Proceedings of the
47th IEEE Conference on Decision and Control
Cancun, Mexico, Dec. 9-11, 2008

Delta Modulation for Multivariable Centralized Linear Networked Controlled Systems

JONATHAN JAGLIN, CARLOS CANUDAS-DE-WIT, CYRILLE SICLET
Control Systems Department, GIPSA-Lab, CNRS.
NeCS, INRIA-CNRS project-team, FRANCE.
carlos.canudas-de-wit@inpg.fr, jonathan.jaglin@lag.ensieg.fr

***Abstract*—This paper investigates the closed-loop properties of multivariable (MIMO) linear systems where the sensed information is centralized and coded on the basis of a Δ-modulation algorithm often used for minimizing the numbers of transmitted bits. In particular we propose a new centralized vector coding algorithm that allows us to extend our previous results in [4] to any type of linear multivariable systems. In addition, we provide an estimation of the stability attraction domain, and we give some simulation results validating the proposed approach.**

***Index Terms*—Delta modulation, Networked controlled systems, NCS, quantized systems.**

I. INTRODUCTION

THis paper deals with the stabilization problem of a linear multivariable system through a communication network where information is transmitted via a particular coding algorithm. Coding algorithms seeking to transmit a minimum number of information bits are appealing in wireless networks since they allows a substantial channel bandwidth reduction. Many of such types of control architecture using that type of codes have been studied in the past. See [6], [11], [10], [8], [1], [13] among others.

Delta modulation (Δ-M) is one alternative to minimize the numbers of bits to be coded. Recent works in [4] have re-adapted the standard form of the delta modulation structure to their use in a feedback setup. Inspired by this approach several variants of [4] have been studied: asynchronous entropy coding [2], energy-aware coding [3], adaptive delta modulation [7], and gain scheduling multi-bit coding [9]. Except for the trivial case of diagonalizable multivariable system that can be reformulated as a set of n-scalar ones, all these works deal exclusively with scalar system.

In this paper, we present a generalization of the delta-modulation coding presented in [4], to MIMO systems. In particular we introduce a vector coding structure for *multi-variable centralized* linear systems. The notion of centralization refers here to the fact that both the encoder-decoder and the control law use the full available information from all sensors. The idea is shown in Fig. 1, where we can see that all the sensed system outputs are collected in a *central* point, then transformed into a different coordinate-basis (using the transform matrix $\boldsymbol{T}_k$) before they are coded using a vector-coding algorithm. At the receiver side, it is similarly assumed that the transmitted information arrives to a central receiver, then decoded, and finally the control is computed using this centralized information. It is worth to notice that *decentralized* case is clearly much more constrained, even in absence of a coding process. A recent work [12] dealing with the case of decentralized multi-controller stability over communication channel illustrates well the fundamental difficulties, and provides an interesting preview on how to handle these problems when information is not centralized.

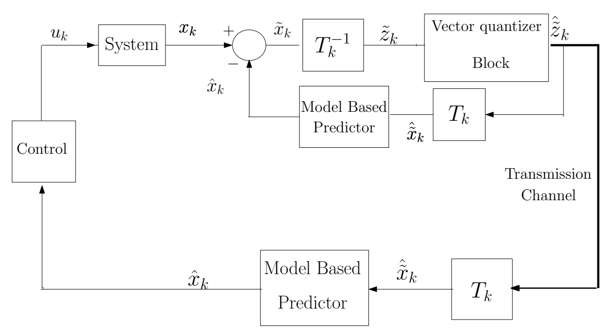

Fig. 1. Schematic representation of the system in our study.

The paper is organized as follows. After formalizing the problem in section II, we introduce in section III the general vector coding algorithm that can be adapted for all different forms of Jordan blocks resulting from the change of coordinate basis. Then, vector coding is performed in the transform domain. Vector coding here refers to the fact that a specific code-word is assigned for specific combinations between states. Closed-loop stability properties resulting from this approach are also exposed here. Section V characterizes the attraction set associated to the previous local stability conditions. This allows a finer estimation of quantization values to be used in the coding process. Finally simulation results are shown in section VI.

II. PROBLEM FORMULATION AND ASSUMPTIONS

The problem considered here is the stabilization of a multivariable system in which sensor signals are centralized, and then transmitted through a digital communication link to the controller. The coding design aims to achieve stability with a minimal information rate, thanks to a judicious coding strategy selection during the quantization step.

978-1-4244-3124-3/08/$25.00 ©2008 IEEE

Let us assume the following:

- the coding process is centralized : a single encoder can be used to encode all the sensed states of the system,
- the encoded information is transmitted through a noiseless perfect transmission channel. Hence delay, errors due to the transmission are not considered,
- the encoder and decoder clocks are assumed synchronized, and samples are assumed to occur at each T_s.

The following notations will be used:

- $x_k = [x_k^1, \ldots, x_k^n]^T \in R^{(n\times 1)}$ is the n-dimensional sensed state vector at instant kT_{s} (each x_k^i corresponds to the $i-th$ sensor) ;
- $u_k = [u_k^1, \ldots, u_k^m]^T \in R^{(m\times 1)}$, is m-dimensional control input vector at instant kT_{s}.

The discretized system is described by:

$$x_{k+1} = \boldsymbol{A}x_k + \boldsymbol{B}u_k \tag{1}$$

$$u_k = -\boldsymbol{K}\hat{x}_k \tag{2}$$

with $\boldsymbol{K}$ such as $\boldsymbol{A}-\boldsymbol{B}\boldsymbol{K}$ is Hurwitz. $\hat{x}_k$ is an estimation of x_k, and $\tilde{x}_k$ denotes the estimation error :

$$\tilde{x}_k = x_k - \hat{x}_k, \tag{3}$$

and, more generally, for a given signal s_k, $\hat{s}_k$ represents an estimated value of s_k and $\tilde{s}_k$ represents the error $s_k - \hat{s}_k$.

