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Network Localization Using Graph Decomposition and Rigidity †

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Abstract— **The sensor network localization problem with distance information is to determine the positions of all the sensors in a network given the positions of some sensors and the distances between some pairs of sensors. One approach to localizing a large network is to divide the network into smaller subnetworks whereby each subnetwork is localized in its own coordinate system. We present two algorithms which use linear algebra methods for computing the actual sensor positions given the local solutions of a collection of subnetworks. We also use graph rigidity theory to characterize collections of subnetworks for which the algorithms are applicable.**

I. Introduction

The sensor network localization problem with distance information is to determine the positions of all the sensors in a network given the positions of some sensors and the distances between some pairs of sensors. A sensor whose position is given is called an *anchor*. A network in $\mathbb{R}^d$ is said to be *localizable* if there exists exactly one position in $\mathbb{R}^d$ corresponding to each non-anchor sensor such that the given inter-sensor distances are satisfied. The authors of [1] use rigidity theory to give the necessary and sufficient conditions for a network to be localizable. Furthermore, for networks in $\mathbb{R}^d$ where $d \leq 2$, the conditions are graph based and can be checked efficiently. The localization problem for a network of n sensors and m known distances has an equivalent formulation as a system of m simultaneous quadratic equations in $2n$ variables. In general, such systems of simultaneous equations are difficult to solve directly, especially when n is large. In [2], it was shown that the process of localizing a network is NP-hard even if the network is known to be localizable.

One approach to localizing a network with a large number of sensors is to first decompose the network into subnetworks. A *local solution* for a subnetwork is any assignment of positions to the subnetwork's sensors so that their positions are correct relative to each other, i.e. the actual positions of the sensors can be obtained from the assigned positions by applying a simple Euclidean transformation to all the assigned positions. Ideally, computing a local solution for a subnetwork would pose a less difficult localization problem than attempting to localize the entire network all at once. Once a local solution is obtained for each subnetwork, the local solutions must be "merged" into a solution for the entire network, i.e. an assignment of positions to the sensors so that *all* the sensors are correctly positioned relative to each other. In general, merging local solutions to obtain a solution for the entire network is difficult. In [3], global optimization techniques are used for merging local solutions, and heuristics are proposed for detecting local minimums. In [4], an algorithm is proposed for computing actual sensor positions from the local solutions by means of "stress matrices."

In this paper, we study the properties of collections of subnetworks whose local solutions can be merged into a solution of the entire network using purely linear algebra methods. We use graph rigidity theory to characterize collections of subnetworks for which we present two linear algebra based algorithms for computing the actual sensor positions from given local solutions of subnetworks in a collection. The first is based on an algorithm proposed in [4] which merges local solutions by means of "stress matrices," and the second is based on the sequential localization algorithm Sweeps proposed in [5], [6], [7].

In Section II, we review the theoretical background of the localization problem, and we give the terms and definitions to be used in the exposition that follows. The two algorithms are presented in Section III and Section IV respectively, together with the graphical characterizations of subnetwork collections for which they are applicable. We conclude in Section V with further characterizations of the "efficiently localizable" networks studied in [8].

II. Background

A *multi-point* $p = \{p_1, \ldots, p_n\}$ in d-dimensional space is a set of n points in $\mathbb{R}^d$ labelled $p_1, \ldots, p_n$. Because we will only be concerned with networks in the plane, we will henceforth restrict our attention to the case of $d = 2$. Two multi-points $p = \{p_1, \ldots, p_n\}$ and $q = \{q_1, \ldots, q_n\}$ of n points in $\mathbb{R}^2$ are *congruent* if $\| p_i - p_j \| = \| q_i - q_j \|$ for all $i, j \in \{1, \ldots, n\}$. A graph with vertex set $\mathcal{V}$ and edge set $\mathcal{E}$ is denoted $(\mathcal{V}, \mathcal{E})$. A *point formation* of n points at a multi-point $p = \{p_1, \ldots, p_n\}$ consists of p and a simple undirected graph $\mathbb{G}$ with vertex set $\mathcal{V} = \{1, \ldots, n\}$, and is denoted by $(\mathbb{G}, p)$. If (i, j) is an edge in $\mathbb{G}$, then the *length of edge* (i, j) in the point formation $(\mathbb{G}, p)$ is the distance between p_i and p_j, i.e. $\| p_i - p_j \|$. Two point formations with the same graph have the same

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edge lengths just in case the length of each edge in the graph is the same in both point formations.

For any multi-point $p = \{p_1, \ldots, p_n\}$ in $\mathbb{R}^2$ and $\varepsilon > 0$, let $\mathscr{B}_p(\varepsilon)$ denote the set of all multi-points $q = \{q_1, \ldots, q_n\}$ in $\mathbb{R}^2$ where $\| p_i - q_i \| < \varepsilon$ for all $i \in \{1, \ldots, n\}$. A point formation $(\mathbb{G}, p)$ is *rigid* in $\mathbb{R}^2$ if there exists $\varepsilon > 0$ such that for all $q \in \mathscr{B}_p(\varepsilon)$, p and q are congruent whenever $(\mathbb{G}, p)$ and $(\mathbb{G}, q)$ have the same edge lengths. Roughly speaking, a rigid point formation is one that cannot be continuously deformed without causing an edge length to change. A graph $\mathbb{G}$ is said to be *rigid* in $\mathbb{R}^2$ if there exists a multi-point p in $\mathbb{R}^2$ and $\varepsilon > 0$ such that $(\mathbb{G}, q)$ is rigid in $\mathbb{R}^2$ for all $q \in \mathscr{B}_p(\varepsilon)$. A set consisting of a finite number of elements from $\mathbb{R}$ is said to be *algebraically independent over the rationals* if its elements do not satisfy any non-zero polynomial with rational coefficients. A multi-point is said to be *generic* if the set consisting of the coordinates of its points is algebraically independent over the rationals. It is known that if a multi-point p is generic, then a point formation $(\mathbb{G}, p)$ is rigid if and only if $\mathbb{G}$ is rigid. A point formation $(\mathbb{G}, p)$ in $\mathbb{R}^2$ is *globally rigid* in $\mathbb{R}^2$ if multi-points p and q are congruent whenever $(\mathbb{G}, p)$ and $(\mathbb{G}, q)$ have the same edge lengths. In other words, edge lengths of a globally rigid point formation uniquely determine all inter-vertex distances. A graph $\mathbb{G}$ is said to be *globally rigid* in $\mathbb{R}^2$ if there exist multi-point p in $\mathbb{R}^2$ and $\varepsilon > 0$ such that $(\mathbb{G}, q)$ is globally rigid in $\mathbb{R}^2$ for all $q \in \mathscr{B}_p(\varepsilon)$. It is known that if a multi-point p in $\mathbb{R}^2$ is generic, then the point formation $(\mathbb{G}, p)$ is globally rigid in $\mathbb{R}^2$ if and only if $\mathbb{G}$ is globally rigid in $\mathbb{R}^2$. There are a number of efficient algorithms for determining if a graph is rigid or globally rigid in $\mathbb{R}^d$ when $d \leq 2$ [9], [10]. Since we are only concerned with networks in the plane, we will in the following refer to graphs or point formations which are (globally) rigid in $\mathbb{R}^2$ as simply (globally) rigid.

