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Robust Stabilization and Performance Recovery of Nonlinear Systems with Input Unmodeled Dynamics

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Abstract— **In this paper we extend the time-scale separation redesign for stabilization and performance recovery of uncertain nonlinear systems proposed in [4] and [5] to systems with input unmodeled dynamics. The class of unmodeled dynamics studied are relative degree zero and minimum phase. We design two sets of high gain filters - the first to estimate the uncertain input to the plant over a fast time-scale, and the second to force this estimate to converge to the nominal input on an intermediate time-scale. The control input then acts over the slow time-scale and guarantees that the closed-loop trajectories approach those of the nominal system.**

I. Introduction

Wide literature exists on various stabilizing control designs for nonlinear systems with unmodeled dynamics. Krstic *et al.* [15] considered scalar nonlinear systems with stable input unmodeled dynamics and developed a dynamic nonlinear damping design which guarantees global boundedness of trajectories by restricting the size of the uncertainty. Arcak and Kokotovic [1] proved a similar result by using a less restrictive assumption that the unmodeled dynamics be relative degree zero and minimum phase. Input-to-state stability (ISS) property of the unmodeled dynamics and small-gain theorems were employed in the designs by Jiang and Mareels [9], Jiang, Teel and Praly [12], and Praly and Wang [20]. Jiang and Arcak [8] proposed a small-gain design and augmented it with observer-based control. Other related designs such as adaptive control of systems with unmodeled dynamics have been presented in papers by Taylor *et al.* [23], and Jiang and Praly [10], [11].

These designs are focused only on stabilization in the presence of unmodeled dynamics and do not take into account the changes in the transient response relative to the nominal control system without unmodeled dynamics. To address the problem of performance recovery, in this paper we adapt the time-scale separation redesigns proposed in two of our earlier papers [4], [5] to nonlinear systems with input unmodeled dynamics. We build two sets of high-gain filters - one for estimating the input signal to the plant over a fast time-scale and the other to force this estimate to converge to the nominal input over an intermediate time-scale. For relative degree zero and minimum phase unmodeled dynamics, we use singular perturbation theory [13], [14] to prove that the trajectories of the redesigned system approach those of the nominal system as the filter gains are increased.

A performance recovery result related to ours has been presented by Mahmoud and Khalil [16] for fast unmodeled dynamics. In contrast to [16], in our design we do not assume that the unmodeled dynamics are fast compared to the speed of the plant. Another design that has similar features to ours is by Jiang and Praly [19]. This result, however, only addresses stability and does not study performance recovery properties. Moreover, the construction of our high-gain filters requires the existence of a relative-degree-one output which is less restrictive than the *rectifiability* assumption in [19].

The rest of the paper is organized as follows. In Section 2 we describe the proposed design. In Section 3 we illustrate the design with an example. Section 4 concludes the paper.

II. Problem Statement and Main Result

We consider a nonlinear system of the form

$$\dot{x} = f(x) + g(x)\,v \quad (1)$$

$$\dot{\xi} = q(\xi,\, u) \quad (2)$$

$$v = \rho(u, \xi) \quad (3)$$

where $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$ are respectively the state and the input for the plant, $\xi \in \mathbb{R}^r$ and $v \in \mathbb{R}^m$ are respectively the state and output for the unmodeled dynamics, $f(x) : \mathbb{R}^n \to \mathbb{R}^n$ and $g(x) : \mathbb{R}^n \to \mathbb{R}^{n\times m}$ are known functions with $f(0) = 0$, while $q(\xi, u) : \mathbb{R}^r \times \mathbb{R}^m \to \mathbb{R}^r$ and $\rho(u, \xi) : \mathbb{R}^m \times \mathbb{R}^r \to \mathbb{R}^m$ are partially known or unknown functions. We assume all functions are continuously differentiable.