Without loss of generality, we suppose that system (1) is expressed in its Jordan's form, such that $\boldsymbol{A}$ is of the form,

$$\boldsymbol{A} = \begin{pmatrix} \boldsymbol{J}_{\lambda_1} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{J}_{\lambda_l} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{J}_{\lambda_\gamma} \end{pmatrix} \tag{4}$$

where we assume that there are α Jordan's blocks, of dimension $\mu_l \times \mu_l$, with multi-valued real eigenvalue, and $\gamma-\alpha$ Jordan's blocks, of dimension $2\mu_l \times 2\mu_l$, with multi-valued complex conjugated eigenvalues.

For the multi-valued real eigenvalue case, the $\boldsymbol{J}_{\lambda_l}$, for $1 \leqslant l \leqslant \alpha$, are of the form,

$$\boldsymbol{J}_{\lambda_l} = \begin{pmatrix} \lambda_l & 1 & 0 & \\ 0 & \lambda_l & 1 & \\ & 0 & \lambda_l & 1 \\ & & 0 & \lambda_l \end{pmatrix} \tag{5}$$

and, for the multi-valued complex conjugated eigenvalues, the $\boldsymbol{J}_{\lambda_l}$, are, for all $\alpha+1 \leqslant l \leqslant \gamma$, of the form,

$$\boldsymbol{J}_{\lambda_l} = \begin{pmatrix} |\lambda_l|\boldsymbol{R}(\theta_l) & I_2 & 0 & \\ 0 & |\lambda_l|\boldsymbol{R}(\theta_l) & I_2 & \\ & 0 & |\lambda_l|\boldsymbol{R}(\theta_l) & I_2 \\ & & 0 & |\lambda_l|\boldsymbol{R}(\theta_l) \end{pmatrix} \tag{6}$$

where $\lambda_l = |\lambda_l|(\cos(\theta_l) + j\sin(\theta_l))$ describes the complex eigenvalues, with magnitude $|\lambda_l|$, and angle θ_l. $\boldsymbol{R}(\theta_l)$ is the rotation matrix associated to the form adopted above, i.e.

$$\boldsymbol{R}(\theta_l) = \begin{pmatrix} \cos(\theta_l) & \sin(\theta_l) \\ -\sin(\theta_l) & \cos(\theta_l) \end{pmatrix} \tag{7}$$

With $\mu_1 + \ldots + \mu_\gamma = n$.

III. Multivariable Δ-Modulation coding strategy

In this section, we present the multivariable coding strategy. This strategy is inspired from the Δ-modulation algorithm studied previously in [4] for the one-dimensional case. The n-dimensional case considered here does not result from the simple extension of the one-dimensional case, but requires a new vector coding strategy, and a particular change of coordinates (matrix T_k) for the multi-valued complex conjugated eigenvalue case. The role of the rotation matrix T_k is to align the direction of the eigenvector (signal oscillation) to the vector quantizer block.

A. Principle of multivariable coding and decoding process

Figure 1 shows the architecture of the proposed differential coding algorithm. It is composed of three main components:

- **The vector quantizer block** transforms the error $\tilde{z}_k$, into a finite codeword set
- **The predictor** , that transforms back the codeword into a system state prediction $\hat{x}_k$
- **The rotation matrix** T_k transforms the estimation error $\tilde{x}_k$ between the signal x_k and its estimated (reconstructed) value $\hat{x}_k$ into a new set of coordinates $\tilde{z}_k$, i.e.

$$\tilde{z}_k = \boldsymbol{T}_k^{-1}\tilde{x}_k \tag{8}$$

Each of these components are explained in detail next.

1) vector quantizer: it maps the transformed vector $\tilde{z}_k$ into the quantized vector $\hat{\tilde{z}}_k$. The multi-level quantizer is constructed as follows:

- we consider M_i (odd or even) subdivisions for each $\tilde{z}^i$ with respective quantization step Δ_i. The partition is centered at the origin,
- This partition generates an hypercube of dimension n with a total of $n_{\mathcal{C}} = \prod_{i=1}^{i=n} M_i$ quantized volumes (see example in Figure 2),
- To each quantized volume is associated a value for the quantized vector $\hat{\tilde{z}}_k$ (see example in the Table I).

The formula used to compute $\hat{\tilde{z}}_k$ is the following:

$\hat{\tilde{z}}_k^i$ is given as:

$$\hat{\tilde{z}}_k^i = \begin{cases} (M_i-1)\Delta_i/2 & \text{if} \quad \mathcal{C}_1 \\ N\Delta_i & \text{if} \quad \mathcal{C}_2 \\ -(M_i-1)\Delta_i/2 & \text{if} \quad \mathcal{C}_3 \end{cases}$$

If M_i is odd, then $\mathcal{C}_i$ are:

$$\begin{aligned} \mathcal{C}_1 &: \tilde{z}_k^i \geqslant (M_i-1)\Delta_i/2 \\ \mathcal{C}_2 &: \tilde{z}_k^i \in [(N-1/2)\Delta_i, (N+1/2)\Delta_i], \\ & \quad (N \in \{-(M_i-1)/2, \ldots, (M_i-1)/2\}) \\ \mathcal{C}_3 &: \tilde{z}_k^i < -(M_i-1)\Delta_i/2 \end{aligned}$$

