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Model reduction by moment matching for nonlinear systems

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***Abstract*— The problem of model reduction by moment matching for nonlinear systems is addressed and solved using the recently introduced notion of moment for nonlinear systems. It is shown that reduced order models can be parameterized by a free mapping which, in turn, can be used so that the model possesses specific properties, *e.g.* it has an asymptotically stable equilibrium or given relative degree, it is minimum phase, it is passive. In addition, a nonlinear enhancement of the notion of Markov parameters is provided. The theory is illustrated by means of simple examples.**

I. INTRODUCTION

The model reduction problem for linear and nonlinear systems has been widely studied over the past decades. This problem has great importance in applications, because *reduced order models* are often used in analysis and design. This is the case, for example, in the study of mechanical systems, which is often based on models derived from a rigid body perspective that neglects the presence of flexible modes and elasticity; in the study of large scale systems, such as integrated circuits or weather forecast models, which relies upon the construction of simplified models that capture the main features of the system. From a theoretical point of view the model reduction problem generates important theoretical questions and requires advanced mathematical tools.

The model reduction problem can be simply, and informally, posed as follows. Given a system, described by means of (linear or nonlinear) differential equations together with an output map, compute a *simpler* system which *approximates* (in a sense to be specified) its behaviour.

There are several ways in which to make precise this problem formulation. To begin with, one could introduce an approximation error given in terms of the (steady-state) response, if it exists, of the system for classes of input signals. Moment matching methods, see *e.g.* [2] for the linear case and the recent paper [5] for the nonlinear case, belong to this class. Alternatively, approximation errors expressed in terms of the H_2 or H_∞ norm of the error system have been considered, see [16], [13], [3], [21]. Finally, approximation errors based on the Hankel operator of the system have been widely considered, see [11], [20], [22], [10], [17].

The concept of simplicity is understood, for linear systems, in terms of the dimension of the system, *i.e.* an approximating system is simpler than the model to approximate if its state-space realization has fewer states. For nonlinear systems this dimensional argument may be inappropriate, as one has to take into consideration also the complexity of the functions involved in the state-space representation. In addition, there are other important issues worth investigating. In particular, one may require that some specific property of the system to be approximated is retained by the reduced order model.

Goal of this work is to show that, exploiting the results in [5], it is possible to construct reduced order models with specific properties, thus partly providing a nonlinear counterpart to the results in [6].

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Notation. Throughout the paper we use standard notation. $\mathbb{R}$, $\mathbb{R}^n$ and $\mathbb{R}^{n\times m}$ denote the set of real numbers, of n-dimensional vectors with real components, and of $n \times m$-dimensional matrices with real entries, respectively. $\mathbb{R}^+$ ($\mathbb{R}^-$) denotes the set of non-negative (non-positive) real numbers, $\mathbb{C}^-$ denotes the set of complex numbers with negative real part.

II. THE NOTION OF MOMENT

In this section we recall some of the results in [5]. Consider a nonlinear, single-input, single-output, continuous-time system described by equations of the form

$$\begin{array}{rcl}\dot{x} & = & f(x,u),\\ y & = & h(x),\end{array} \tag{1}$$

with $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}$, $y(t) \in \mathbb{R}$ and $f(\cdot,\cdot)$ and $h(\cdot)$ smooth mappings, a signal generator described by the equations

$$\begin{array}{rcl}\dot{\omega} & = & s(\omega),\\ \theta & = & l(\omega),\end{array} \tag{2}$$

with $\omega(t) \in \mathbb{R}^\kappa$, $\theta(t) \in \mathbb{R}$ and $s(\cdot)$ and $l(\cdot)$ smooth mappings, and the interconnected system

$$\begin{array}{rcl}\dot{\omega} & = & s(\omega),\\ \dot{x} & = & f(x,l(\omega)),\\ y & = & h(x).\end{array} \tag{3}$$

Suppose, in addition, that $f(0,0) = 0$, $s(0) = 0$, $l(0) = 0$ and $h(0) = 0$. The signal generator captures the requirement that one is interested in studying the behaviour of system (1) only in *specific circumstances*. However, for this to make sense and to provide a generalization of the notion of moment, we need the following assumptions and definitions.

Assumption 1: There is a unique mapping $\pi(\cdot)$, locally[1] defined in a neighborhood of $\omega = 0$, which solves the partial differential equation

$$f(\pi(\omega), l(\omega)) = \frac{\partial \pi}{\partial \omega} s(\omega). \tag{4}$$

Assumption 1 implies that the interconnected system (3) possesses an invariant manifold, described by the equation

[1] All statements are local, although global versions can be easily given.

