta_{b_i}\right)+$
$+\frac{1}{2}\left(\delta_{b_0}(bu_1+au_2)m(m+1)h+|\Delta u'(0)|\sum_{i=1}^{m}\delta_{a_i}\right)$,
- For $j=1,2,\ldots,m$,
$\frac{\mu_j}{M}>\delta_{b_j}\left(u_1\sum_{i=0}^{m}(|a_i|+\delta_{a_i})+\frac{1}{2}(bu_1+au_2)m(m+1)h\right)+$
$+\delta_{a_j}\left(u_1\sum_{i=0}^{m}\delta_{b_i}+\frac{1}{2}|\Delta u'(0)|\sum_{i=0}^{m}|a_i|+\right.$
$\left.+u_4\sum_{i=1}^{m}\left(\sum_{l=i}^{m}|a_l|\right)+\frac{1}{2}(bu_2+au_3)m(m+1)h\right)+$
$+|a_j|\left(u_1\sum_{i=0}^{m}\delta_{b_i}+\frac{1}{2}|\Delta u'(0)|\sum_{i=0}^{m}\delta_{a_i}\right)+\sum_{l=j}^{m}|a_l|\left(u_4\sum_{i=1}^{m}\delta_{a_i}\right)$,
- For $j=1,2,\ldots,m$,
$\frac{\nu_j}{M}>(bu_1+au_2)\sum_{i=0}^{m}\delta_{b_i}+(bu_2+au_3)\sum_{i=1}^{m}\delta_{a_i}$,

where a, b, u_1, u_3 and u_3 are defined in (7), and $\Delta u'(0)$ is the jump size at the point $\tau=0$.

Remark 2: The uncertain parameters $\Delta_{b_j}, j=0,1,\ldots,m$, may be time varying or may even depend on $y(t-jh)$, $j=0,1,\ldots,m$. The only assumption one does really need is that the parameters are continuous with respect to these arguments and that they satisfy (3) for all values of the arguments.

V. Examples

To illustrate the effectiveness of our main result, two numerical examples are presented.

Example 1: Scalar neutral type time delay equations arise in the study of electrical networks such as the classical transmission line presented in [1]. The system consists of a long electrical cable of length l, one end of which is connected to a power source E with resistance R, while the other end is connected to an oscillating circuit formed by a condenser C_1 and a nonlinear element, the volt-ampere characteristic of which is $i=g(v)$, see Figure 1. Let L and C denote the linear inductance and capacitance of the cable, respectively. This system is described by the following nonlinear neutral type time delay equation reported in [1], [2] and [5]:

$$\dot{x}(t)-K\dot{x}(t-\frac{2l}{s})=-\frac{1}{C_1z}x(t)-\frac{K}{C_1z}x(t-\frac{2l}{s})-$$
$$-g(x(t)-Kx(t-\frac{2l}{s}))\tag{8}$$

Here $s=(LC)^{-1/2}$, $z=(L/C)^{1/2}$ and $K=(z-R)(z+R)$ are dimensionless quantities.

In this note, we consider the linear part of equation (8) as the nominal equation. Selecting $l=0.1006$, $L=0.2$,

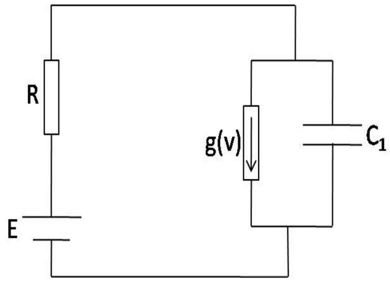

Fig. 1. Transmission line network

$C = 0.1$, $R = 0.12$ and $C_1 = 0.1$ we obtain the scalar neutral type time delay equation

$$\dot{x}(t) - 0.8436\dot{x}(t-0.0285) = -7.0711x(t) - \\ -5.9649x(t-0.0285).$$

As shown in [2] this equation is exponentially stable.

We model the nonlinear dynamics of (8) as parameter uncertainty: we consider the linear neutral type time delay equation with uncertain coefficients

$$\dot{x}(t) + (-0.8436 + \delta_{a_1})\dot{x}(t-0.0285) = \\ (-7.0711 + \delta_{b_0})x(t) + (-5.9649 + \delta_{b_1})x(t-0.0285). \tag{9}$$

Here we assume that

$$|\delta_{a_1}| \leq \delta, \quad |\delta_{b_0}| \leq \delta, \quad \text{and} \quad |\delta_{b_1}| \leq \delta.$$

For the choice $\mu_0 = 0.4$, $\mu_1 = 1.8$, and $\nu_1 = 1$, the functions $u(\tau)$, $u'(\tau)$ and $u''(\tau)$ are respectively sketched on Figures 2, 3 and 4.