If M_i is even, the conditions $\mathcal{C}_i$ are:

$$\begin{aligned} \mathcal{C}_1 &: \tilde{z}_k^i \geqslant (M_i-1)/2\Delta_i \\ \mathcal{C}_2 &: \tilde{z}_k^i \in [N\Delta_i, (N+1)\Delta_i], \\ & \quad (N \in \{-(M_i-1)/2, \ldots, (M_i-1)/2\}) \\ \mathcal{C}_3 &: \tilde{z}_k^i < -(M_i-1)\Delta_i/2 \end{aligned}$$

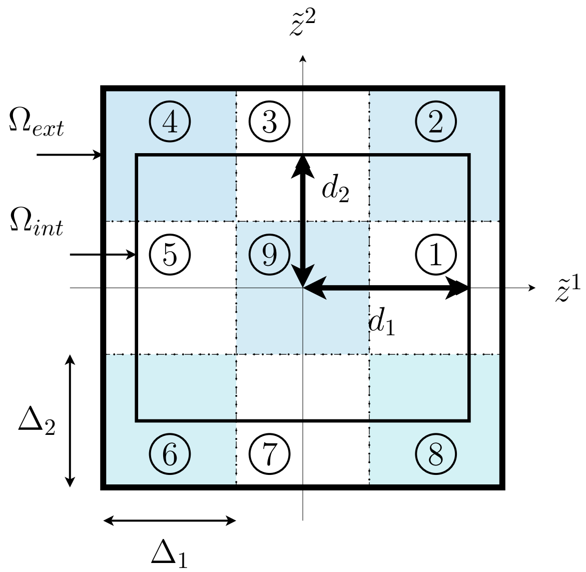

Fig. 2. Evolution of $\tilde{z}_k$ where $\tilde{z}_0$ begins in $\Omega^{\text{ext}} = \{[-1.5\Delta_1, 1.5\Delta_1] \times [-1.5\Delta_2, 1.5\Delta_2]\}$ and $\tilde{z}_k \in \Omega^{\text{int}} = \{[-d_1, d_1] \times [-d_2, d_2]\}$ and the dots delimit the nine subdivisions of the space.

Remark 1: Note that quantizer vector $\hat{\tilde{z}}_k^i$ is associated to a codeword of dimension $n_{\mathcal{C}}$ that can be coded directly into $R = \lceil \log_2(n_{\mathcal{C}}) \rceil$ bits, where $\lceil . \rceil$ denotes the ceil function.

2) Predictor: The estimation of the signal $\hat{x}_k$ is computed thanks to a model-based predictor:

$$\begin{aligned} \hat{x}_{k+1} &= (\boldsymbol{A} - \boldsymbol{B}\boldsymbol{K})\hat{x}_k + \boldsymbol{A}\hat{\tilde{x}}_k \\ &= (\boldsymbol{A} - \boldsymbol{B}\boldsymbol{K})\hat{x}_k + \boldsymbol{A}\boldsymbol{T}_k\hat{\tilde{z}}_k \end{aligned} \tag{9}$$

where the last expression results from the use of the inverse transformation matrix, i.e.

$$\tilde{x}_k = \boldsymbol{T}_k \tilde{z}_k \tag{10}$$

Due to the particular nature of this transformation (rotation matrix) its inverse always exists. Thus, using equations (8), (10) and (9), we get :

$$\tilde{z}_{k+1} = \boldsymbol{T}_{k+1}^{-1} \boldsymbol{A} \boldsymbol{T}_k (\tilde{z}_k - \hat{\tilde{z}}_k) \tag{11}$$

Note that, as this predictor is used at both the encoder and the decoder side, their respective initial conditions $\hat{x}_0$ and $\hat{\tilde{z}}_0$ are assumed to be the same.

3) Transformation matrix $\boldsymbol{T}_k$: The selection of this matrix for the general case is quite involved. In what follows we present two examples: a trivial choice ($\boldsymbol{T}_k = \boldsymbol{I}$), and an other where its choice depends on the eigenvalues position in the complex plane. The general case will be treated in detail in section IV.

B. Example 1: two-dimensional system with a real eigenvalue

Consider a system of the form (1), with

$$\boldsymbol{A} = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$$

TABLE I
CODING STRUCTURE RELATED TO FIGURE 2

Codeword	Value of $\hat{\tilde{z}}_k$	Codeword	Value of $\hat{\tilde{z}}_k$
1	$(\Delta_1, 0)$	6	$(-\Delta_1, -\Delta_2)$
2	(Δ_1, Δ_2)	7	$(0, -\Delta_2)$
3	$(0, \Delta_2)$	8	$(\Delta_1, -\Delta_2)$
4	$(-\Delta_1, \Delta_2)$	9	$(0, 0)$
5	$(-\Delta_1, 0)$		

and some $\boldsymbol{B}$ such that $(\boldsymbol{A}, \boldsymbol{B})$ is controllable. Then, we can take $\boldsymbol{T}_k = I_2$, where $\boldsymbol{I}_n$ denotes the n-entry identity matrix, which leads, with $\tilde{x}_k = \tilde{z}_k$, to

$$\tilde{z}_{k+1} = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} (\tilde{z}_k - \hat{\tilde{z}}_k)$$