A network with n sensors is modelled by a point formation $(\mathbb{G}, p)$ where each sensor corresponds to exactly one vertex of $\mathbb{G}$, and vice versa, with (i, j) being an edge of $\mathbb{G}$ if i and j are distinct and the distance between the corresponding sensors is known, and $p = \{p_1, \ldots, p_n\}$ where p_i is the position of the sensor corresponding to vertex i. We say that $\mathbb{G}$ is the graph of the network, and p is the multi-point of the network. It is known that if the multi-point of a network in $\mathbb{R}^2$ is generic, then the network is localizable if and only if it has 3 non-collinear anchors and the graph of the network is globally rigid. A number of efficient algorithms exist for determining if a graph is globally rigid in $\mathbb{R}^2$ [10], [9]. Hence, for a network in $\mathbb{R}^2$ with a generic multi-point, it can be efficiently determined if the network is localizable by counting the number of its sensors and analyzing its graph. Since *almost all* multi-points are generic, we will without loss of generality restrict our attention to those networks with generic multi-points. In particular, this implies no two sensors occupy the same point and no three sensors are collinear in the networks we consider.

By an *assignment* of a network with sensor set $\mathscr{V}$ is meant any function $\alpha : \mathscr{V} \to \mathbb{R}^2$ where $\alpha(u) \neq \alpha(v)$ for all $u, v \in \mathscr{V}$, $u \neq v$. An assignment is *consistent* if for all sensors $u, v \in \mathscr{V}$ such that the distance between u and v is known, say as d_{uv}, then $\| \alpha(u) - \alpha(v) \| = d_{uv}$. By a *sub-assignment* of a network is meant any function that is the restriction of an assignment of the network to some nonempty subset of its domain. A sub-assignment $\beta : \mathscr{U} \to \mathbb{R}^2$ is *consistent* if for all sensors $u, v \in \mathscr{U}$ such that the distance between u and v is known, say as d_{uv}, then $\| \beta(u) - \beta(v) \| = d_{uv}$. For any sub-assignment β, let $\mathscr{D}(\beta)$ denote the domain of β, and for any $\mathscr{U} \subset \mathscr{D}(\beta)$, let $\beta(\mathscr{U})$ denote the set $\{\beta(u) \mid u \in \mathscr{U}\}$. Given two sub-assignments α and β, we write $\alpha \sim \beta$ if there does not exist $u \in \mathscr{D}(\alpha) \cap \mathscr{D}(\beta)$ such that $\alpha(u) \neq \beta(u)$.

For a graph $\mathbb{G} = (\mathscr{V}, \mathscr{E})$, and any subset $\mathscr{U}$ of $\mathscr{V}$, let $\mathbb{G}(\mathscr{U})$ denote the graph induced in $\mathbb{G}$ by vertices in $\mathscr{U}$. For any subset $\mathscr{F}$ of edge set $\mathscr{E}$, let $V(\mathscr{F})$ denote the set of all vertices some edge in $\mathscr{F}$ is incident on, and let $\mathbb{G}(\mathscr{F})$ denote the graph $(V(\mathscr{F}), \mathscr{F})$. We call $\mathbb{G}(\mathscr{F})$ the graph induced in $\mathbb{G}$ by edges in $\mathscr{F}$. A subgraph of $\mathbb{G}$ is said to be *proper* if it is not equal to $\mathbb{G}$, and a subgraph of $\mathbb{G}$ with the same vertex set as $\mathbb{G}$ is called a *spanning* subgraph of $\mathbb{G}$. Given a subgraph $\mathbb{H} = (\mathscr{U}, \mathscr{F})$ of $\mathbb{G}$ and a set $\mathscr{T} = \{(u, v) | u, v \in \mathscr{V}, u \neq v\}$, let $\mathbb{H} \cup \mathscr{T}$ denote the graph $(\mathscr{U} \cup V(\mathscr{T}), \mathscr{F} \cup \mathscr{T})$. For subgraphs $\mathbb{G}_i = (\mathscr{V}_i, \mathscr{E}_i)$ and $\mathbb{G}_j = (\mathscr{V}_j, \mathscr{E}_j)$ of $\mathbb{G}$, let the union of $\mathbb{G}_i$ and $\mathbb{G}_j$, denoted $\mathbb{G}_i \cup \mathbb{G}_j$, be the graph denoted by $(\mathscr{V}_i \cup \mathscr{V}_j, \mathscr{E}_i \cup \mathscr{E}_j)$. A subgraph $\mathbb{G}_i$ of $\mathbb{G}$ is said to be *edge distinct* from subgraph $\mathbb{G}_j$ of $\mathbb{G}$ if there is at least one edge in $\mathbb{G}_i$ which is not in $\mathbb{G}_j$. For any matrix A, let A' denote the transpose of A.

A. Local Solutions

Let $\mathbb{N}$ denote a localizable network in $\mathbb{R}^2$ with sensors labelled 1 through n, and positioned at $\pi(1), \ldots, \pi(n)$, respectively. Let $\mathbb{G} = (\mathscr{V}, \mathscr{E})$ denote the graph of $\mathbb{N}$. For each $(u, v) \in \mathscr{E}$, let d_{uv} denote the known distance between sensors u and v. Since $\mathbb{N}$ is localizable and the multi-point of $\mathbb{N}$ is assumed to be generic, we have that $\mathbb{G}$ must be globally rigid in $\mathbb{R}^2$. Let $\mathbb{N}_1, \ldots, \mathbb{N}_s$, $s \geq 2$, be subnetworks of $\mathbb{N}$, and suppose a local solution is computed for each $\mathbb{N}_i$ using only the known inter-sensor distances in $\mathbb{N}_i$. For each $i \in \{1, \ldots, s\}$, let $\mathscr{V}_i$ denote the set of sensors in $\mathbb{N}_i$. It is easy to show that $\mathbb{G}(\mathscr{V}_i)$ must be globally rigid in order for a local solution of $\mathbb{N}_i$ to be computable using just the known distances among sensors in $\mathbb{N}_i$. For each $\mathbb{N}_i$, let $\mathbb{G}_i$ denote any spanning subgraph of $\mathbb{G}(\mathscr{V}_i)$ which is also globally rigid. By definition, the vertex set of $\mathbb{G}_i$ must be $\mathscr{V}_i$, and let $\mathscr{E}_i$ denote the edge set of $\mathbb{G}_i$.