$x = \pi(\omega)$. The (well-defined) dynamics of the system restricted to the invariant manifold are described by $\dot{\omega} = s(\omega)$, *i.e.* are a copy of the dynamics of the signal generator (2).

Assumption 2: The signal generator (2) is observable, *i.e.* for any pair of initial conditions $\omega_a(0)$ and $\omega_b(0)$, such that $\omega_a(0) \neq \omega_b(0)$, the corresponding output trajectories $l(\omega_a(t))$ and $l(\omega_b(t))$ are such that $l(\omega_a(t)) - l(\omega_b(t))) \not\equiv 0$.

Definition 1: Consider the system (1) and the signal generator (2). Suppose Assumptions 1 and 2 hold. The function $h(\pi(\omega))$, with $\pi(\omega)$ solution of equation (4), is *the moment of system (1) at* $s(\omega)$.

Definition 2: Consider the system (1) and the signal generator (2). Suppose Assumption 1 holds. Let the signal generator (2) be such that $s(\omega) = 0$ and $l(\omega) = \omega$. Then the function $h(\pi(\omega))$ is the 0-moment of system (1) at $s^\star = 0$.

We are now ready to recall one of the main results of [5].

Theorem 1: Consider system (1) and the signal generator (2). Assume Assumption 2 holds. Assume the zero equilibrium of the system $\dot{x} = f(x, 0)$ is locally exponentially stable and system (2) is Poisson stable[2]. Assume $\omega(0) \neq 0$.

Then Assumption 1 holds and the moment of system (1) at $s(\omega)$ is in one-to-one relation with the (locally well-defined) steady-state response of the output of the interconnected system (3).

Remark 1: If the equilibrium $x = 0$ of system $\dot{x} = f(x, 0)$ is unstable, it is still possible to define the moment of system (1) at $s(\omega)$, provided the equilibrium $x = 0$ is hyperbolic and the system (2) is Poisson stable, although it is not possible to establish a relation with the steady-state response of the interconnected system (3).

Remark 2: While for linear systems it is possible to define moments for linear signal generators yielding unbounded trajectories, this may be difficult, or impossible, for nonlinear systems.

Example 1: Consider a linear system described by equations of the form

$$\begin{aligned} \dot{x} &= Ax + Bu, \\ y &= Cx, \end{aligned} \qquad (5)$$

with $x(t) \in I\!R^n$, $n > 3$, $u(t) \in I\!R$, $y(t) \in I\!R$ and the nonlinear signal generator (2) with $\omega = [\omega_1, \omega_2, \omega_3]'$,

$$s(\omega) = \begin{bmatrix} \dfrac{I_2 - I_3}{I_1}\omega_2\omega_3 \\ \dfrac{I_3 - I_1}{I_2}\omega_3\omega_1 \\ \dfrac{I_1 - I_2}{I_3}\omega_1\omega_2 \end{bmatrix}, \qquad (6)$$

with $I_1 > 0$, $I_2 > 0$, $I_3 > 0$, $I_i \neq I_j$, for $i \neq j$, and

$$l(\omega) = L\omega = \begin{bmatrix} L_1 & L_2 & L_3 \end{bmatrix} \omega,$$

with $L_1 L_2 L_3 \neq 0$. This signal generator, which describes the evolution of the angular velocities of a free rigid body in space, is Poisson stable and, under the stated assumption on L, observable [18], [4].

Suppose system (5) is asymptotically stable. The moment of system (5) at $s(\omega)$ can be computed as follows. Let

$$\pi(\omega) = \sum_{i \geq 1} \pi_i(\omega),$$

with

$$\pi_i(\omega) = \begin{bmatrix} \pi_i^1(\omega) \\ \vdots \\ \pi_i^n(\omega) \end{bmatrix}$$

and $\pi_i^j(\omega)$ a homogeneous polynomial of degree i in ω. Then equation (4) yields

$$\pi_1(\omega) = -A^{-1}BL\omega, \quad \pi_i(\omega) = -A^{-i}BL\frac{d^{i-1}\omega}{dt^{i-1}}, \cdots .$$

Hence, the moment of system (5) at $s(\omega)$ is given by

$$C\pi(\omega) = -CA^{-1}\left(BL\omega + \cdots + A^{-i+1}BL\frac{d^{i-1}\omega}{dt^{i-1}} \cdots\right),$$

which is a polynomial series in ω.

III. The Markov parameters of a nonlinear system

In this section we provide a nonlinear counterpart of the notion of moment at $s = \infty$ of a linear system. Note that, since for linear systems this notion is associated with the impulse response, it is not possible to use the results in Section II, or in [5].