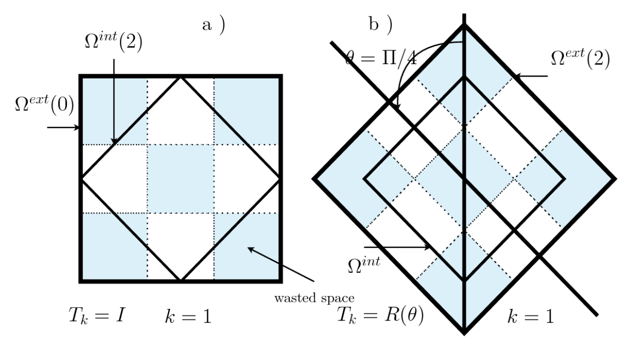

Fig. 3. Evolution of $\tilde{z}_k$, in the first figure, we choose that $\tilde{z}_0 \in \Omega^{\text{ext}}(0)$, $\tilde{z}_1 \in \Omega^{\text{int}}$ but if we code the signal $\tilde{z}_{k+1}$ with $T_k = I_2$ we loose some space and, to ensure that $\Omega^{\text{int}} \subset \Omega^{\text{ext}}$, the constrained eigenvalue is $|\lambda| < 3/\sqrt{2}$. The second figure shows a forced rotation which permits better performances

Let us choose $M_i = 3$ subdivisions per signal, with a different step for each one; a quantization step of $\Delta_1 > 0$ for $\tilde{z}_k^1$, and $\Delta_2 > 0$ for $\tilde{z}_k^2$. This partition is shown in Figure 2, and the associated coding strategy in Table I.

Now if we assume that $|\lambda| < 3$, and that the quantization steps are chosen such that

$$\Delta_2 < \Delta_1(3 - |\lambda|) \tag{12}$$

then it is easy to show that if the $\tilde{z}_0$ is initiated inside the centered rectangle set Ω^{ext}, then the evolution of $\tilde{z}_k$ will enter (in one step) inside the set Ω^{int} as defined in Figure 2.

To see that, note that if $\tilde{z}_k \in \Omega^{\text{ext}}$, then we have $|\tilde{z}_k^i - \hat{\tilde{z}}_k^i| \leqslant \frac{\Delta_i}{2}, \forall i \in \{1, 2\}$. Now, from error equation in $\tilde{z}_k$, we have that $|\tilde{z}_{k+1}^1| < |\lambda|\frac{\Delta_1}{2} + \frac{\Delta_2}{2} = d_1$, and that $|\tilde{z}_{k+1}^2| < |\lambda|\frac{\Delta_2}{2} = d_2$. This defines the set Ω^{int}. From here it is obviously needed that $\Omega^{\text{int}} \subset \Omega^{\text{ext}}$, which leads to the condition (12).

C. Example 2: two-dimensional system with complex conjugate eigenvalues

Consider a system of the form (1), with

$$\boldsymbol{A} = |\lambda| \boldsymbol{R}(\pi/4)$$

with $\boldsymbol{R}(\pi/4)$ is defined in (7), and $\boldsymbol{B}$ such that the pair $(\boldsymbol{A}, \boldsymbol{B})$ is controllable. Suppose that we take $\boldsymbol{T}_k = \boldsymbol{I}_2$, which gives $\tilde{x}_k = \tilde{z}_k$ and from (3) we get

$$\tilde{z}_{k+1} = |\lambda| \boldsymbol{R}(\pi/4)(\tilde{z}_k - \hat{\tilde{z}}_k)$$

As in the former example, let us choose $M_i = 3$ subdivisions per signal, with quantization steps $\Delta_1, \Delta_2 > 0$.

We suppose that the initial condition at $k = 0$ $\tilde{z}_0 \in \Omega^{\text{ext}}$ defined in the Figure 3 a), thus at $k = 1$ we obtain $\tilde{z}_1 \in \Omega^{\text{int}}$ (Figure 3 b)). It can be proved following similar steps as in Example 1 that Ω^{ext} is an invariant set if $|\lambda| < \frac{M_1}{\sqrt{2}}$ with $\Delta_1 = \Delta_2$. This condition is more conservative than the one obtained in Example III-B, where we only require that $|\lambda| < M_1$. It is also possible to retrieve the same result by redefining the transform matrix $\boldsymbol{T}_k$ as shown below

Let us choose $\boldsymbol{T}_k$ such that

$$\boldsymbol{T}_k = \boldsymbol{R}(k\pi/4)$$

Then $\tilde{z}_k = \boldsymbol{R}(-k\pi/4)\tilde{x}_k$ with $\boldsymbol{R}(\pi/4)^{-1} = \boldsymbol{R}(-\pi/4)$. Equation (3) becomes

$$\begin{aligned} \tilde{z}_{k+1} &= \boldsymbol{R}(-(k+1)\pi/4)|\lambda|\boldsymbol{R}(\pi/4)\boldsymbol{R}(-k\pi/4)^{-1}(\tilde{z}_k - \hat{\tilde{z}}_k) \\ &= |\lambda|\boldsymbol{R}(-(k+1)\pi/4)\boldsymbol{R}(\pi/4)\boldsymbol{R}(k\pi/4)(\tilde{z}_k - \hat{\tilde{z}}_k) \\ &= \tilde{z}_{k+1} = |\lambda|\boldsymbol{I}_2(\tilde{z}_k - \hat{\tilde{z}}_k) \end{aligned}$$

Hence, we obtain a fully decoupled system and it is straight forward to show that if $\tilde{z}_0$ begins in the set Ω^{ext}, it is necessary that $\Omega^{\text{int}} \subset \Omega^{\text{ext}}$ to ensure that Ω^{ext} is an invariant set, this condition leads to $|\lambda| < 3$ and a independent choice of Δ_1 and Δ_2. The generalization of this result needs an other transformation to have the same properties as real eigenvalues system.