For each $i \in \{1, \ldots, s\}$, let the computed local solution of $\mathbb{N}_i$ be denoted by sub-assignment α_i. In the following, we present two algorithms for computing a consistent assignment of the entire network $\mathbb{N}$ from the given local solutions, using purely *linear algebra* methods. Since $\mathbb{N}$ is localizable, it follows that for all consistent assignments α of $\mathbb{N}$, it must be the case that $\| \alpha(u) - \alpha(v) \| = \| \pi(u) - \pi(v) \|$ for all $u, v \in \mathscr{V}$. Hence, any consistent assignment α of $\mathbb{N}$ is a solution of the entire network $\mathbb{N}$, and the actual sensor positions can be obtained from $\alpha(\mathscr{V})$ by a Euclidean transformation which can be easily computed by solving a linear system of equations using the anchor positions. For each $i \in \{1, \ldots, s\}$,

we will without loss of generality assume the multi-point consisting of the points in $\alpha_i(\mathscr{D}(\alpha_i))$ is generic since *almost all* multi-points are generic, and in particular, the multi-point of $\mathbb{N}$ is assumed to be generic.

III. Stress Matrices

In [4], it was shown that a "stress matrix" which satisfies a maximal rank condition can be used to compute a consistent assignment of $\mathbb{N}$ using strictly linear algebra methods. In this section, we use graph rigidity theory to characterize the properties of subnetwork collections for which the local solutions can be used to obtain a stress matrix of maximal rank. More specifically, we will do so by considering the kernel of the transpose of the "rigidity matrix" associated with the point formation modelling the network.

Let $(\mathbb{H},q)$ be any point formation in $\mathbb{R}^2$, and suppose $\mathbb{H}$ has m edges and k vertices. Let $\{1,\ldots,k\}$ denote the vertex set of $\mathbb{H}$, and for each vertex i of $\mathbb{H}$, let q_i denote the point in q corresponding to vertex i, and let q_i^1 and q_i^2 denote the x and y coordinates of q_i respectively. Let $\mathscr{F}$ denote the edge set of $\mathbb{H}$, and order the edges of $\mathbb{H}$ lexicographically, i.e. edge (u,v), $u<v$, precedes edge (y,z), $y<z$, if either $u<y$ or $u=y, v<z$. Let ω be a $1\times m$ row vector of real numbers. For each edge $(u,v)\in\mathscr{F}$, let ω_{uv} denote the ith element of ω if (u,v) is the ith edge in the lexicographical ordering of the edges of $\mathbb{H}$. The row vector ω is called a *stress* of $(\mathbb{H},q)$ if the following holds for each vertex v of $\mathbb{H}$: $\sum_{(u,v)\in\mathscr{F}}\omega_{uv}(q_u-q_v)=0$. Suppose ω is a stress of $(\mathbb{H},q)$. A *stress matrix* of $(\mathbb{H},q)$ corresponding to ω, and denoted $\Omega(\omega)$, is the $k\times k$ matrix defined as follows. For $i,j\in\{1,\ldots,k\}$ where $(i,j)\in\mathscr{F}$, let $\Omega(\omega)_{ij}=\omega_{ij}$. For $i,j\in\{1,\ldots,k\}$ where $i\neq j$ and $(i,j)\notin\mathscr{F}$, let $\Omega(\omega)_{ij}=0$. For each $i\in\{1,\ldots,k\}$, let $\Omega(\omega)_{ii}=-\sum_{j\neq i}\Omega(\omega)_{ij}$. Clearly, a point formation has infinitely many stresses and stress matrices associated with it.

The *rigidity matrix* of $(\mathbb{H},q)$, denoted M, is the $m\times 2k$ matrix defined as follows. For each $i\in\{1,\ldots,m\}$, associate the ith row of M with the ith edge in the lexicographical ordering of the edges of $\mathbb{H}$. For the ith row of M, where row i is associated with edge (u,v), let $M_{i,2u-1}=q_u^1-q_v^1$, $M_{i,2u}=q_u^2-q_v^2$, $M_{i,2v-1}=q_v^1-q_u^1$, $M_{i,2v}=q_v^2-q_u^2$, and let $M_{ij}=0$ for all $j\notin\{2u-1,2u,2v-1,2v\}$. It is known that the rank of the rigidity matrix of any point formation in $\mathbb{R}^2$ with k points is at most $2k-3$. Furthermore, when the multi-point q is generic, it is known that the point formation $(\mathbb{H},q)$ is rigid if and only if its rigidity matrix M has rank $2k-3$. Let w be any $1\times m$ row vector where $wM=0$. We say the ith element of w *corresponds to* edge (u,v) if the ith row of M is associated with edge (u,v) of $\mathbb{H}$. A $1\times m$ row vector ω is a stress of $(\mathbb{H},q)$ if and only if $\omega M=0$ [11]. Since each stress of $(\mathbb{H},q)$ corresponds to a stress matrix of $(\mathbb{H},q)$, it follows that each vector w where $wM=0$, or equivalently w in the kernel of M', corresponds to a stress matrix of $(\mathbb{H},q)$, which we denote by $\Omega(w)$. The following is a simple observation on the kernel of M':

Lemma 1: Let $(\mathbb{H},q)$ be any point formation in $\mathbb{R}^2$ where $\mathbb{H}$ is rigid and q is generic, and let $\mathscr{V}$ and $\mathscr{F}$ denote the vertex and edges sets of $\mathbb{H}$ respectively. The nullity of the transpose of the rigidity matrix of $(\mathbb{H},q)$ is $|\mathscr{F}|-(2|\mathscr{V}|-3)$.