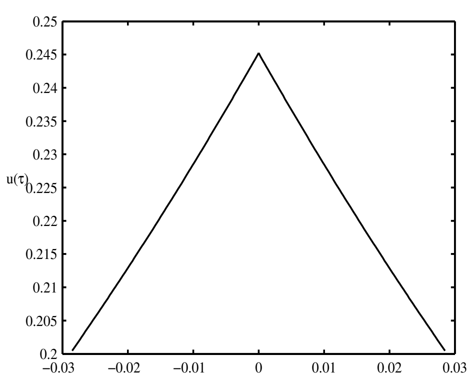

Fig. 2. Function $u(\tau)$

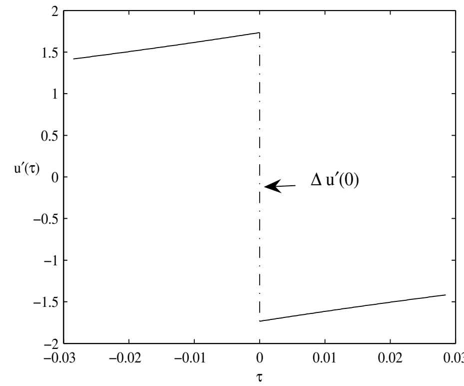

Fig. 3. Function $u'(\tau)$

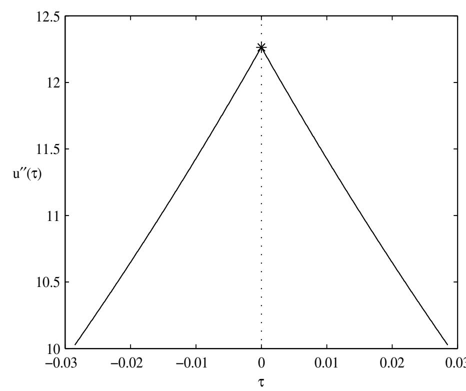

Fig. 4. Function $u''(\tau)$

One can readily compute the values $a = 0.8436$, $b = 7.0711$, $u_1 = 0.2452$, $u_2 = 1.734$, $u_3 = 12.26$, and $u_4 = 3.4682$. Then, it follows from Theorem 1 that the perturbed equation (9) remains stable for all perturbations if $\delta \leq 0.0154$.

Example 2: Now, let us consider the following scalar neutral equation describing dynamics in mechanical systems under measurements in contact with rigid environment, see [7]

$$\dot{x}(t) - c\dot{x}(t-\tau_1) = -dx(t-\tau_2), \tag{10}$$

where c, d are reals, such that $|c| < 1$, and $d > 0$.

The scalar equation (10) with $|c| < 1$ and $d > 0$ is exponentially stable for any delay value $\tau_1 \in \mathbb{R}_+$ and any $\tau_2 \in [0, \overline{\tau}_2]$, where $\overline{\tau}_2 < \frac{1-|c|}{d}$, see [6].

We consider the nominal equation

$$\dot{x}(t) + 0.2\dot{x}(t-0.5) = -x(t-0.5),$$

and we consider the neutral type time delay equation with uncertain coefficients

$$\dot{x}(t) + (0.2 + \delta_{c_1})\dot{x}(t-0.5) = (-1 + \delta_{d_1})x(t-0.5). \tag{11}$$

Here, we assume that

$$|\delta_{c_1}| \leq \delta', \quad \text{and} \quad |\delta_{d_1}| \leq \delta'.$$

For the choice $\mu_0 = 0.4$, $\mu_1 = 1.8$, and $\nu_1 = 1$, the functions $u(\tau)$, $u'(\tau)$ and $u''(\tau)$ are respectively sketched on Figures 5, 6 and 7.

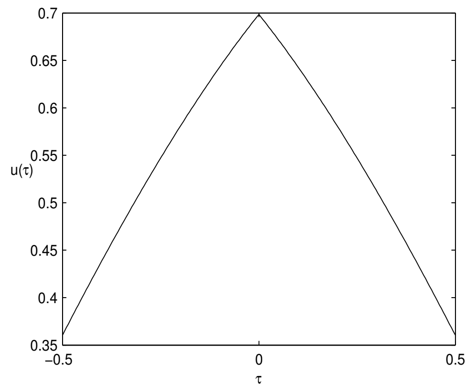

Fig. 5. Function $u(\tau)$

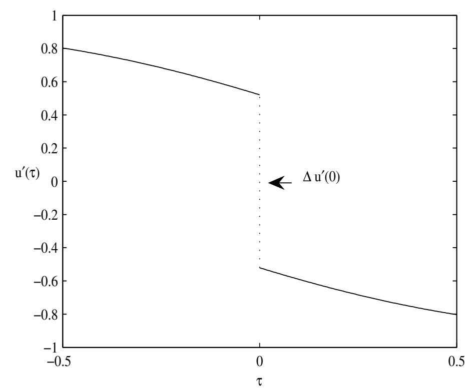

Fig. 6. Function $u'(\tau)$

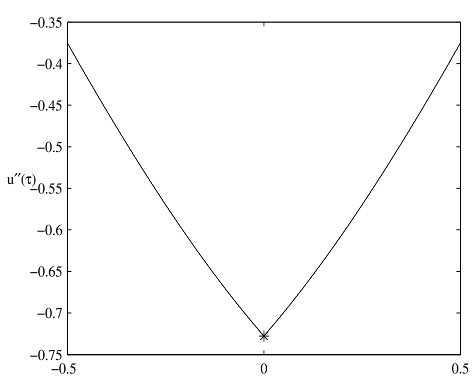

Fig. 7. Function $u''(\tau)$

One can readily compute the values $a = 0.2$, $b = 1$, $u_1 = 0.6986$, $u_2 = 0.8027$, $u_3 = 0.7277$, and $u_4 = 1.0417$. Then, it follows from Theorem 1 that the perturbed equation (11) remains stable for all perturbations if $\delta' \leq 0.1214$.

VI. Conclusions

The robust stability of a class of neutral type time delay equations with multiple delays and bounded uncertain coefficients is analyzed. It is shown how Lyapunov-Krasovskii functionals of complete type are used to derive sufficient conditions that depend on the Lyapunov function of the nominal equation. The main result is illustrated by a classical example of transmission line networks and by an example with delay in force feedback.

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