When the ξ-dynamics are ignored so that $v = u$, we refer to (1) as the nominal system. To design a control input u which stabilizes the equilibrium of (1) and also recovers the nominal closed loop trajectories in the presence of the unmodeled dynamics (2), we make the following assumptions:

Assumption 1: There exists a C^1 feedback control law $u = \alpha(x)$ such that the origin of the nominal closed loop system

$$\dot{x} = f(x) + g(x)\alpha(x) \quad (4)$$

is globally asymptotically stable with a positive definite, radially unbounded C^2 Lyapunov function $V_1(x)$ satisfying

$$\frac{\partial V_1}{\partial x}[f(x) + g(x)\alpha(x)] \le -\beta_1(\|x\|) \quad \forall x \in \mathbb{R}^n, \quad (5)$$

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where $\beta_1(\cdot)$ is a class-$\mathcal{K}_\infty$ function, further restricted in Assumption 3 below. □

Assumption 2: There exists a known C^1 function $S(\cdot) : \mathbb{R}^m \to \mathbb{R}^m$ such that $\forall(\omega,\xi) \in \mathbb{R}^m \times \mathbb{R}^r$,

$$\frac{1}{2}\left(\frac{\partial\rho(S(\omega),\xi)}{\partial\omega} + \left(\frac{\partial\rho(S(\omega),\xi)}{\partial\omega}\right)^T\right) \geq kI, \tag{6}$$

where, k is a positive number independent of ω and ξ, and I is the $m \times m$ identity matrix. □

When $m = 1$ and $\rho(u,\xi)$ is strictly increasing in u with a uniform lower bound on its slope, Assumption 2 holds with $S(\omega) = \omega$. If $\rho(u,\xi) = \bar{\rho}(\xi) + Ku$ where $K \in \mathbb{R}^{m\times m}$ is an unknown constant matrix, then Assumption 2 means that a known matrix $S \in \mathbb{R}^{m\times m}$ exists such that

$$KS + S^TK^T > 0. \tag{7}$$

Assumptions similar to (7) are used in MIMO model reference adaptive control as a generalization of the SISO condition that the sign of the *high-frequency gain* K be known [6], [17]. It follows from [18, Theorem 5.4.5, pg 143] that Assumption 2 guarantees the existence of the functional inverse of $\rho(S(\omega),\xi)$ with respect to ω. Denoting this inverse by $\vartheta(\cdot,\xi)$ we note that $\omega = \vartheta(v,\xi)$ satisfies

$$\rho(S(\vartheta(v,\xi)),\xi) = v. \tag{8}$$

The following assumption implies that the unmodeled dynamics (2)-(3) are minimum-phase.

Assumption 3: Let $\vartheta(v,\xi)$ be the inverse of $\rho(S(\cdot),\xi)$ as in (8). Setting the output $v = \alpha(x)$, where $\alpha(\cdot)$ is defined in Assumption 1, the inverse dynamics of (2)-(3):

$$\dot{\xi} = q(\xi, S(\vartheta(\alpha(x),\xi))) \tag{9}$$

is ISS with respect to x seen as an input, with a C^2 positive definite, radially unbounded Lyapunov function $V_2(\xi)$ satisfying

$$\frac{\partial V_2}{\partial\xi}\, q(\xi, S(\vartheta(\alpha(x),\xi))) \leq -\beta_2(\|\xi\|) + \beta_3(\|x\|) \tag{10}$$

where $\beta_2(\cdot)$ and $\beta_3(\cdot)$ are class-$\mathcal{K}_\infty$ functions satisfying[1]

$$\begin{aligned} r^2 &= \mathcal{O}(\beta_2(r)) && (11)\\ r^2 &= \mathcal{O}(\beta_1(r)) && (12)\\ \beta_3(r) &= \mathcal{O}(\beta_1(r)) && (13) \end{aligned}$$

as $r \to 0^+$, and $\beta_1(\cdot)$ is as defined in Assumption 1. □

The conditions (11)-(13) on the class-$\mathcal{K}_\infty$ functions are imposed to achieve *asymptotic* stability with our high-gain redesign. They can be removed in applications where *practical* asymptotic stability (convergence to a small ball around the origin whose radius can be reduced arbitrarily) is satisfactory. The next assumption will be useful for obtaining an estimate of the signal v in (1) from measurements of x.