For a linear system, described by the equations (5), the k-moments at $s = \infty$ are defined as $\eta_k(\infty) = CA^kB$, *i.e.* the first $k+1$ moments at $s = \infty$ coincide with the first $k+1$ Markov parameters [2]. To obtain a nonlinear counterpart of this notion recall that[3]

$$CA^kB = \frac{d^k}{dt^k}(Ce^{At}B)\bigg|_{t=0} = y_I^{(k)}(0) = y_{F,B}^{(k)}(0),$$

where $y_I(\cdot)$ denotes the impulse response of the system and $y_{F,B}(\cdot)$ denotes the free output response from $x(0) = B$.

Consider now a nonlinear affine system[4] described by equations of the form

$$\begin{aligned} \dot{x} &= f(x) + g(x)u, \\ y &= h(x), \end{aligned} \qquad (7)$$

[2]See [14, Chapter 8] for the definition of Poisson stability.

[3]In this section we use the notation $y^{(i)}(t)$ to denote the i-th order time derivative of $y(\cdot)$ at time t. Moreover, time derivatives at $t = 0$ are computed at $t = 0^+$.

[4]We focus on affine systems, since for non-affine systems the impulse response, and its derivatives, may not be well-defined. To illustrate this statement consider the system $\dot{x} = u^2$, $y = x$, with $x(t) \in I\!R$, $u(t) \in I\!R$ and $y(t) \in I\!R$. Setting $u(t) = \delta_0(t)$, where $\delta_0(t)$ denotes the Dirac δ-function, and integrating, yields

$$y_I(t) = \int_0^t \delta_0^2(\tau)d\tau = \delta_0(0).$$

See [7, Chapter 10], and references therein, for an in-depth discussion on the above issue.

with $x(t) \in I\!R^n$, $u(t) \in I\!R$, $y(t) \in I\!R$ and $f(\cdot)$, $g(\cdot)$ and $h(\cdot)$ smooth mappings. Integrating the first of equations (7), with $x(0) = 0$ and $u(t) = \delta_0(t)$, evaluating for $t = 0$, and substituting in the second equation yields $y_I(0) = y_I^0(0) = h(g(0))$ and[5] $y_I^{(k)}(0) = L_f^k h \circ g(0)$, for $k \geq 0$. It is therefore natural to define the k-moment at $s = \infty$, for $k \geq 0$, of the nonlinear system (7) as $\eta_k(\infty) = y_I^{(k)}(0)$. Note finally that, for the considered class of nonlinear systems, $y_I^{(k)}(0) = y_{F,g(0)}^{(k)}(0)$, where $y_{F,g(0)}(t)$ denotes the free output response of the system from $x(0) = g(0)$.

These considerations allow to derive a reduced order model which matches the $0, \cdots, k-1$ moments at $s = \infty$ of system (7). For, consider a linear system described by equations of the form (5), with $x(t) \in I\!R^k$, and A, B and C such that $CA^iB = y_I^{(i)}(0)$, for $i = 0, \cdots, k-1$. (Note that the matrices A, B and C can be computed using standard realization algorithms, *e.g.* Ho-Kalman realization algorithm. In addition, these matrices are not uniquely defined, hence it is possible, for example, to assign the eigenvalues of the reduced order model.) As a consequence of the discussion above, the linear system thus constructed is a model of the nonlinear system achieving moment matching at $s = \infty$. Note that the computation of such a reduced order model does not require the solution of any partial differential equation, but simply regularity of the nonlinear system.

IV. Moment matching

Analogously to the linear case, we now introduce the notion of reduced order model and characterize the solution of the model reduction problem by moment matching.

Definition 3: The system

$$\begin{array}{rcl} \dot{\xi} &=& \phi(\xi, u), \\ \psi &=& \kappa(\xi), \end{array} \tag{8}$$

with $\xi(t) \in I\!R^\nu$, is a *model at $s(\omega)$ of system (1)* if system (8) has the same moment at $s(\omega)$ as (1). In this case, system (8) is said to *match* the moment of system (1) at $s(\omega)$. Furthermore, system (8) is a reduced order model of system (1) if $\nu < n$.

Lemma 1: Consider the system (1), the system (8) and the signal generator (2). Suppose Assumptions 1 and 2 hold. System (8) *matches* the moments of (1) at $s(\omega)$ if the equation

$$\phi(p(\omega), l(\omega)) = \frac{\partial p}{\partial \omega} s(\omega) \tag{9}$$

has a unique solution $p(\cdot)$ such that

$$h(\pi(\omega)) = \kappa(p(\omega)), \tag{10}$$

where $\pi(\cdot)$ is the (unique) solution of equation (4).