It is easy to show that any stress matrix $(\mathbb{H},q)$ has rank at most $k-3$ when $k\geq 4$. From [12], [11], it is known that when $k\geq 4$ and the multi-point q is generic, there exists vector w where $wM=0$ and $\Omega(w)$ has rank $k-3$ if and only if $\mathbb{H}$ is globally rigid.

Let α be any consistent assignment of $\mathbb{N}$, and let $p=\{\alpha(1),\ldots,\alpha(n)\}$. Without loss of generality, suppose p is generic. Let R denote the rigidity matrix of the point formation $(\mathbb{G},p)$. Since $\mathbb{G}$ is globally rigid and p is generic, there exists vector w where $wR=0$ and $\Omega(w)$ has rank $n-3$ when $n\geq 4$. To avoid degenerate cases, suppose $n\geq 4$ and $|\mathscr{V}_i|\geq 4$ for each $i\in\{1,\ldots,s\}$. Suppose that without knowing R, we are given a vector w for which $wR=0$ and $\Omega(w)$ has rank $n-3$. In [4], it was shown that such a $\Omega(w)$ can be used to compute a consistent assignment of $\mathbb{N}$ using strictly linear algebra methods; the proof is included for completeness in the Appendix. Hence, if a vector w is known where $wR=0$ and the corresponding stress matrix $\Omega(w)$ has rank $n-3$, then a consistent assignment of $\mathbb{N}$ can be computed using only linear algebra methods. However, without knowing some consistent assignment of $\mathbb{N}$ to begin with, and therefore, without knowing R, it is not obvious how such a w can be obtained. In the next section, we give conditions on $\mathbb{G}_i$, $i\in\{1,\ldots,s\}$, and $\mathbb{G}$ so that such a w can be computed using just the given local solutions α_i, $i\in\{1,\ldots,s\}$.

A. Graphical Conditions

For $i\in\{1,\ldots,s\}$, let R_i denote the rigidity matrix of the point formation $(\mathbb{G}_i,\alpha_i(\mathscr{V}_i))$ where each vertex $u\in\mathscr{V}_i$ corresponds to the point $\alpha_i(u)$. For $i\in\{1,\ldots,s\}$, and vector x where $xR_i=0$, the stress matrix of $(\mathbb{G}_i,\alpha_i(\mathscr{V}_i))$ corresponding to x has rank at most $|\mathscr{V}_i|-3$. By assumption, $|\mathscr{V}_i|\geq 4$ and $\mathbb{G}_i$ is globally rigid for all $i\in\{1,\ldots,s\}$. Hence, there exists a vector x where $xR_i=0$ and the stress matrix of $(\mathbb{G}_i,\alpha_i(\mathscr{V}_i))$ corresponding to x, i.e. $\Omega(x)$, has rank $|\mathscr{V}_i|-3$ [12]. Clearly, R is not known, but each R_i, $i\in\{1,\ldots,s\}$, can be directly obtained using $\mathbb{G}_i$ and the given α_i. In the following, we will give conditions on $\mathbb{G}_i$, $i\in\{1,\ldots,s\}$, so that a vector w, where $wR=0$ and $\Omega(w)$ has rank $n-3$, can be computed using just R_i, $i\in\{1,\ldots,s\}$.

Consider any R_i where $i\in\{1,\ldots,s\}$. Given a vector x where $xR_i=0$, let x_e denote the element of x corresponding to edge e in $\mathbb{G}_i$. We define the "extension" of x, denoted x^+, as follows. Let m denote the number of edges in $\mathbb{G}$, i.e. $m=|\mathscr{E}|$, and let x^+ be a $1\times m$ vector whose elements are defined as follows. Order the edges of $\mathbb{G}$ lexicographically, and for each $k\in\{1,\ldots,m\}$, let e_k denote the kth edge in the ordering. Let the kth element of x^+ be zero if e_k is not an edge in $\mathbb{G}_i$. If e_k is an edge in $\mathbb{G}_i$, then let the kth element of x^+ be x_{e_k}. When x^+ is thus defined, we say that x^+ (of R) is *extended* from x (of R_i).

Lemma 2: If x is a vector such that $xR_i=0$ for some $i\in\{1,\ldots,s\}$, then $x^+R=0$.

For $i \in \{1,\ldots,s\}$, let x_i be any vector where $x_i R_i = 0$, and let w be any linear combination of x_i^+, $i \in \{1,\ldots,s\}$. From lemma 2, we have that $wR = 0$, which implies w corresponds to a stress matrix of $(\mathbb{G}, p)$ which is denoted by $\Omega(w)$. For each row i of R, if the ith row of R corresponds to edge e, then we write w_e for the ith element of w. We now give a graphical necessary condition for stress matrix $\Omega(w)$ of $(\mathbb{G}, p)$ to have rank $n-3$:

Lemma 3: If $\Omega(w)$ has rank $n-3$, then the graph induced in $\mathbb{G}$ by the edge set $\{e \mid e \in \mathscr{E},\ w_e \neq 0\}$ is globally rigid and is a spanning subgraph of $\mathbb{G}$.

It is a direct consequence of Lemma 3 that if $\Omega(w)$ has rank $n-3$, then the union of $\mathbb{G}_i$, $i \in \{1,\ldots,s\}$, must be a spanning globally rigid subgraph of $\mathbb{G}$.

A globally rigid graph is said to be *minimally globally rigid* if the removal of any edge causes the graph to not be globally rigid. A minimally globally rigid graph of $n \geq 4$ vertices has exactly $2n-2$ edges, and any globally rigid graph has a spanning subgraph which is also minimally globally rigid. The following can be shown using Lemmas 1 and 3:

Lemma 4: Suppose $|\mathscr{V}_i| \geq 4$ and $\mathbb{G}_i$ is minimally globally rigid for some $i \in \{1,\ldots,s\}$. If w is a nonzero vector where $wR_i = 0$, then the stress matrix of $(\mathbb{G}_i, \alpha_i(\mathscr{V}_i))$ corresponding to w, denoted $\Omega(w)$, has rank $|\mathscr{V}_i| - 3$, and each element of w is nonzero.

A natural question to ask is if a globally rigid subgraph of at least four vertices can be obtained by removing vertices from a minimally globally rigid graph. A simple consequence of lemma 4 shows the answer to be negative:

Lemma 5: If $\mathbb{H}$ is a minimally globally rigid graph of at least four vertices, then $\mathbb{H}$ does not contain any proper subgraph of at least four vertices which is globally rigid.