Assumption 4: There exists a function $h(x) : \mathbb{R}^n \to \mathbb{R}^m$ such that the $m \times m$ matrix

$$\gamma(x) := L_g h(x) = \frac{\partial h}{\partial x} g(x) \tag{14}$$

is nonsingular for all x. □

Assumption 4 is less restrictive than a similar condition in reference [19] which requires that $\gamma(x) = a(x)I$ for some scalar, positive function $a(x)$. With Assumption 4 we note that the variable $y = h(x)$ satisfies

$$\dot{y} = L_f h(x) + \gamma(x)\, v. \tag{15}$$

Mimicking (15) we build the filter

$$\dot{\hat{y}} = L_f h(x) - \frac{\hat{y} - y}{\mu}, \quad \hat{y}(0) = y(0) \tag{16}$$

where $\mu > 0$. From (1), (3), (15) and (16), the variable

$$l := \frac{\hat{y} - y}{\mu} \tag{17}$$

satisfies

$$\mu\, \dot{l} = -l - \gamma(x)\, \rho(u,\xi), \quad l(0) = 0. \tag{18}$$

When μ is small, l evolves in a faster time-scale than $(x,\, \xi)$, and reaches a small neighborhood of the manifold

$$l = -\gamma(x)\, \rho(u,\xi), \tag{19}$$

which means that an estimate for the input signal $v = \rho(u,\xi)$ is given by

$$\hat{v} = -\gamma(x)^{-1} l. \tag{20}$$

The following dynamic control law makes use of this estimate and, as we prove in Theorem 1 below, guarantees recovery of nominal system trajectories when the two small parameters $\mu > 0$ and $\epsilon > 0$ are tuned appropriately:

$$\begin{aligned} \epsilon\,\dot{\omega} &= \alpha(x) - \hat{v} && (21)\\ u &= S(\alpha(x) + \omega). && (22) \end{aligned}$$

Since (21) makes use of the estimate generated by the filter in (16), the speed of convergence of l to a neighborhood of the manifold (19) must be faster compared to the speed of ω; that is $\mu \ll \epsilon$. Because the two time-scales are dependent on each other, we assign

$$\epsilon = \epsilon_1 \tag{23}$$

$$\mu = \epsilon_1\, \epsilon_2 \tag{24}$$

where ϵ_1 and ϵ_2 are now independent small parameters.

The following theorem shows that the redesign (22) recovers the performance of the nominal system and enlarges the region of attraction arbitrarily as $(\epsilon_1, \epsilon_2) \to 0$.

Theorem 1: Suppose Assumptions 1 *to* 4 *hold. Then, given compact sets* $\Omega_{(x,\xi)} \in \mathbb{R}^{n+r}$ *and* $\Omega_\omega \in \mathbb{R}^m$ *of initial conditions, there exists a pair* $(\epsilon_1^*, \epsilon_2^*)$ *such that for all* $0 < \epsilon_1 < \epsilon_1^*,\ 0 < \epsilon_2 < \epsilon_2^*$ *and for all* $(x(0), \xi(0)) \in \Omega_{(x,\xi)}$

[1]The order of magnitude notation $\mathcal{O}$ is defined as follows: $\beta(r) = \mathcal{O}(\tilde{\beta}(r))$ if $\beta(r)/\tilde{\beta}_1(r)$ is well-defined and continuous for $r > 0$, and if there exist positive constants $\bar{k}$ and $\bar{c}$ such that $|\beta(r)| \leq \bar{k}|\tilde{\beta}(r)|,\ \forall\, |r| < \bar{c}$.