[5] $L_f h(\cdot)$ denotes the Lie derivative of the smooth function $h(\cdot)$ along the smooth vector field $f(\cdot)$, as defined in [14, Chapter 1].

V. Model reduction by moment matching

To construct a reduced order model it is necessary to determine mappings $\phi(\cdot,\cdot)$, $\kappa(\cdot)$ and $p(\cdot)$ such that equations (9) and (10) hold, where $\pi(\cdot)$ is the solution of equation (4). To solve this problem we make the following assumption.

Assumption 3: There exists mappings $\kappa(\cdot)$ and $p(\cdot)$ such that $k(0) = 0$, $p(0) = 0$, $p(\cdot)$ is locally C^1, equation (10) holds and

$$\det \frac{\partial p(\omega)}{\partial \omega}(0) \neq 0,$$

i.e. the mapping $p(\cdot)$ possesses a local inverse $p^{-1}(\cdot)$.

Remark 3: Assumption 3 holds selecting $p(\omega) = \omega$ and $k(\omega) = h(\pi(\omega))$.

A direct computation shows that a family of reduced order models, all achieving moment matching, provided equation (9) has a unique solution $p(\omega)$, is described by

$$\begin{array}{rcl} \dot{\xi} &=& \phi(\xi) + \dfrac{\partial p(\omega)}{\partial \omega}\delta(\xi)u, \\ \psi &=& \kappa(\xi), \end{array}$$

where $\kappa(\cdot)$ and $p(\cdot)$ are such that Assumption 3 holds, $\delta(\xi) = \tilde{\delta}(p^{-1}(\xi))$, where $\tilde{\delta}(\cdot)$ is a free mapping, and

$$\phi(\xi) = \left[\frac{\partial p(\omega)}{\partial \omega} \left(s(\omega) - \delta(p(\omega))l(\omega) \right) \right]_{\omega = p^{-1}(\xi)}.$$

In particular, selecting $p(\omega) = \omega$ yields a family of reduced order models described by

$$\begin{array}{rcl} \dot{\xi} &=& s(\xi) - \delta(\xi)l(\xi) + \delta(\xi)u, \\ \psi &=& h(\pi(\xi)), \end{array} \tag{11}$$

where $\delta(\cdot)$ is any mapping such that the equation

$$s(p(\omega)) - \delta(p(\omega))l(p(\omega)) + \delta(p(\omega))l(\omega) = \frac{\partial p}{\partial \omega} s(\omega) \tag{12}$$

has the unique solution $p(\omega) = \omega$.

Similarly to what discussed for the linear case in [6], it is possible to use the *parameter* $\delta(\cdot)$ to achieve specific properties of the reduced order model. In what follows, we implicitly assume that the $\delta(\cdot)$ achieving a specific property is such that equation (12) has a unique solution.

Note that, in the nonlinear case, it is not possible to obtain the simple characterizations given in [6] and to address and solve the same problems. On the contrary, there are problems that are interesting only in the nonlinear framework.

A. Matching with asymptotic stability

Consider the problem of achieving model reduction by moment matching with a reduced order model, described by equations of the form (11), such that the model has an asymptotically stable equilibrium. Such a reduced order model can be constructed selecting, if possible, the free mapping $\delta(\cdot)$ such that the zero equilibrium of the system (recall that $s(0) = 0$ and $l(0) = 0$)

$$\dot{\xi} = s(\xi) - \delta(\xi)l(\xi)$$

is locally asymptotically stable. This is possible, for example, if the pair

$$\left(\frac{\partial l(\xi)}{\partial \xi}(0), \frac{\partial s(\xi)}{\partial \xi}(0)\right)$$

is observable, or detectable. Note, however, that this is not necessary.

B. Matching with prescribed relative degree

Consider the problem of selecting the mapping $\delta(\cdot)$ in system (11) such that the reduced model has a given relative degree $r \in [1, \nu]$ at some point ξ_0. For such a problem the following fact holds.

Theorem 2: Consider the following statements.

(RD1) For all $r \in [1, \nu]$ there exists a $\delta(\cdot)$ such that system (11) has relative degree r at ξ_0.

(RD2) The codistribution

$$d\mathcal{O}_\nu = \mathrm{span}\{dh(\pi(\xi)), \cdots, dL_s^{\nu-1}h(\pi(\xi))\}$$

has dimension ν at ξ_0.

(RD3) The system (11) is locally observable at ξ_0.

(RD4) The system (3) is locally observable at ξ_0.