Obviously, the set of all vectors u where $uR = 0$ is just the kernel of R'. In the following, we use edge distinct minimally globally rigid graphs to give some graphical properties of collection of subnetworks from which we can "easily" obtain a basis of the kernel of R'.

Lemma 6: Suppose each $\mathbb{G}_i$ is edge distinct from $\bigcup_{j\neq i}\mathbb{G}_j$, and is minimally globally rigid. If $s = |\mathscr{E}| - (2|\mathscr{V}| - 3)$, and for $i \in \{1,\ldots,s\}$, x_i is any nonzero vector such that $x_i R_i = 0$, then $\{x_i^+ \mid i \in \{1,\ldots,s\}\}$ is a basis for the space of all vectors w where $wR = 0$.

The next lemma considers a case where not all of the $\mathbb{G}_i$ are edge distinct from $\bigcup_{j\neq i}\mathbb{G}_j$:

Lemma 7: Suppose each $\mathbb{G}_i$ is minimally globally rigid, and $s = |\mathscr{E}| - (2|\mathscr{V}| - 3)$. Suppose $\mathbb{G}_k$ is not edge distinct from $\bigcup_{j\neq k}\mathbb{G}_j$, but $\mathbb{G}_i$ is edge distinct from $\bigcup_{j\neq i}\mathbb{G}_j$ for all $i \neq k$. If for $i \in \{1,\ldots,s\}$, x_i is a nonzero vector such that $x_i R_i = 0$, then $\{x_i^+ \mid i \in \{1,\ldots,s\}\}$ is a basis for the space of all vectors w where $wR = 0$.

As noted previously, there exists vector w where $wR = 0$ and $\Omega(w)$ has rank $n-3$. Lemmas 6 and 7 immediately suggest a way to compute such a w using just R_i, $i \in \{1,\ldots,s\}$, and without knowing R. Suppose the premise of either Lemma 6 or 7 is satisfied. For each $i \in \{1,\ldots,s\}$, first compute nonzero vector x_i where $x_i R_i = 0$, and let x_i^+ denote the vector extended from x_i. Note that each x_i and x_i^+, $i \in \{1,\ldots,s\}$, can be computed from just R_i. From lemmas 6 and 7, we have that each vector w where $wR = 0$, and stress matrix $\Omega(w)$ of $(\mathbb{G}, p)$ has rank $n-3$, must be a linear combination of vectors in $\{x_i^+ \mid i \in \{1,\ldots,s\}\}$. Hence, there are elements $a_i \in \mathbb{R}$, $i \in \{1,\ldots,s\}$, such that the stress matrix $\Omega(\sum_{i\in\{1,\ldots,s\}} a_i x_i^+)$ of $(\mathbb{G}, p)$ has rank $n-3$, which implies there is a nonzero minor of $\Omega(\sum_{i\in\{1,\ldots,s\}} a_i x_i^+)$ of order $n-3$. This minor corresponds to a polynomial P with integer coefficients whose variables take on values from the entries of $\Omega(\sum_{i\in\{1,\ldots,s\}} a_i x_i^+)$. Note that P cannot be trivial, i.e. P cannot be the zero polynomial, since P is nonzero for the elements of $\Omega(\sum_{i\in\{1,\ldots,s\}} a_i x_i^+)$. Hence, for *almost all* linear combinations of vectors in $\{x_1^+,\ldots,x_s^+\}$, i.e. $u = \sum_{i\in\{1,\ldots,s\}} c_i x_i^+$ for $c_i \in \mathbb{R}$, P is nonzero for the elements of $\Omega(u)$, in which case $\Omega(u)$ must have rank $n-3$ since it has a nonzero minor of order $n-3$.

We now give some non-graphical conditions for obtaining a maximally ranked stress matrix from the given local solutions. Let $\mathscr{B}_i$, $i \in \{1,\ldots,s\}$, denote a basis for the space of all vectors x where $xR_i = 0$, and let $d_i = |\mathscr{B}_i|$. Let $\mathscr{B}_i^+$ denote the set of vectors x^+ of R extended from vectors x of R_i in $\mathscr{B}_i$: $\mathscr{B}_i^+ = \{x^+ \mid x \in \mathscr{B}_i\}$, and let $\mathscr{B}^+ = \bigcup_{i\in\{1,\ldots,s\}} \mathscr{B}_i^+$. The set $\mathscr{B}^+$ is directly computable using the given local solutions α_i, $i \in \{1,\ldots,s\}$. From Lemma 2, we have that if w is a linear combination of vectors in $\mathscr{B}^+$, then $wR = 0$. Let m denote the number of edges in $\mathbb{G}$. It follows from Lemma 1 that $\mathscr{B}^+$ is a spanning set of the space of all vectors w where $wR = 0$ if and only if $\mathscr{B}^+$ has $m-(2n-3)$ linearly independent vectors. As shown above, once a basis or spanning set of the space of all vectors w where $wR = 0$ is known, then it is easy to obtain a vector w such that the stress matrix of $(\mathbb{G}, p)$ corresponding to w has rank $n-3$. A non-graphical necessary condition for $\mathscr{B}^+$ to be a spanning set for the space of all vectors w where $wR = 0$ is that $\sum_{i\in\{1,\ldots,s\}} d_i \geq m-(2n-3)$. Let w be any linear combination of vectors in $\mathscr{B}^+$. By construction, we have that $wR = 0$. We now give a non-graphical sufficiency condition for $\Omega(w)$ to be a stress matrix of $(\mathbb{G}, p)$ with rank $n-3$:

Lemma 8: If the set consisting of the elements of w is algebraically independent over the rationals, then $\Omega(w)$ has rank $n-3$.