and $\omega(0) \in \Omega_\omega$, the controller (16), (17), (20), (21) and (22) guarantees boundedness of $x(t)$, $\xi(t)$, $\omega(t)$ and $\hat{y}(t)$, and convergence of $x(t)$, $\xi(t)$ to the origin. In addition, given any $\zeta > 0$, there exist $\epsilon_1^{**} > 0$, $\epsilon_2^{**} > 0$ such that for all $0 < \epsilon_1 < \epsilon_1^{**}$, $0 < \epsilon_2 < \epsilon_2^{**}$ and for all $(x(0), \xi(0)) \in \Omega_{(x,\xi)}$, $\omega(0) \in \Omega_\omega$, the solution $\bar{x}(t)$ of the nominal system (4) and $x(t, \epsilon_1, \epsilon_2)$ of the system (1)-(3) with the redesigned controller (16), (17), (20), (21) and (22) satisfy

$$||x(t, \epsilon_1, \epsilon_2) - \bar{x}(t)|| \leq \zeta \quad \forall t \geq 0. \tag{25}$$

Proof : We define the off-manifold variables

$$\tilde{l} = l + \gamma(x)\rho(S(\alpha(x) + \omega), \xi) \tag{26}$$
$$\tilde{\omega} = \omega + \alpha(x) - \vartheta(\alpha(x), \xi) \tag{27}$$

and stack the states as $\chi =$col(x, ξ) so that the closed loop system with the controller (22) can be written as

$$\dot{\chi} = \begin{bmatrix} \tilde{f}(x) - g(x)\,\Delta(\chi, \tilde{\omega}) \\ q(\xi, S(\vartheta(\alpha(x), \xi) + \tilde{\omega})) \end{bmatrix} \tag{28}$$
$$\epsilon_1\,\dot{\tilde{\omega}} = \Delta(\chi, \tilde{\omega}) + \gamma(x)^{-1}\tilde{l} + \epsilon_1\,\nu_1(\chi, \tilde{\omega}) \tag{29}$$
$$\begin{aligned}\epsilon_1\,\epsilon_2\,\dot{\tilde{l}} &= -\tilde{l} + \epsilon_1\,\epsilon_2\,\nu_2(\chi, \tilde{\omega}) \\ &\quad + \epsilon_2\,\nu_3(\chi, \omega)\,(\Delta(\chi, \tilde{\omega}) + \gamma(x)^{-1}\tilde{l})\end{aligned} \tag{30}$$

where

$$\begin{aligned}\tilde{f}(x) &:= f(x) + g(x)\alpha(x) \\ \Delta(\chi, \tilde{\omega}) &:= \alpha(x) - \rho(S(\vartheta(\alpha(x), \xi) + \tilde{\omega}), \xi) \\ \nu_1(\chi, \tilde{\omega}) &:= \left[\frac{\partial}{\partial \chi}(\alpha(x) - \vartheta(\alpha(x), \xi))\right]\dot{\chi} \\ \nu_2(\chi, \tilde{\omega}) &:= \left[\frac{\partial}{\partial \chi}(\gamma(x)\rho(S(\alpha(x) + \omega), \xi))\right]\dot{\chi} \\ \nu_3(\chi, \omega) &:= \frac{\partial}{\partial \omega}(\gamma(x)\rho(S(\alpha(x) + \omega), \xi)).\end{aligned}$$

The key properties of the function $\Delta(\chi, \tilde{\omega})$ are given by the following Lemma.

Lemma 1: The function $\Delta(\chi, \tilde{\omega})$ satisfies

1) $\Delta(\chi, 0) = 0$
2) $\tilde{\omega}^T\,\Delta(\chi, \tilde{\omega}) \leq -k\|\tilde{\omega}\|^2$

where k is as in Assumption 2.
Proof : See proof of Lemma 1 in [5].

For the reduced system (obtained by setting $\epsilon_1 = \epsilon_2 = 0$ along the trajectories of (28)-(30)) we need a Lyapunov function satisfying (31) below. The following Lemma shows that its existence follows from Assumptions 1 and 3.