IV. Construction of the transform matrix $\boldsymbol{T}_k$: general case

Consider a system of the form (1), with $\boldsymbol{A}$ defined in (4) and $\boldsymbol{B}$ such that $(\boldsymbol{A}, \boldsymbol{B})$ is controllable. The error equation:

$$\tilde{x}_{k+1} = \boldsymbol{A}(\tilde{x}_k - \hat{\tilde{x}}_k)$$

As we have assumed that $\boldsymbol{A}$ is a block diagonal matrix, the associated stability properties can be analyzed separately for $\boldsymbol{J}_{\lambda_l}$. In the following paragraph, we will first deal with the case of real eigenvalues $1 \leqslant l \leqslant \alpha$ and latter we will focus on the complex conjugate case $\alpha + 1 \leqslant l \leqslant \gamma$.

To simplify the notation, we only note $\tilde{x}_k$ instead of $\tilde{x}_k(l) \in R^{\mu_l}$, $\boldsymbol{J}_\lambda = \boldsymbol{J}_{\lambda_l}$ and $\mu = \mu_l$.

A. Case of multiple-valued real eigenvalues

Lemma 1: Case of multiple real eigenvalues. Assuming that $\hat{\tilde{z}}_k$ is computed thanks to the quantization procedure given in section III-A1, and suppose that

$$\tilde{z}_0 \in \Omega_{ext} = \{\tilde{z} \in \mathbf{R}^\mu : |\tilde{z}^i| \leqslant M_i \frac{\Delta_i}{2}, 1 \leqslant i \leqslant \mu\}$$

and the quantization steps satisfy the equations

$$|\lambda| + \frac{\Delta_{i+1}}{\Delta_i} \leqslant M_i, \; 1 \leqslant i \leqslant \mu - 1 \tag{13}$$

Then

i) Ω^{ext} is an invariant set

ii) $\tilde{z}_k \in \Omega^{\text{int}}$, $\forall k \geqslant 1$ where
$\Omega^{\text{int}} = \{\tilde{z} \in \mathbf{R}^\mu : |\tilde{z}^i| \leqslant |\lambda|\Delta_i/2 + \Delta_{i+1}/2$
$\forall i : 1 \leqslant i \leqslant \mu - 1$ and $|\tilde{z}^\mu| \leqslant \lambda\Delta_\mu/2\}$

Proof: According to (5):

$$\tilde{z}^i_{k+1} = \lambda(\tilde{z}^i_k - \hat{\tilde{z}}^i_k) + (\tilde{z}^{i+1}_k - \hat{\tilde{z}}^{i+1}_k) \tag{14}$$

$$\tilde{z}^\mu_{k+1} = \lambda(\tilde{z}^\mu_k - \hat{\tilde{z}}^\mu_k) \tag{15}$$

Given that $\hat{\tilde{z}}^{i+1}_k$ is quantized by the procedure given in section III-A1, we have $|\tilde{z}^{i+1}_k - \hat{\tilde{z}}^{i+1}_k| \leqslant \frac{\Delta_{i+1}}{2}$. Then using (13), for $1 \leqslant l \leqslant \mu - 1$, we get

$$\begin{aligned} |\tilde{z}^i_{k+1}| \leqslant |\lambda||\tilde{z}^i_k - \hat{\tilde{z}}^i_k| + |\tilde{z}^{i+1}_k - \hat{\tilde{z}}^{i+1}_k| &\leqslant |\lambda|\frac{\Delta_i}{2} + \frac{\Delta_{i+1}}{2} \\ &\leqslant M_i \frac{\Delta_i}{2} \end{aligned} \tag{16}$$

Finally, (13) implies that $|\lambda| < M_i$, so that

$$|\tilde{z}^\mu_{k+1}| \leqslant M_i \frac{\Delta_\mu}{2} \tag{17}$$

■

B. Case of complex conjugate eigenvalues.

We now consider the case where $\lambda \in \mathbb{C}$ for $\alpha + 1 \leqslant l \leqslant \gamma$. So, let us introduce matrices $\boldsymbol{W}(\theta)$ and $\boldsymbol{Q}(\theta)$ defined by

$$\boldsymbol{W}(\theta) = \begin{pmatrix} \boldsymbol{R}(\theta) & 0 \\ 0 & \boldsymbol{R}(\theta) \end{pmatrix} \tag{18}$$

$$\boldsymbol{Q}(\theta) = \begin{pmatrix} \boldsymbol{R}(-\theta) & \mathbf{0} \\ \mathbf{0} & \boldsymbol{R}(-\mu\theta) \end{pmatrix}. \tag{19}$$

It can be shown after a few calculations that

$$\boldsymbol{Q}^{-1}(\theta)\boldsymbol{W}^{-1}((k+1)\theta)\boldsymbol{J}_\lambda\boldsymbol{W}(k\theta)\boldsymbol{Q}(\theta) = \begin{pmatrix} |\lambda|I_2 & I_2 & 0 & 0 \\ 0 & |\lambda|I_2 & I_2 & 0 \\ \vdots & \ddots & |\lambda|I_2 & 0 \\ & \dots & 0 & |\lambda|I_2 \end{pmatrix} \hat{=} \check{\boldsymbol{J}}_\lambda$$

Let us choose $\boldsymbol{T}_k = \boldsymbol{W}(k\theta)\boldsymbol{Q}(\theta)$. Then, as in the case of real-valued eigenvalues, we have

$$\tilde{z}_{k+1} = \check{\boldsymbol{J}}_\lambda(\tilde{z}_k - \hat{\tilde{z}}_k) \tag{20}$$

and $\check{\boldsymbol{J}}_\lambda$ is a block diagonal matrix, so that we can consider separately each block again.