Then (RD1)$\Leftrightarrow$(RD2)$\Rightarrow$(RD3) $\Leftarrow$(RD4).

Remark 4: Note that although the implication (RD3)$\Rightarrow$(RD2) does not hold in general, (RD3) implies that the codistribution $d\mathcal{O}_\nu$ has dimension ν for all ξ in an open and dense set around ξ_0 (see [18, Corollary 3.35]).

C. Matching with prescribed zero dynamics

Consider the problem of selecting the mapping $\delta(\cdot)$ in system (11) such that the reduced model has zero dynamics with specific properties. To simplify the study of this problem we assume that condition (RD2) holds, which allows to obtain a special form for system (11).

Lemma 2: Consider system (11). Assume condition (RD2) holds. Then there exists a coordinates transformation $\chi = \Xi(\xi)$, locally defined around ξ_0, such that, in the new coordinates, system (11) is described by equations of the form

$$\begin{array}{rcl} \dot{\chi}_1 &=& \chi_2 + \tilde{\delta}_1(\chi)(v - \tilde{l}(\chi)), \\ \dot{\chi}_2 &=& \chi_3 + \tilde{\delta}_2(\chi)(v - \tilde{l}(\chi)), \\ &\vdots& \\ \dot{\chi}_\nu &=& \tilde{f}(\chi) + \tilde{\delta}_\nu(\chi)(v - \tilde{l}(\chi)) \\ \psi &=& \chi_1, \end{array}$$

where $[\tilde{\delta}_1(\chi), \cdots, \tilde{\delta}_\nu(\chi)]' = \delta(\Xi^{-1}(\chi))$, $\tilde{l}(\chi) = l(\Xi^{-1}(\chi))$, and $\tilde{f}(\chi) = L_s^\nu h(\pi(\Xi^{-1}(\chi)))$.

As a consequence of the result established in Lemma 2 we have the following statement.

Proposition 1: Consider system (11). Assume condition (RD2) holds and ξ_0 is an equilibrium of system (11). Then, for all $r \in [1, \nu - 1]$, there is a $\delta(\cdot)$ such that system (11) has relative degree r and its zero dynamics have a locally exponentially stable equilibrium. In addition, there is a coordinates transformation, locally defined around ξ_0, such that, in the new coordinates, the zero dynamics of system (11) are described by equations of the form

$$\begin{array}{rcl} \dot{z}_1 &=& z_2 + \hat{\delta}_1(z)z_1, \\ \dot{z}_2 &=& z_3 + \hat{\delta}_2(z)z_1, \\ &\vdots& \\ \dot{z}_{\nu-r} &=& \hat{f}_{\nu-r}(z) + \hat{\delta}_{\nu-r}(z)z_1, \end{array}$$

where the $\hat{\delta}_i(\cdot)$ are free functions and

$$\hat{f}(z) = \tilde{f}(\chi)\Big|_{\chi=[0,\cdots,0,z_1,\cdots,z_{\nu-r}]'}.$$

D. Matching with a passivity constraints

Consider now the problem of selecting the mapping $\delta(\cdot)$ such that system (11) is lossless or passive. For such a problem the following fact holds.

Theorem 3: Consider the following statements.

(P1) The family of reduced order models (11) contains, locally around ξ_0, a lossless (passive, respectively) system with a differentiable storage function.

(P2) There exists a differentiable function $V(\cdot)$, locally positive definite around ξ_0, such that

$$V_\xi s(\xi) = h(\pi(\xi)), \ \ (V_\xi s(\xi) \le h(\pi(\xi)), \text{ respectively}), \tag{13}$$

locally around ξ_0.

(P3) There exists a differentiable function $V(\cdot)$, locally positive definite around ξ_0, such that equation (13) holds and

$$V_{\xi\xi}(\xi_0) > 0. \tag{14}$$

Then (P1)$\Rightarrow$(P2), (P3)$\Rightarrow$(P2), and (P3)$\Rightarrow$(P1).

E. Matching for linear systems at $s(\omega)$

In this section we consider the model reduction problem for linear systems at $s(\omega)$, *i.e.* we consider the case in which the signal generator is a nonlinear system. For such a problem, under suitable assumptions, it is possible to obtain in an explicit way a formal description of reduced order models, as detailed in the following statement.