Lemma 2 : Assumptions 1 and 3 imply that there exists a positive definite, radially unbounded C^2 Lyapunov function $V_4(\chi)$ satisfying

$$\frac{\partial V_4}{\partial \chi}\begin{bmatrix} \tilde{f}(x) \\ q(\xi, S(\vartheta(\alpha(x), \xi))) \end{bmatrix} \leq -\|\chi\|^2, \quad \forall \chi. \tag{31}$$

Proof: From (12)-(13) in Assumption 3, we can choose $\tilde{\beta}_1(r) \in \mathcal{K}_\infty$ such that

$$r^2 = \mathcal{O}(\tilde{\beta}_1(r)) \tag{32}$$
$$\tilde{\beta}_1(r) = \mathcal{O}(\beta_1(r)) \tag{33}$$
$$\tilde{\beta}_1(r) \geq 2\beta_3(r). \tag{34}$$

Considering equation (5) in Assumption 1 and equation (33) above, using the result in Theorem 2 in [21] we can find a positive definite, radially unbounded C^2 Lyapunov function $\tilde{V}_1(x)$ satisfying

$$\frac{\partial \tilde{V}_1}{\partial x}\,\tilde{f}(x) \leq -\tilde{\beta}_1(\|x\|) \quad \forall x. \tag{35}$$

Defining

$$V_3(\chi) = \tilde{V}_1(x) + V_2(\xi) \tag{36}$$

where $V_2(\xi)$ is the Lyapunov function in Assumption 3, from (10) and (34) we see that along the trajectories of (28)-(30) with $\epsilon_1 = \epsilon_2 = 0$, $V_3(\cdot)$ satisfies

$$\dot{V}_3(\chi) \leq -\frac{1}{2}\tilde{\beta}_1(\|x\|) - \beta_2(\|\xi\|) \tag{37}$$
$$\leq -\beta_4(\|\chi\|) \quad \forall \chi \tag{38}$$

for some $\beta_4(\cdot) \in \mathcal{K}_\infty$, which, from (11) and (32), satisfies

$$r^2 = \mathcal{O}(\beta_4(r)) \quad as\ r \to 0^+. \tag{39}$$

Therefore, applying Theorem 2 of [21] again, we can find a positive definite, radially unbounded C^2 Lyapunov function $V_4(\chi)$ satisfying (31). □

Coming back to the proof of Theorem 1, we consider the augmented Lyapunov function

$$W(\chi, \tilde{l}, \tilde{\omega}) = V_4(\chi) + \frac{\tilde{l}^T\tilde{l}}{2} + \frac{\tilde{\omega}^T\tilde{\omega}}{2}, \tag{40}$$

and take its time derivative along (28)-(30) to get

$$\begin{aligned}\dot{W} &= \frac{\partial V_4^T}{\partial \chi}\begin{bmatrix} \tilde{f}(x) - g(x)\,\Delta(\chi, \tilde{\omega}) \\ q(\xi, S(\vartheta(\alpha(x), \xi) + \tilde{\omega})) \end{bmatrix} + \frac{\tilde{\omega}^T\Delta(\chi, \tilde{\omega})}{\epsilon_1} \\ &\quad + \frac{1}{\epsilon_1}\tilde{\omega}^T\gamma(x)^{-1}\tilde{l} + \tilde{\omega}^T\nu_1(\chi, \tilde{\omega}) \\ &\quad - \frac{1}{\epsilon_1\epsilon_2}\,\tilde{l}^T\,\tilde{l} + \tilde{l}^T\,\nu_2(\chi, \tilde{\omega}) \\ &\quad + \frac{1}{\epsilon_1}\tilde{l}^T\,\nu_3(\chi, \tilde{\omega})(\Delta(\chi, \tilde{\omega}) + \gamma(x)^{-1}\tilde{l}).\end{aligned} \tag{41}$$