For each $\mathbb{N}_i$, $i \in \{1,\ldots,s\}$, all inter-sensor distances in $\mathbb{N}_i$ are implicitly known from the given local solution α_i. In the following, for each $i \in \{1,\ldots,s\}$, let $\mathbb{H}_i$ be any globally rigid graph with vertex set $\mathscr{V}_i$ such that $\bigcup_{i\in\{1,\ldots,s\}} \mathbb{H}_i$ is globally rigid and $\bigcup_{i\in\{1,\ldots,s\}} \mathscr{V}_i = \mathscr{V}$. Let $\mathbb{H} = \bigcup_{i\in\{1,\ldots,s\}} \mathbb{H}_i$. For each $i \in \{1,\ldots,s\}$, let $\bar{R}_i$ denote the rigidity matrix of the point formation $(\mathbb{H}_i, \alpha_i(\mathscr{V}_i))$, and let $\bar{R}$ denote the rigidity matrix of the point formation $(\mathbb{H}, p)$. Clearly, $\bar{R}$ is unknown, and each $\bar{R}_i$ can be obtained directly from $\mathbb{H}_i$ and the given local solution α_i. Furthermore, if w is a vector where $w\bar{R} = 0$ and the stress matrix $\Omega(w)$ of $(\mathbb{H}, p)$ has rank $n-3$, then

$\Omega(w)$ can be used to obtain the sensor positions of $\mathbb{N}$ via the linear algebra based methods outlined in the Appendix. It is straightforward to show that the results in this section apply virtually verbatim when $\mathbb{G}_i$, R_i, $i \in \{1,\ldots,s\}$, $\mathbb{G}$ and R are replaced by $\mathbb{H}_i$, $\bar{R}_i$, $i \in \{1,\ldots,s\}$, $\mathbb{H}$ and $\bar{R}$ respectively.

IV. Sequential merging

The merging of local solutions into a global solutions is equivalent to solving a system of non-linear equations which is in general difficult. In this section we present a "sequential" merging algorithm which processes the local solutions according to a specified order by solving a sequence of systems of linear equations. We then characterize the graph properties of subnetworks whose local solutions can be merged into a solution of the network by the algorithm. To avoid degenerate cases, suppose that the network contains at least four sensors, and each subnetwork $\mathbb{N}_i$, $i \in \{1,\ldots,s\}$, contains at least two sensors.

We will begin with some definitions to be used in describing the sequential algorithm. Consider two sub-assignments β and δ where $\beta \sim \delta$. If $\beta(u) \neq \delta(v)$ for all $u \in \mathscr{D}(\beta)$, $v \in \mathscr{D}(\delta)$ and $u \neq v$, then define $e(\beta,\delta)$ as the sub-assignment β^+ where $\beta^+(u) = \beta(u)$ for all $u \in \mathscr{D}(\beta)$ and $\beta^+(v) = \delta(v)$ for all $v \in \mathscr{D}(\delta) - \mathscr{D}(\beta)$. If $\beta(u) = \delta(v)$ for some $u \in \mathscr{D}(\beta)$, $v \in \mathscr{D}(\delta)$ and $u \neq v$, then define $e(\beta,\delta)$ as the empty function $\emptyset \to \mathbb{R}^2$.

Consider a sub-assignment α where no three points of $\alpha(\mathscr{D}(\alpha))$ are collinear, and let $\mathscr{U}$ be any subset of $\mathscr{V}$. For $\mathscr{W} \subset \mathscr{V}$, let $\mathscr{I}(\mathscr{W},\mathscr{U})$ be the set of all vertices $u \in \mathscr{U}$ such that either $u \in \mathscr{W}$ or u is adjacent to three or more vertices of $\mathscr{W}$ in $\mathbb{G}$. Since no three points of $\alpha(\mathscr{D}(\alpha))$ are collinear, it follows that there is at most one sub-assignment δ where $\mathscr{D}(\delta) = \mathscr{D}(\alpha) \cup \mathscr{I}(\mathscr{D}(\alpha),\mathscr{U})$ whose restriction to $\mathscr{D}(\alpha)$ is equal to α and $\| \delta(u) - \delta(v) \| = d_{uv}$ for all u,v where $u \in \mathscr{I}(\mathscr{D}(\alpha),\mathscr{U})$, $v \in \mathscr{D}(\alpha)$ and $(u,v) \in \mathscr{E}$. When such a δ exists, then let $\bar{e}(\alpha,\mathscr{U}) = \delta$; otherwise, let $\bar{e}(\alpha,\mathscr{U})$ denote the empty function $\emptyset \to \mathbb{R}^2$.

Let $\mathbb{E}$ denote the space of all Euclidean transformations in $\mathbb{R}^2$. Let $\mathscr{S}$ be a set consisting of a finite number of sub-assignments with the same domain, and let $\mathscr{D}(\mathscr{S})$ denote the domain of any sub-assignment in $\mathscr{S}$. Let α be a sub-assignment denoting the local solution of some subnetwork, and define $g(\mathscr{S},\alpha)$ as:

$$\begin{aligned} g(\mathscr{S},\alpha) = \{ & e(\bar{e}(\beta,\mathscr{D}(\alpha)), L\circ\alpha) \mid \beta \in \mathscr{S},\ L \in \mathbb{E}, \\ & \mathscr{D}(\bar{e}(\beta,\mathscr{D}(\alpha))) \neq \emptyset,\ \bar{e}(\beta,\mathscr{D}(\alpha)) \sim L\circ\alpha, \\ & \mathscr{D}(e(\bar{e}(\beta,\mathscr{D}(\alpha)), L\circ\alpha)) \neq \emptyset \} \end{aligned}$$

Lemma 9 gives a simple condition on the domains of sub-assignments in $\mathscr{S}$ and the domain of α so as to guarantee that $g(\mathscr{S},\alpha)$ consists of a finite number of sub-assignments:

Lemma 9: Suppose $|\mathscr{D}(\alpha)| \geq 2$ and there is a sub-assignment in $\mathscr{S}$ which is the restriction of a consistent assignment of $\mathbb{N}$ to $\mathscr{D}(\mathscr{S})$. Then $g(\mathscr{S},\alpha)$ is finite if and only if $\mathscr{I}(\mathscr{D}(\mathscr{S}),\mathscr{D}(\alpha)) \geq 2$.

For $i \in \{1,\ldots,s\}$, let $\mathbb{G}_i = \mathbb{G}(\mathscr{V}_i)$, and let $\mathbb{G}_{o(1)},\ldots,\mathbb{G}_{o(s)}$ denote some ordering of $\mathbb{G}_1,\ldots,\mathbb{G}_s$. We now describe a sequential merging algorithm which computes a sequence of sub-assignment sets $\mathscr{B}(i)$, $i = 1,\cdots,s$, by processing the local solutions α_i, $i \in \{1,\ldots,s\}$, in the ordering $o(1),\ldots,o(s)$. For $i \in \{1,\ldots,s\}$, let $\mathscr{V}_{1,i} = \bigcup_{j\leq i} \mathscr{V}_{o(j)}$. Let $\mathscr{B}(1) = \{\alpha_{o(1)}\}$, and define $\mathscr{B}(i)$, $i \in \{2,\ldots,s\}$, as:

$$\mathscr{B}(i) = g(\mathscr{B}(i-1), \alpha_{o(i)}) \tag{1}$$

It can be shown from Lemma 9 and the assumption that $\alpha_{o(1)}$ is a local solution that $\mathscr{B}(i)$ is finite for all $i \in \{1,\ldots,s\}$ if and only if $|\mathscr{I}(\mathscr{V}_{1,j-1},\mathscr{V}_{o(j)})| \geq 2$ for all $j \in \{2,\ldots,s\}$. It is straightforward to show that if each $\mathscr{B}(i)$ is finite, then each $\mathscr{B}(i+1)$ can be computed by solving a finite number of systems of linear equations. Henceforth, assume the ordering $\mathbb{G}_{o(1)},\ldots,\mathbb{G}_{o(s)}$ specified above satisfies $|\mathscr{I}(\mathscr{V}_{1,j-1},\mathscr{V}_{o(j)})| \geq 2$ for all $j \in \{2,\ldots,s\}$.