Proposition 2: Consider the linear system (5), with $x(t) \in \mathbb{R}^n$ and $\sigma(A) \subset \mathbb{C}^-$. Consider the signal generator (2), with $\omega(t) \in \mathbb{R}^\nu$, $n > \nu$ and $l(\omega) = L\omega$. Assume that the signal generator is Poisson stable and that $s(\cdot)$ can be expressed, locally around $\omega = 0$, as a formal power series, *i.e.*

$$s(\omega) = \sum_{i \ge 1} s^{[i]}(\omega),$$

where $s^{[i]}(\cdot)$ denotes a polynomial vector field which is homogeneous of degree i. Suppose in addition that $s^{[1]}(\omega) = 0$.

Then a family of reduced order models achieving moment matching at $s(\omega)$ is described by the equations

$$\begin{array}{rcl} \dot{\xi} &=& s(\xi) - \delta(\xi)L\xi + \delta(\xi)u, \\ \psi &=& C\pi(\xi), \end{array}$$

with $\delta(\cdot)$ a free mapping and

$$\pi(\xi) = \sum_{i\geq 1} \pi^{[i]}(\xi), \qquad (15)$$

where $\pi^{[1]}(\xi) = -A^{-1}BL\xi$ and

$$\pi^{[k]}(\xi) = A^{-1} \sum_{i=1}^{k-1} \frac{\partial \pi^{[i]}(\xi)}{\partial \xi} s^{[k-i+1]}(\xi), \qquad (16)$$

for $k \geq 2$.

Example 2 (Example 1 continued): Exploiting the results in Proposition 2 and the discussion in Example 1, we infer that a reduced order model for a linear asymptotically stable system at the $s(\omega)$ in equation (6) is given by

$$\begin{array}{rcl} \dot{\xi} & = & s(\xi) - \delta(\xi)L\omega + \delta(\xi)u, \\ \psi & = & -CA^{-1}\left(BL\omega + \cdots + A^{-i+1}BL\frac{d^{i-1}\omega}{dt^{i-1}}\cdots\right). \end{array}$$

Simulations have been run selecting $I_1 = 1$, $I_2 = 2$, $I_3 = 3$, $L_1 = 1$, $L_2 = 1/2$, $L_3 = 1/3$ and $\delta(\cdot) = \text{diag}(1/I_1, 1/I_2, 1/I_3)L'$, which yields a reduced order model with a globally asymptotically stable equilibrium at $\xi = 0$ (see [1] for a proof of this fact).

The linear system, that has to be reduced, is a randomly selected asymptotically stable system of dimension 15. The initial condition of the signal generator has been selected as $\omega(0) = \frac{1}{5}[1\ 1\ 1]'$.

The linear system and the reduced order model, both driven by the signal generator, have been numerically integrated from zero initial conditions. Figure 1 displays the output $y(t)$ of the linear system when driven by the signal generator, and the signals $\psi^I(t)$, $\psi^{II}(t)$ and $\psi^{III}(t)$, obtained by truncating the formal power series defining $\psi(t)$ to the first, second and third order term, respectively. Note that, in steady-state,

$$\begin{array}{r} \max(|y(t) - \psi^I(t)|) = 0.2765 > \\ \max(|y(t) - \psi^{II}(t)|) = 0.1644 > \\ \max(|y(t) - \psi^{III}(t)|) = 0.0764, \end{array}$$

which shows that the approximation error decreases by adding terms in the formal power series defining the output of the reduced order model.

F. Matching for nonlinear systems at $S\omega$

In this section we consider the model reduction problem for nonlinear systems at $s(\omega) = S\omega$, *i.e.* we consider the case in which the signal generator is a linear system.

This problem is of particular interest since, exploiting the discussion in Section V, we infer that the reduced order models have a very simple description, *i.e.* a family of reduced order models is given by the equations

$$\begin{array}{rcl} \dot{\xi} & = & (S - \delta(\xi)L)\xi + \delta(\xi)u, \\ \psi & = & h(\pi(\omega)), \end{array}$$

where $\delta(\cdot)$ is a free mapping. In particular, selecting $\delta(\xi) = \Delta$, for some constant matrix Δ, we have that the family of reduced order models is described by a linear differential equation with a nonlinear output map. This structure has two main advantages. The former is that the matrix Δ can be selected to achieve additional goals, such as to assign the eigenvalues or the relative degree of the reduced order model (provided additional assumptions on the output map holds). The latter is that the computation of (an approximation of) the reduced order model boils down to the computation of (an approximation of) the output map $h(\pi(\omega))$. This computation can be carried out in the spirit of the results in [12, Section 4.2 and 4.3].