Since $l(0) = 0$ in (17), we get $\tilde{l}(0) = \gamma(x(0))\rho(S(\alpha(x(0)) + \omega(0)), \xi(0))$ which means that for the given compact set $(\Omega_\chi := \Omega_{(x,\xi)}) \times \Omega_\omega$ of initial conditions $(x(0),\, \xi(0))$, $\omega(0)$, we can find a corresponding compact set $\Omega_{\tilde{l}}$ of initial conditions $\tilde{l}(0)$. We then find a level set Ω_c of W such that

$$\Omega_\chi \times \Omega_{\tilde{\omega}} \times \Omega_{\tilde{l}} \subseteq \Omega_c \tag{42}$$

and positive numbers L_1, L_2, L_3, L_4, L_5 and L_6 such that on this level set Ω_c

$$\begin{aligned} \left\| \frac{\partial V_4}{\partial \chi} \right\| &\leq L_1 \|\chi\| && (43)\\ \|\tilde{f}(x)\| &\leq L_2 \|x\| && (44)\\ \|\gamma(x)^{-1}\| &\leq L_3 && (45)\\ \|g(x)\| &\leq L_4 && (46)\\ \|\Delta(\chi,\tilde{\omega})\| &\leq L_5 \|\tilde{\omega}\| && (47)\\ \|\nu_3(\chi,\omega)\| &\leq L_6 && (48) \end{aligned}$$

hold. In particular, (47) follows from the first property of $\Delta(\chi, \tilde{\omega})$ stated in Lemma 1. Also, since $\alpha(\cdot)$ and $\vartheta(\cdot,\cdot)$ are at least C^1 in their first arguments, there exist positive numbers L_7, L_8 such that

$$\begin{aligned} \nu_1(\chi,\tilde{\omega}) &\leq L_7 (\|\chi\| + \|\tilde{\omega}\|) && (49)\\ \nu_2(\chi,\tilde{\omega}) &\leq L_8 (\|\chi\| + \|\tilde{\omega}\|) && (50) \end{aligned}$$

hold in Ω_c.

Substituting inequalities (43)-(50) in (41) we get

$$\begin{aligned} \dot{W} \leq\; & -\|\chi\|^2 + \frac{\partial V_4^T}{\partial \chi} \begin{bmatrix} -g(x)\,\Delta(\chi,\tilde{\omega}) \\ \psi(\chi,\tilde{\omega}) \end{bmatrix} + L_7 \|\chi\| \, \|\tilde{\omega}\| \\ & - \left(\frac{k}{\epsilon_1} - M_1 \right) \|\tilde{\omega}\|^2 - \left(\frac{1}{\epsilon_1 \epsilon_2} - \frac{M_2}{\epsilon_1} \right) \|\tilde{l}\|^2 \\ & + \left(\frac{M_3}{\epsilon_1} + M_4 \right) \|\tilde{\omega}\| \|\tilde{l}\| + M_4 \|\chi\| \, \|\tilde{l}\| \qquad (51) \end{aligned}$$

where $M_1 := L_7$, $M_2 := L_3 L_6$, $M_3 = L_6 L_5 + L_3$, $M_4 := L_8$ and

$$\begin{aligned} \psi(\chi,\tilde{\omega}) \;:=\;& q(\xi, S(\vartheta(\alpha(x),\xi) + \tilde{\omega})) \\ & -q(\xi, S(\vartheta(\alpha(x),\xi))). \qquad (52) \end{aligned}$$

Since $\psi(\chi, 0) = 0$, we apply [7, Proof of Lemma 9.2.1] to get

$$\psi(\chi,\tilde{\omega}) := \tilde{\psi}(\chi,\tilde{\omega})\,\tilde{\omega} \qquad (53)$$

where $\tilde{\psi}(\cdot,\cdot)$ satisfies

$$\|\tilde{\psi}(\chi,\tilde{\omega})\| \leq L_9 \qquad (54)$$

in Ω_c, L_9 being a positive number. Therefore, the second term on the RHS of (51) can be bounded as

$$\left\| \frac{\partial V_4^T}{\partial \chi} \begin{bmatrix} -g(x)\,\Delta(\chi,\tilde{\omega}) \\ \psi(\chi,\tilde{\omega})) \end{bmatrix} \right\| \leq L_{10} \|\chi\| \|\tilde{\omega}\| \qquad (55)$$

where $L_{10} = \min(L_1 L_4 L_5,\, L_1 L_9)$. Substituting (55) in (51), and referring to the proof of Theorem 1 in [5], it can be shown that for all $\epsilon_1 \in (0, \epsilon_1^*)$ and $\epsilon_2 \in (0, \epsilon_2^*)$ where