Then, considering separately even indices and odd indices, we exactly recover the results of the case of real-valued eigenvalues. Indeed, if we denote $\tilde{z}^e_k = [\tilde{z}^2_k, \tilde{z}^4_k, \ldots, \tilde{z}^{2\mu}_k]$ and $\tilde{z}^o_k = [\tilde{z}^1_k, \tilde{z}^3_k, \ldots, \tilde{z}^{2\mu-1}_k]$, we have

$$\tilde{z}^o_{k+1} = F(|\lambda|)(\tilde{z}^o_k - \hat{\tilde{z}}^o_k); \quad \tilde{z}^e_{k+1} = F(|\lambda|)(\tilde{z}^e_k - \hat{\tilde{z}}^e_k)$$

Where $F(|\lambda|)$ is the matrix given by the structure of Eq 5, namely sum of diagonal matrix $|\lambda|$ plus the superior diagonal with only 1.

Lemma 2: Case of multiple complex eigenvalues. Assuming that $\hat{\tilde{z}}_k$ is computed thanks to the quantization procedure given in section III-A1, and suppose that

$$\tilde{z}_0 \in \Omega^{\text{ext}} = \{\tilde{z} \in \mathbf{R}^{2\mu} : |\tilde{z}^i| \leqslant M_i \frac{\Delta_i}{2}\}$$
$$|\lambda| + \Delta_{i+2}/\Delta_i \leqslant M_i, \quad \forall i: \ 1 \leqslant i \leqslant 2\mu - 2 \qquad (21)$$

Then we ensure that

i) Ω^{ext} is an invariant set
ii) $\tilde{z}_k \in \Omega^{\text{int}}$, $\forall k \geqslant 1$ where $\Omega^{\text{int}} = \{\tilde{z} \in \mathbf{R}^{2\mu} : |\tilde{z}^i| \leqslant |\lambda|\Delta_i/2 + \Delta_{i+2}/2$ $1 \leqslant i \leqslant 2\mu - 2$ and else $|\tilde{z}^i| \leqslant |\lambda|\Delta_i/2\}$

Proof: The proof is identical to the demonstration of Lemma 1 in the case of real-valued eigenvalues. ■

C. General case: combined real and complex eigenvalues

Theorem 1: Suppose the system (2)

$$x_{k+1} = \boldsymbol{A}x_k + \boldsymbol{B}u_k$$

with the pair $(\boldsymbol{A}, \boldsymbol{B})$ controllable.
And a channel rate R bounded by

$$\prod_{l=1, |\lambda_l|>1}^{n} \lceil |\lambda_l| \rceil < 2^R$$

Then, the coding structure that ensures that x_k is bounded, is realized with the Delta-modulation coding explained in section III where $\tilde{z}_k = \boldsymbol{T}_k^{-1}\tilde{x}_k$ with $\boldsymbol{T}_k$ defined as

$$\boldsymbol{T}_k = \begin{pmatrix} \boldsymbol{I}_{\mu_1} & 0 & 0 & 0... \\ 0 & \boldsymbol{I}_{\mu_\alpha} & 0 & 0... \\ 0 & 0... & \boldsymbol{W}_\iota(k\theta_\iota)\boldsymbol{Q}_\iota(k\theta_\iota) & 0 \\ 0 & 0... & 0 & \boldsymbol{W}_\gamma(k\theta_\gamma)\boldsymbol{Q}_\gamma(\theta_\gamma) \end{pmatrix} \qquad (22)$$

with $\alpha + 1 = \iota$.
Then $\tilde{z}_{k+1} = \breve{\boldsymbol{J}}(\tilde{z}_k - \hat{\tilde{z}}_k)$ and where $\boldsymbol{A} = \boldsymbol{T}_{k+1}\breve{\boldsymbol{J}}\boldsymbol{T}_k^{-1}$

$$\breve{\boldsymbol{J}} = \begin{pmatrix} \boldsymbol{J}_1 & 0 & \dots & 0 \\ \vdots & \boldsymbol{J}_\alpha & \dots & 0 \\ \vdots & 0 & \breve{\boldsymbol{J}}_\iota & 0 \\ 0 & \dots & 0 & \breve{\boldsymbol{J}}_\mu \end{pmatrix}$$

with the properties for M_i and Δ_i given in lemma 1 for real eigenvalues and lemma 2 for complex eigenvalues.

Proof: For each signal with instable open loop, one of the condition is $|\lambda_l| < M_i$, it is sufficient that $\lceil |\lambda_l| \rceil < M_i$ with $R = \log_2 \prod_{i=1}^{n} M_i$. If we multiply for all the coefficients, the result becomes

$$\prod_{l=1, |\lambda_l|>1}^{n} \lceil |\lambda_l| \rceil < 2^R$$

Using the previous lemmas, we ensure that $\tilde{x}$ is bounded.

$$x_{k+1} = (\boldsymbol{A} - \boldsymbol{B}\boldsymbol{K})x_k + \boldsymbol{A}\tilde{x}_k$$

With the following system where $\boldsymbol{A} - \boldsymbol{B}\boldsymbol{K}$ has its eigenvalues strictly inferior than 1, the authors of [4] have shown that the cascade system ensures that x_k is bounded. ■

V. Domain of attraction, new tuning rules for Δ_i

The aim of this section is twofold. Firstly assuming the use of the tuning rule (13), we provide a less conservative method to estimate the attraction domain (named $\mathcal{B} \supset \Omega^{\text{ext}}$). Secondly, assuming the same attraction domain Ω^{ext}, we provide a new tuning rule for the Δ_i that,compared to previous rule given in (13), results in smaller values for Δ_i. As a consequence, the system precision can be improved.