For $\mathscr{U},\mathscr{W} \subset \mathscr{V}$, let $\mathscr{E}(\mathscr{U},\mathscr{W})$ denote the set of all edges $(u,w) \in \mathscr{E}$ where $u \in \mathscr{U}$ and $w \in \mathscr{W}$. Let $\mathbb{G}_{o(1)} + \mathbb{G}_{o(2)}$ denote the union of $\mathbb{G}_{o(1)}$, $\mathbb{G}_{o(2)}$ and edges in $\mathscr{E}(\mathscr{V}_{o(1)},\mathscr{I}(\mathscr{V}_{o(1)},\mathscr{V}_{o(2)}) - \mathscr{V}_{o(1)})$. For $i \in \{3,\ldots,s\}$, let $\mathbb{G}_{o(1)} + \cdots + \mathbb{G}_{o(i)}$ denote the union of $\mathbb{G}_{o(i)}$, $\mathbb{G}_{o(1)} + \cdots + \mathbb{G}_{o(i-1)}$ and $\mathscr{E}(\mathscr{V}_{1,i-1},\ \mathscr{I}(\mathscr{V}_{1,i-1},\mathscr{V}_{o(i)}) - \mathscr{V}_{1,i-1})$. Using the '+' operation and globally rigid graphs, we can now give the necessary and sufficient condition for each element of $\mathscr{B}(s)$ to be a consistent assignment of $\mathbb{N}$:

Lemma 10: Each element of $\mathscr{B}(s)$ is a consistent assignment of $\mathbb{N}$ if and only if $\mathbb{G}_{o(1)} + \ldots + \mathbb{G}_{o(s)}$ is globally rigid and $\mathscr{V}_{1,s} = \mathscr{V}$.

As can be seen from equation 1, the complexity of computing each $\mathscr{B}(i)$, $i \in \{2,\ldots,s\}$, is entirely dependent on $|\mathscr{B}(i-1)|$. In the following, we consider a special case where $|\mathscr{B}(i)| = 1$ for all $i \in \{1,\ldots,s\}$. We say that the ordering $\mathbb{G}_{o(1)},\ldots,\mathbb{G}_{o(s)}$ is a *super-trilateration* ordering if $|\mathscr{I}(\mathscr{V}_{1,j-1},\mathscr{V}_{o(j)})| \geq 3$ for all $j \in \{2,\ldots,s\}$. If $\mathbb{G}_{o(1)},\ldots,\mathbb{G}_{o(s)}$ is a super-trilateration ordering and $\mathscr{V}_{1,s} = \mathscr{V}$, then it is straightforward to show using Lemma 10 that $|\mathscr{B}(i)| = 1$ for all $i \in \{1,\ldots,s\}$, and $\mathscr{B}(s)$ consists of a consistent assignment of $\mathbb{N}$. For each vertex $v \in \mathscr{V}$, let $\mathscr{N}(v) = \{u \mid u \in \mathscr{V}$ *and* $(u,v) \in \mathscr{E}\}$. The graph $\mathbb{G}$ is said to be *locally globally rigid* if for each $v \in \mathscr{V}$, the graph $\mathbb{G}(\mathscr{N}(v) \cup \{v\})$ contains at least four vertices and is globally rigid.

Lemma 11: If $\mathbb{G}$ is locally globally rigid, and for each $v \in \mathscr{V}$, there is some $\mathbb{G}_i$, $i \in \{1,\ldots,s\}$, such that $\mathbb{G}_i = \mathbb{G}(\mathscr{N}(v) \cup \{v\})$, and for each $i \in \{1,\ldots,s\}$, $\mathbb{G}_i = \mathbb{G}(\mathscr{N}(v) \cup \{v\})$ for some $v \in \mathscr{V}$, then there exists a super-trilateration ordering of $\mathbb{G}_1,\ldots,\mathbb{G}_s$ and $\mathscr{V}_{1,s} = \mathscr{V}$.

V. Efficiently Localizable Networks

A graph is said to have a *trilateration ordering* if its vertices can be ordered as $v_1,v_2,v_3,\ldots,v_n$ so that v_1,v_2,v_3 induce a complete graph, and each v_i, $i > 3$, is adjacent to three or more vertices v_j where $j < i$ [8]. Graphs with trilateration orderings are globally rigid. A network with three anchors and whose graph has a trilateration ordering is efficiently localizable in that it can be localized by solving a number of linear systems of equations that is polynomial in the number of sensors. In the following, we characterize

a subclass of graphs with trilateration orderings using the notions of chordal and split graphs. [1] The following is an easily shown property of split graphs:

Lemma 12: If each vertex of a split graph has degree at least three, then the graph has a trilateration ordering.

For each integer $k > 0$, let $\mathscr{C}(k)$ denote the set of all chordal graphs on k vertices, let $\mathscr{H}(k)$ denote the set of graphs $\mathbb{H}$ in $\mathscr{C}(k)$ such that each vertex of $\mathbb{H}$ has degree at least three, and let $\mathscr{T}(k)$ denote the set of all graphs in $\mathscr{H}(k)$ which has a trilateration ordering.

Lemma 13: As $k \to \infty$, $\frac{|\mathscr{T}(k)|}{|\mathscr{H}(k)|} \to 1$.