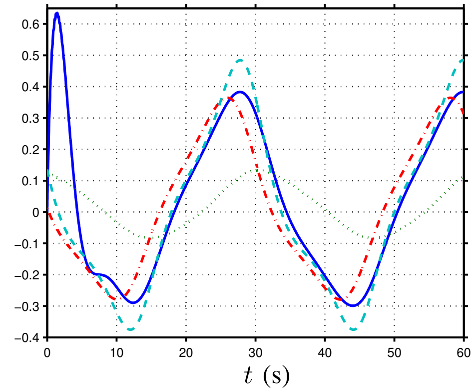

Fig. 1. Time histories of the output of the driven linear system and of the *approximating* outputs of the driven reduced order model: $y(t)$ (solid), $\psi^{I(t)}$ (dotted), $\psi^{II(t)}$ (dash-dotted) and $\psi^{III}(t)$ (dashed).

We complete this section discussing the model reduction problem with 0-moment matching at $s^\star = 0$, *i.e.* the model reduction problem at $s(\omega) = 0$. This problem can be solved, under specific assumptions, without solving any partial differential equation, as detailed in the following statement.

Proposition 3 (0-moment matching at $s^\star = 0$): Consider system (1) and the signal generator $\dot{\omega} = 0$, $\theta = \omega$. Assume the zero equilibrium of the system $\dot{x} = f(x, 0)$ is locally exponentially stable. Then the zero moment of system (1) is (locally) well-defined and given by $h(\pi(\omega))$, with $\pi(\cdot)$ the unique solution of the algebraic equation $f(\pi(\omega), \omega) = 0$. Finally, a reduced order model, for which the zero equilibrium is locally asymptotically stable is given by

$$\begin{array}{rcl} \dot{\xi} & = & -\delta(\xi)(\xi - u), \\ \psi & = & h(\pi(\xi)), \end{array}$$

with $\delta(\cdot)$ such that $\delta(0) > 0$.

Example 3 (See [5]): The averaged model of the DC–to–DC Ćuk converter is given by the equations [19]

$$\begin{array}{rcl} L_1 \dfrac{d}{dt} i_1 & = & -(1-u)\, v_2 + E, \\ C_2 \dfrac{d}{dt} v_2 & = & (1-u)\, i_1 + u\, i_3, \\ L_3 \dfrac{d}{dt} i_3 & = & -u\, v_2 - v_4, \\ C_4 \dfrac{d}{dt} v_4 & = & i_3 - G\, v_4, \\ y & = & v_4, \end{array} \qquad (17)$$

where $i_1(t) \in I\!R^+$ and $i_3(t) \in I\!R^-$ describe currents, $v_2(t) \in I\!R^+$ and $v_4(t) \in I\!R^-$ voltages, L_1, C_2, L_3, C_4, E and G positive parameters and $u(t) \in (0,1)$ a continuous control signal which represents the slew rate of a PWM circuit used to control the switch position in the converter.

The 0-moment of the system at $s^\star = 0$ is

$$h(\pi(\omega)) = \frac{\omega}{\omega - 1}E,$$

and a locally asymptotically stable reduced order model achieving moment matching at $s^\star = 0$ is

$$\begin{aligned} \dot{\xi} &= -\delta(\xi)(\xi - u), \\ \psi &= E\frac{\xi}{\xi - 1}, \end{aligned} \tag{18}$$

with $\delta(0) > 0$, which is well-defined if $\xi \neq 1$. This is consistent with the fact that the 0-moment at $s^\star = 0$ is defined for $\omega \neq 1$.

Simulations have been run to assess the properties of the reduced order model. The parameters have been selected as in [19], the input signal is piecewise constant, with jumps every 0.05 seconds. The reduced order model is described by equations (18), where the function $\delta(\cdot)$ depends upon the input signal u and it is equal to the real part of the slowest eigenvalue of the system (17) (which is a linear system for constant u).

Figure 2 displays the output $y(t)$ of the averaged model of the Ćuk converter and the output $\psi(t)$ of the reduced order model. The figure shows that the reduced order model provides a good *static* approximation of the behaviour of the system but does not capture its dynamic (under-damped) behaviour. The dynamic behaviour can be captured constructing a two dimensional model, which is (in the spirit of the model (18)) a linear system with a nonlinear output map. Since such a model is required to match only one moment, it is possible to assign its eigenvalues at the location of the dominant modes of system (17) with u fixed. The output $\psi^{2d}(t)$ of this two dimensional reduced order model is also displayed in Figure 2. Note that this signal may provide a better approximation of $y(t)$.