$$\begin{aligned} \epsilon_1^* &= \min \left[\frac{k}{2\left(M_1 + M_5^2 + \frac{M_4 k}{2M_3} \right)},\; \frac{M_3^2 + 2kM_2}{M_4 M_3 + 2kM_4^2} \right], \\ \epsilon_2^* &= \frac{k}{2M_2 k + M_3^2} \qquad (56) \end{aligned}$$

with $M_5 := L_7 + L_{10}$, $\dot{W}(\chi, \tilde{l}, \tilde{\omega})$ in (51) is negative definite in Ω_c. This concludes the first part of the theorem.

The proof for the second part of the theorem is analogous to that of Theorem 1 in [5] and, hence, will be skipped to save space. It follows that there exists a pair $(\epsilon_1^{**} > 0,\ \epsilon_2^{**} > 0)$ such that for all $\epsilon_1 \in (0, \epsilon_1^{**})$ and $\epsilon_2 \in (0, \epsilon_2^{**})$, we have

$$\left\| \begin{matrix} x(t,\epsilon_1,\epsilon_2) - \bar{x}(t) \\ \xi(t,\epsilon_1,\epsilon_2) - \bar{\xi}(t) \end{matrix} \right\| \leq \zeta. \qquad (57)$$

Since the reduced system is a cascade of $\dot{\bar{x}}$ and $\dot{\bar{\xi}}$ and the $\bar{x}$-subsystem is unaffected by $\bar{\xi}$, hence from (57) we conclude that for all $t \geq 0$

$$\| x(t,\epsilon_1,\epsilon_2) - \bar{x}(t) \| \leq \zeta.$$

■

Remark 1: In Theorem 1 we require that $\hat{y}(0) = y(0)$ as stated in (16) to avoid peaking. This point has also been suggested in [22]. However, it is not difficult to show that the proof holds true for small errors between $\hat{y}(0)$ and $y(0)$. In implementation $\hat{y}(0) = y(0)$ can be achieved by resetting the filter initial conditions. Alternatively the restriction $\hat{y}(0) = y(0)$ can be removed by saturating the controller (22) as in the high-gain observer literature [3].

Remark 2: If, in (1)-(3) the output is of the form

$$v = \delta(\xi) + \phi(u) \qquad (58)$$

where $\delta(\cdot)$ is unknown while $\phi(\cdot)$ is known and invertible, then it suffices to design only one filter. Since by Assumption 4, $y = h(x)$ satisfies

$$\dot{y} = L_f h(x) + \gamma(x)(\delta(\xi) + \phi(u)), \qquad (59)$$

the filter is constructed as

$$\dot{\hat{y}} = L_f h(x) + \gamma(x)\phi(u) - \frac{\hat{y} - y}{\epsilon}, \quad \hat{y}(0) = y(0) \qquad (60)$$

where $\epsilon > 0$ is a small parameter. Then $l := (\hat{y} - y)/\epsilon$ satisfies

$$\epsilon\,\dot{l} = -\gamma(x)\,\delta(\xi) - l, \quad l(0) = 0 \qquad (61)$$

and an estimate for the input uncertainty, computed over the fast time scale, is obtained as

$$\hat{\delta} = -\gamma(x)^{-1} l, \qquad (62)$$

and the control input is designed directly as

$$u = \phi^{-1}(\alpha(x) + \gamma(x)^{-1} l). \qquad (63)$$

III. Example

We consider the plant with nonlinear unmodeled dynamics given in [2]:

$$\begin{aligned} \dot{x}_1 &= x_1^3 + x_2 && (64)\\ \dot{x}_2 &= x_2^2 + (1 + x_1^2)v && (65)\\ \dot{\xi}_1 &= -2\xi_1 - \xi_1^3 + \xi_2 && (66)\\ \dot{\xi}_2 &= -2\xi_2 + u && (67)\\ v &= \xi_1 + \frac{10\xi_2^3 - \xi_2}{1 + \xi_2^2} + u && (68) \end{aligned}$$