A. Characterization of $\mathcal{B}$

Let assume that the Δ_i are tuned following the rule in (13), and denote $\mathcal{B}$ the new estimation of the attraction domain with $\Omega^{\text{ext}} \subset \mathcal{B} \subset \mathbf{R}^n$. Let $\mathcal{B}$ be defined as the compositions of the sub-sets $\mathcal{B}_{\lambda_l}$,

$$\mathcal{B} = \mathcal{B}_{\lambda_1} \times ... \times \mathcal{B}_{\lambda_w} \qquad (23)$$

where the $\mathcal{B}_{\lambda_l}$ describes the attraction domain for the l-th Jordan's block, $\breve{\boldsymbol{J}}_{\lambda_l}$, under consideration,

$$\tilde{z}_{k+1} = F(|\lambda_l|)(\tilde{z}_k - \hat{\tilde{z}}_k)$$

This decomposition simplifies the analysis by looking at each block separately instead of considering the whole system together. Therefore, we only need to focus on a single block $\mathcal{B}_{\lambda_l}$, and repeat the same analysis for other block when needed.

Inspired by the Jordan block structure, assume in turn that $\mathcal{B}_{\lambda_l} = \mathcal{H}_{\lambda_l,1} \times ... \times \mathcal{H}_{\lambda_l,\mu_l}$ where each subset, $\mathcal{H}_{\lambda_l,1}$, correspond to a domain associated to each of the Jordan block components. For simplicity reasons, we omit the subindex λ_l in the sequel. Hence, we simply note $\mathcal{B} = \mathcal{H}_1 \times ... \times \mathcal{H}_{\mu_l}$.

Theorem 2: Assume that $\hat{\tilde{z}}_k$ is computed thanks to the quantization procedure given in section III-A1, and that Δ_i are tuned following the rule in (13), and suppose that

$$\tilde{z}_0 \in \mathcal{B} = \{\tilde{z} \in \mathbf{R}^\mu : |\tilde{z}^i| \leqslant \gamma_i\}$$

with, for $1 \leqslant i \leqslant \mu - 1$,

$$\begin{array}{lll} \gamma_i & = & \min\left((M-1)\Delta_i/2 + \varepsilon^i_{\text{max}}, (|\lambda||\hat{\tilde{z}}^i_k| - \varepsilon^{i+1}_{\text{max}})/(|\lambda| - 1)\right) \\ \varepsilon^{i+1}_{\text{max}} & \leqslant & \min\left((M - |\lambda|)\Delta_i/2, (M - 1 - |\lambda|)\Delta_i/2 + \varepsilon^i_{\text{max}}\right) \end{array}$$

Then:

i) $\mathcal{B}$ is an invariant set, i.e. $\tilde{z}_k \in \mathcal{B} \, \forall k \geqslant 0$.
ii) $\exists k_1 > 0$, such that, $\tilde{z}_k \in \Omega^{\text{int}}$, $\forall k \geqslant k_1$. where Ω^{int} is the same set as defined in Lemma 1-*ii)*.

Details of the proof are given in [5]. Note that this analysis allows us to obtain a bigger attraction domain than the one obtained in section IV. To see this, note that $\varepsilon^i_{\text{max}} \geqslant \Delta_i/2$, which implies that $\gamma_i \geqslant M\Delta_i/2$, and therefore we have that

$$\mathcal{B} \supset \Omega^{\text{ext}}$$

B. Tuning policies for Δ_i

Assume now that the attraction domain $\bar{\Omega}^{\text{ext}}$, is given by

$$\bar{\Omega}^{\text{ext}} = \{\tilde{z} \in \mathbf{R}^\mu : |\tilde{z}^i| \leqslant \delta_i, 1 \leqslant i \leqslant \mu\}$$

where δ_i are arbitrary values specified by the user. Note that the specification above imposes, in the previous tuning method, that $M_i\frac{\Delta_i}{2}=\delta_i$, whereas theorem 3 below will show that the new values $\bar{\Delta}_i<\Delta_i=\frac{2\delta_i}{M_i}$ leading to a smaller convergence set $\bar{\Omega}^{\rm int}\subset\Omega^{\rm int}$, where $\Omega^{\rm int}$ is the same set as defined in Lemma 1-ii).

Theorem 3: Suppose that $\tilde{z}_0\in\bar{\Omega}^{\rm ext}$, and let the following rule to be applied to select the coding levels,for $1\leqslant i\leqslant\mu-1$,

$$\begin{aligned}\bar{\Delta}_i &= 2\frac{|\lambda|-1}{|\lambda|(M-1)}\delta_i+2\frac{\delta_{i+1}-(M-1)\bar{\Delta}_{i+1}/2}{|\lambda|(M-1)}\\ \bar{\Delta}_\mu &= \delta_\mu(2(|\lambda|-1))/|\lambda|\end{aligned}$$

Then:

$i)$ $\bar{\Omega}^{\rm ext}$ is an invariant set, and

$ii)$ $\exists k_1>0$, such that, $\tilde{z}_k\in\bar{\Omega}^{\rm int}$, $\forall k\geqslant k_1$, where $\bar{\Omega}^{\rm int}\subset\Omega^{\rm int}$ is given as:

$$\bar{\Omega}^{\rm int}=\left\{\tilde{z}\in\mathbf{R}^\mu:\left\{\begin{array}{ll}|\tilde{z}^i|\leqslant|\lambda|\bar{\Delta}_i/2+\bar{\Delta}_{i+1}/2 & 1\leqslant i\leqslant\mu-1\\ |\tilde{z}^\mu|\leqslant|\lambda|\bar{\Delta}_\mu/2 & i=\mu\end{array}\right.\right.$$

Proof: Property $i)$ can be shown following the same proof as in part $i)$ of Theorem 2. For the Property $ii)$ the convergence of $\tilde{z}_k$ towards the set $\bar{\Omega}^{\rm int}$ in finite time also follows the same lines as the proof of Theorem 2 and is omitted here.