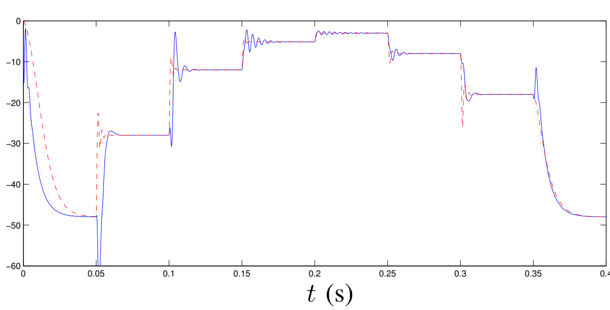

Fig. 2. Time histories of the output of the averaged model of the Ćuk converter and of the *approximating* outputs of the reduced order models: $y(t)$ (solid), $\psi(t)$ (dotted), $\psi^{2d}(t)$ (dashed).

VI. Summary

The recently developed notion of moment for nonlinear systems has been exploited to derive reduced order models achieving moment matching and with pre-specified properties. In addition, a nonlinear counterpart of the notion of Markov parameters has been developed. The theory has been illustrated by simple examples.

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References

[1] D. Aeyels and M. Szafranski. Comments on the stabilizability of the angular velocity of a rigid body. *Systems and Control Letters*, 10:35–39, 1988.
[2] A.C. Antoulas. *Approximation of Large-Scale Dynamical Systems*. SIAM Advances in Design and Control, 2005.
[3] A.C. Antoulas and A. Astolfi. H_∞ norm approximation. In *Unsolved problems in Mathematical Systems and Control Theory, V. Blondel and A. Megretski Editors*, pages 267–270. Princeton University Press, 2004.
[4] A. Astolfi. Output feedback control of the angular velocity of a rigid body. *Systems and Control Letters*, 36(3):181–192, 1999.
[5] A. Astolfi. Model reduction by moment matching (semi-plenary presentation). In *IFAC Symposium on Nonlinear Control System Design, Pretoria, S. Africa*, pages 95–102, 2007.
[6] A. Astolfi. A new look at model reduction by moment matching for linear systems. In *46th Conference on Decision and Control, New Orleans, MS*, pages 4361–4366, 2007.
[7] A. Bressan and B. Piccoli. *Introduction to the Mathematical Theorey of Control*, volume 2. AIMS Series on Applied Mathematics, 2007.
[8] C. I. Byrnes, A. Isidori, and J. C. Willems. Passivity, feedback equivalence, and the global stabilization of minimum phase nonlinear systems. *IEEE Trans. Autom. Control*, 36:1228–1240, November 1991.
[9] J. Carr. *Applications of center manifold theory*. Springer Verlag, 1998.
[10] K. Fujimoto and J.M.A. Scherpen. Nonlinear input-normal realizations based on the differential eigenstructure of Hankel operators. *IEEE Trans. Autom. Control*, 50:2–18, 2005.
[11] K. Glover. All optimal Hankel-norm approximations of linear multivariable systems and their L^∞-error bounds. *International Journal of Control*, 39:1115–1193, 1984.
[12] J. Huang. *Nonlinear Output Regulation: Theory and Applications*. SIAM Advances in Design and Control, 2004.
[13] X.-X. Huang, W.-Y. Yan, and K.L. Teo. H_2 near optimal model reduction. *IEEE Trans. Autom. Control*, 46:1279–1285, 2001.
[14] A. Isidori. *Nonlinear Control Systems, Third Edition*. Springer Verlag, 1995.
[15] A. Isidori and C. I. Byrnes. Output regulation of nonlinear systems. *IEEE Trans. Autom. Control*, 35(2):131–140, 1990.
[16] D. Kavranoglu and M. Bettayeb. Characterization of the solution to the optimal H_∞ model reduction problem. *Systems and Control Letters*, 20:99–108, 1993.
[17] A.J. Krener. Model reduction for linear and nonlinear control systems. In *45th Conference on Decision and Control, San Diego, CA*, 2006. Bode Lecture.
[18] H. Nijmeijer and A. J. Van der Schaft. *Nonlinear Dynamical Control Systems*. Springer Verlag, 1989.
[19] H. Rodriguez, R. Ortega, and A. Astolfi. Adaptive partial state feedback control of the DC-to-DC Ćuk converter. In *American Control Conference, Portland*, pages 5121–5126, 2005.
[20] J.M.A. Scherpen. Balancing for nonlinear systems. *Syst. and Contr. Lett.*, 21:143–153, 1993.
[21] J.M.A. Scherpen. H_∞ balancing for nonlinear systems. *Int. J. Rob. Nonl. Contr.*, 6:645–668, 1996.
[22] J.M.A. Scherpen and A.J. van der Schaft. Normalized coprime factorizations and balancing for unstable nonlinear systems. *Int. Journal of Control*, 60:1193–1222, 1994.