For each integer k, let $\mathscr{S}(k)$ denote the set of all split graphs on k vertices. Lemma 13 is a consequence of lemma 12 and the following well known result on the relationship between chordal and split graphs from [13]: $\frac{|\mathscr{S}(k)|}{|\mathscr{C}(k)|} \to 1$ as $k \to \infty$. So, given a network $\mathbb{N}$ of n sensors with graph $\mathbb{G}$ and three anchors, Lemma 13 implies that in the limit as $n \to \infty$, if $\mathbb{G}$ is chordal and each vertex of $\mathbb{G}$ has degree at least three, then it will *almost always* be the case that $\mathbb{N}$ will be not only localizable but "efficiently" localizable in that its graph $\mathbb{G}$ will have a trilateration ordering.

VI. Appendix

In the following, let w be a row vector such that $wR = 0$ and $\Omega(w)$ has rank $n-3$. Let $\mathbf{1}$ be the $n \times 1$ vector of all ones, and let $\mathbf{x}$ and $\mathbf{y}$ denote $n \times 1$ vectors whose ith elements are the x and y coordinates of $\alpha(i)$, respectively, i.e. $\alpha(i) = \begin{pmatrix} \mathbf{x}(i) & \mathbf{y}(i) \end{pmatrix}'$. From [11], we have that: $\mathbf{1}$, $\mathbf{x}$ and $\mathbf{y}$ are in the kernel of $\Omega(w)$. Note that $\{\mathbf{1}, \mathbf{x}, \mathbf{y}\}$ is a linearly independent set since the multi-point of $\mathbb{N}$ is assumed to be generic. Let $\mathbf{s}$ and $\mathbf{t}$ be any two linearly independent vectors in the kernel of $\Omega(w)$ which are both orthogonal to $\mathbf{1}$, and let $\mathbf{s}(i)$ and $\mathbf{t}(i)$ denote the ith elements of $\mathbf{s}$ and $\mathbf{t}$, respectively. Let $q(i)$ denote the 2×1 vector consisting of $\mathbf{s}(i)$ and $\mathbf{t}(i)$: $q(i) = \begin{pmatrix} \mathbf{s}(i) & \mathbf{t}(i) \end{pmatrix}'$. Let δ denote the assignment where $\delta(i) = q(i)$ for each $i \in \mathscr{V}$. In the following, we compute a 2×2 matrix L satisfying equation 2 using $\mathbf{s}$, $\mathbf{t}$ and the known inter-sensor distances:

$$d_{ij} = \| Lq(i) - Lq(j) \|, \quad \forall (i,j) \in \mathscr{E} \tag{2}$$

Clearly, if an L satisfying 2 can be computed, then $L \circ \delta$, must be a consistent assignment of the entire network $\mathbb{N}$.

Since $\mathbf{s}$ and $\mathbf{t}$ were chosen to be orthogonal to $\mathbf{1}$, there are $a,b,c,d \in \mathbb{R}$ such that $\mathbf{s} = a\mathbf{x} + b\mathbf{y}$ and $\mathbf{t} = c\mathbf{x} + d\mathbf{y}$. If we let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then we get $q(i) = A\alpha(i), \quad \forall i \in \{1,\ldots,n\}$. Note that A must be invertible since $\mathbf{s}$ and $\mathbf{t}$ are linearly independent. So A^{-1} exists, and $\alpha(i) = A^{-1}q(i), \quad \forall i \in \{1,\ldots,n\}$.

[1] A graph is said to be *chordal* if each of its subgraphs which is also a cycle with at least four edges contains at least one "chord." Where by a *chord* of a cycle is meant any edge not in the cycle but which is incident on two vertices of the cycle. A graph $\mathbb{S}$ is called a *split* graph if its vertex set is the disjoint union of subsets $\mathscr{C}$ and $\mathscr{I}$ where the vertices in $\mathscr{C}$ induce a complete graph in $\mathbb{S}$, and no two vertices in $\mathscr{I}$ are adjacent to each other. Split graphs are also chordal graphs.

For each $(i,j) \in \mathscr{E}$, we can write $\| \alpha(i) - \alpha(j) \|^2 = (q(i)' - q(j)')(A^{-1})'A^{-1}(q(i) - q(j))$. Since α is a consistent assignment, we have that $\| \alpha(i) - \alpha(j) \|^2 = d_{ij}^2$ for all $(i,j) \in \mathscr{E}$. By letting $M = (A^{-1})'A^{-1}$, we can write $d_{ij}^2 = (q(i)' - q(j)')M(q(i) - q(j))$. From this equation, and the fact that M is symmetric, we can obtain $r_{ij} \begin{pmatrix} M_{11} & M_{12} & M_{22} \end{pmatrix}' = d_{ij}^2$, where r_{ij} is the row vector $\begin{pmatrix} (\mathbf{s}(i)-\mathbf{s}(j))^2 & 2(\mathbf{t}(i)-\mathbf{t}(j))(\mathbf{s}(i)-\mathbf{s}(j)) & (\mathbf{t}(i)-\mathbf{t}(j))^2 \end{pmatrix}$. Since $\mathbb{G}$ is globally rigid, it must have at least three edges, so let (i_1,j_1), (i_2,j_2), (i_3,j_3) be any three edges of $\mathbb{G}$. Let D be the 3×3 matrix whose kth row is $r_{i_k j_k}$ for $k = 1,2,3$. By letting $d = \begin{pmatrix} d_{i_1 j_1}^2 & d_{i_2 j_2}^2 & d_{i_3 j_3}^2 \end{pmatrix}'$, we can write $D \begin{pmatrix} M_{11} & M_{12} & M_{22} \end{pmatrix}' = d$. Since the multi-point of $\mathbb{N}$ is generic, the rank of D is three for *almost all* vectors $\mathbf{s}$ and $\mathbf{t}$ in the kernel of $\Omega(w)$. So without loss of generality, suppose D has rank three, in which case there is exactly one solution to $Dx = d$, namely, $x = D^{-1}d$. Hence, we can solve for M_{11}, M_{12}, and M_{22} using equation $D \begin{pmatrix} M_{11} & M_{12} & M_{22} \end{pmatrix}' = d$. Since M is the product of a nonsingular matrix, i.e. A^{-1}, and its transpose, we have that M is both symmetric and positive definite. Therefore, one can compute the Cholesky decompositin of M. Let $L'L$ denote the Cholesky decomposition of M: $M = L'L$. Using the Cholesky decomposition of M, we get that $d_{ij}^2 = (q(i)' - q(j)')L'L(q(i) - q(j)), \quad \forall (i,j) \in \mathscr{E}$, which implies $d_{ij}^2 = \| Lq(i) - Lq(j) \|^2, \quad \forall (i,j) \in \mathscr{E}$. Hence, we have computed a matrix L for which equation 2 holds.

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