where $x = (x_1, x_2)$ are the plant states, $\xi = (\xi_1, \xi_2)$ are the unmodeled states and u is the input. The nominal system is feedback linearizable with the change of variables $z_1 = x_1$, $z_2 = \dot{x}_1$, and the corresponding nominal control is defined as

$$\alpha(x) = \frac{x_2^2 - 3x_1^2(x_1^3 + x_2)}{1 + x_1^2} - k_1 z_1 - k_2 z_2 \quad (69)$$

where $k_1 > 0, k_2 > 0$ are design constants. Hence, a quadratic Lyapunov function constructed in z-coordinates satisfies Assumption 1 with a locally quadratic $\beta_1(\|x\|)$. Assumption 2 holds with $S(\omega) = \omega$ as the high-frequency gain is positive. Assumption 3 is satisfied with the Lyapunov function

$$V_2(\xi) = \xi_1^2 + \xi_2^2 \quad (70)$$

which yields

$$\begin{aligned} \dot{V}_2 &= -4\xi_1^2 - 2\xi_1^4 - \xi_2^2(4 - \frac{2}{1+\xi^2}) - \frac{20\,\xi_2^4}{1+\xi^2} + 2\xi_2\, v \\ &\leq -\|\xi\|^2 + v^2. \end{aligned} \quad (71)$$

Thus, the class-$\mathcal{K}_\infty$ function $\beta_2(\cdot)$ in equation (10) of Assumption 3, in this case, is $\beta_2(\|\xi\|) = \|\xi\|^2$. Setting $v = \alpha(x)$ in (71), we obtain a locally quadratic class-$\mathcal{K}_\infty$ function $\beta_3(\|x\|)$ as in (10). Also, since both $\beta_1(\cdot)$ and $\beta_3(\cdot)$ are locally quadratic, condition (13) of Assumption 3 is satisfied.

Since the high-frequency gain is unity, for this example we apply the two-time-scale design discussed in Remark 1. The filter is designed as

$$\dot{\hat{x}}_2 = x_2^2 + (1 + x_1^2)u - \frac{\hat{x}_2 - x_2}{\epsilon}, \qquad \hat{x}_2(0) = x_2(0) \quad (72)$$

and the control input is

$$u = \alpha(x) + \frac{l}{1 + x_1^2} \quad (73)$$

where $l = (\hat{x}_2 - x_2)/\epsilon$. Figure 1 shows the nominal trajectory recovery for x_1 as ϵ is reduced.

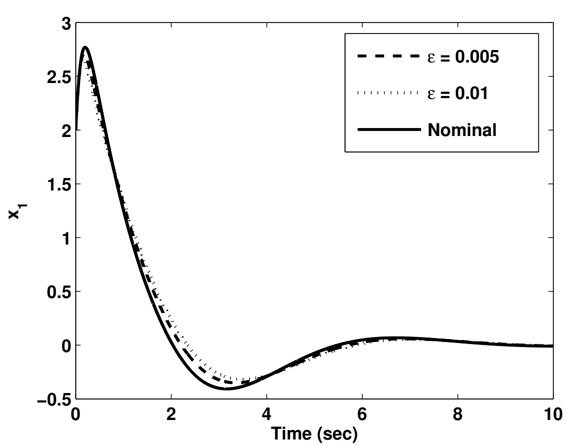

Fig. 1. Nominal vs controlled responses of x_1 for $\epsilon_1 = 0.005$ and 0.01

IV. Conclusion

In this paper we presented a redesign that achieves stabilization and nominal performance recovery of nonlinear systems with input unmodeled dynamics. The effectiveness of this extended redesign in recovering performance has been illustrated by applying it to an example. One demerit of the proposed design is the rapid rate of growth of the control input due to its dependence on the filter gain. This is typical to any high gain design and calls in for a trade-off between the practical choice for the gain to maintain a limited control effort and the closeness of trajectories. An extension of this design to high relative degree input unmodeled dynamics is currently being pursued.

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