Finally the fact that $\bar{\Omega}^{\rm int}\subset\Omega^{\rm int}$ follows by first observing that both sets $\bar{\Omega}^{\rm int}$, and $\Omega^{\rm int}$ have the same upper bound structure, and hence it is sufficient to prove that $\bar{\Delta}_i<\Delta_i$. This last inequality follows from inspection comparing the definition of the $\bar{\Delta}_i$ given in the theorem with the ones resulting from the imposed constraints to the previous tuning method, i.e. $\Delta_i=\frac{2\delta_i}{M_i}$. ■

VI. Simulation Results

The aim of this section, is to compare the precision improvements that the second tuning method can provide. For this, we consider a second order system:

$$\boldsymbol{A}=\begin{pmatrix}1.1 & 1\\ 0 & 1.1\end{pmatrix},\qquad \boldsymbol{B}=\begin{pmatrix}0.2\\ 0.3\end{pmatrix}$$

The controller K is designed to have its closed-loop eigenvalues located at $(0.5,0.6)$ with objective to regulate the output to $x_2^{\rm ref}=1$. The desired attraction domain for the estimation error is specified as $(\delta_1,\delta_2)=(0.62,0.52)$, we have $x_0=(0.6,0.5)$, $\hat{x}_0=(0,0)$. $M_1=M_2=2$.

Hence with the two strategies, we obtain $\Delta_1=0,62$, $\Delta_2=0,52$ and $\bar{\Delta}_1=0,35$, $\bar{\Delta}_2=0,057$.

Figure 4 shows the time-evolution of the resulting closed-loop signals. In both runs, the initial condition are the same, and as it was expected the second method provides smaller values for the coding gains which implies better signal reconstruction quality and better regulation precision.

VII. Conclusion

In this paper, we have investigated the closed-loop properties of multivariable (MIMO) linear systems where the sensed information is centralized and coded on the basis of a Δ-modulation algorithm intended to be used for minimizing the number of transmitted bits.

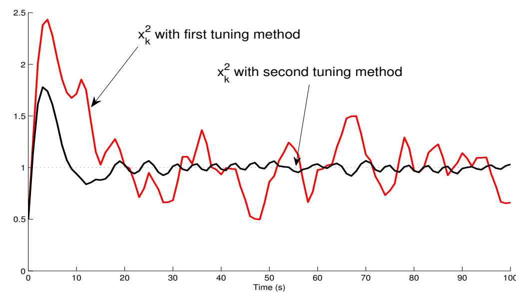

Fig. 4. Time-evolution of the closed-loop state x_k^2 using two different tuning methods discussed in this paper. The impact on the first state is less effective than the second state, that is why we only show the second

The key feature allowing this result was based on the idea of performing the differential coding in a time-varying rotation coordinates associated to canonical Jordan forms.

We have also shown that this fixed-gain simple and methodic coding strategy results in a ultimately uniformly (local) stability. We have also provided an estimation of the attraction domain, and a new method to tune the coding gains, resulting in closed-loop precision improvements. Simulation results validate the proposed approach.

References

[1] Brockett R.-W. and Liberzon D. Quantized feedback stabilization of linear systems. *IEEE Transactions on Automatic Control*, 45(7):1279–1289, July 2000.

[2] C. Canudas-de-Wit, J. Jaglin, and K. Crisanto Vega. Entropy coding in networked controlled systems. *ECC'07 KOS, Greece*, 2007.

[3] C. Canudas-de-Wit, J.Jaglin, and C Siclet. Energy-aware 3-level coding and control co-design for sensor network systems. *Conference on Control Application, Singapore*, 2007.

[4] C. Canudas-de-Wit, F. Rubio, J. Fornes, and F. Gomez-Estern. Differential coding in networked controlled linear systems. *American Control Conference. Silver Anniversary ACC. Minneapolis, Minnesota USA*, June 2006.

[5] J. Jaglin C. Canudas-de-Wit and C. Siclet. Delta modulation for mimo centralized linear networked controlled systems. *Inernal report http://necs.inrialpes.fr/publications.htm*, 2008.

[6] N. Elia and S.-K. Mitter. Stabilization of linear systems with limited information. *IEEE Transaction on Automatic Control*, 46(9):1384–1400, September 2001.

[7] F. Rubio F. Gomez-Estern, C. Canudas-de-Wit and J. Fornes. Adaptive delta-modulation coding in networked controlled systems. *Submitted to American Control Conference, New York, USA USA*, June 2007.

[8] Hespanha J.-P., Ortega A., and Vasudevan L. Towards the control of linear systems with minimum bit–rate. In *15th Int. Symp. Mathematical Theory of Networks and Systems (MTNS)*, Notre Dame, IL, USA, 2002.

[9] C.T. Abdallah I. Lopez and C. Canudas-de-Wit. Compensation schemes for a delta-modulation-based ncs. *ECC'07 KOS, Greece*, 2007.

[10] K. Li and J. Baillieul. Robust quantization for diginal finite communication bandwidth (dfcb) control. *IEEE Transaction on Automatic Control*, 49(9):1573–1584, September 2004.

[11] D. Liberzon. On stabilization of linear systems with limited information. *IEEE Transaction on Automatic Control*, 48(2):304–307, February 2003.

[12] T. Basar S. Yuksel. Decentralized multi controller stability over communication channels. *European Control Conference, Kos, Greece USA*, 2007.

[13] S. Tatikonda and S. K. Mitter. Control under communications constaints. *IEEE Transaction on Automatic Control*, 49(7):1056–1068, July 2004.