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A nonlinear dynamic inversion computational approach applied to the exact tracking problem for the spherical pendulum

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Abstract— **In this paper we present a new computational approach for nonlinear dynamic inversion based on an "homotopy technique" applied to the exact tracking problem for the spherical inverted pendulum on a periodic curve in $\mathbb{R}^3$, non necessarily planar.**

INTRODUCTION

In this paper we present a new computational approach for nonlinear dynamic inversion based on an "homotopy technique" applied to the exact tracking problem for a spherical inverted pendulum on a periodic curve in $\mathbb{R}^3$, non necessarily planar.

In recent literature there has been considerable interest in the control of the spherical pendulum with particular emphasis on the stabilization problem, see for instance [1], [2] and [3] for a more general system, called 3d pendulum. However fewer results are related to the tracking problem. An approximate tracking problem in considered in [4] where it is presented a way-point approach that allows the inverted pendulum base to remain in a neighborhood of a path defined by line segments, avoiding pendulum overturns. A recent result regarding the exact tracking problem is presented in [5], where an interesting application of the nonlinear stable inversion technique proposed in [6] is used to find a reference trajectory for system dynamics when the pendulum base follows a planar curve.

The basic idea of the proposed nonlinear inversion method is the following. We assume that at the initial time the base is at a given point of a periodic reference trajectory γ and we determine the force applied to the base that drives the base all along the given reference trajectory. The resulting unstable motion of the mass with respect to the base is governed by the system internal dynamics for the time-dependent orientation vector ζ (see figure 1). We show that, under suitable hypotheses, there exists a particular initial state ζ_0 such that the internal dynamics have a periodic solution in which ζ remains close to the vertical axis, that is the pendulum does not overturn (see figure 4). This initial state is found through an homotopy method.

Homotopy methods have received considerable attention also in nonlinear control theory: see for instance [7] for the feedback stabilization of linear systems, [8] for least squares estimation, [9] for boundary value problems in optimal control and [10] for an a method for finding an input signal that drives a nonlinear system to a given state.

The idea of our approach is the following: we see the reference trajectory γ as the value that the family of periodic curves $\{s\gamma\}$ assumes at $s = 1$ and we write the internal dynamics for the general curve $s\gamma$. In this way we obtain a family of differential systems dependent on s, which gives exactly the spherical pendulum internal dynamics when $s = 1$. For $s = 0$ the system has the trivial identically null periodic solution which corresponds to the unstable vertical equilibrium of the spherical pendulum when the base is fixed at the origin of $\mathbb{R}^3$ (in fact for $s = 0$ the curve $s\gamma$ collapses to the origin).

With a technique based on a fixed point problem of a Poincaré map, if some hypotheses are verified, by the use of the Implicit Function Theorem, we find that there exists an $\bar{s}$ such that the null solution may be morphed to a family of periodic solutions for the internal dynamics for the curve $s\gamma$, for every $s : 0 \leq s < \bar{s}$. Therefore we find a periodic solution of our system if we can show that $\bar{s} > 1$. Remark that $\bar{s}$ is found by determining the maximal interval of existence (see (11)) of a suitable differential system whose solution is exactly the curve of initial data of the periodic solutions.

We have used this method to face the exact tracking problem for some well-known two dimensional nonminimum phase systems such as the VTOL (see [11]), the planar inverted pendulum (see [12]), the motorcycle and the CTOL aircraft (see [13]) and this method has allowed us to find sufficient conditions for the feasibility of a trajectory.

This paper is a first step for extending this method to non-minimum phase systems with general n-dimensional internal dynamics and we are working on finding results analogous to the 2-dimensional case.

The numerical implementation of the method presented in section II is challenging because it requires solving a time-varying linear system with both positive and negative eigenvalues, which is often ill-conditioned. The integration method proposed is section IV faces this problem and has given good results in simulation.

The following notations will be used: $\forall a, b \in \mathbb{R}$, $a \wedge b = \min\{a,\ b\}$, $a \vee b = \max\{a,\ b\}$ and $[a, b] = \{x \in \mathbb{R} | a \leq x \leq b\}$, $]a, b[= \{x \in \mathbb{R} | a < x < b\}$; $\forall \theta \in [0, 2\pi[$, $\tau(\theta) = (\cos\theta,\ \sin\theta)^T$; $\forall x \in \mathbb{R}^2$, $\arg x = \theta$, where $\theta \in [0, 2\pi[$ is such that $x = \|x\|\tau(\theta)$; $\forall x, y \in \mathbb{R}^3$, $x \times y$ denotes the vector cross product; $\forall x = (x_1, \ldots, x_n)^T$, $y = (y_1, \ldots, y_n)^T \in \mathbb{R}^n$, $\langle x\ ,\ y\rangle = \sum_{i=1}^n x_i y_i$, $\|x\| = \sqrt{\langle x\ ,\ x\rangle}$; if I is a real interval, $\forall f : I \to \mathbb{R}^n$, $\|f\|_\infty = \sup_{x \in I}\{\|f(x)\|\}$.

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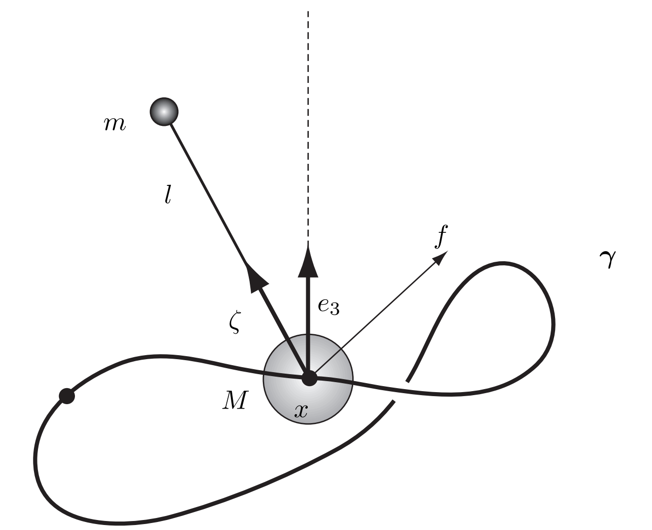

Fig. 1. Spherical pendulum constrained to follow a given periodic γ in the space.

I. Problem formulation

Consider a spherical inverted pendulum of mass m linked to a moving base of mass M through a massless rod of length l , in Figure 1 the pendulum is represented as the smaller sphere and the base as the bigger one. It is supposed that during the motion the force $f \in \mathbb{R}^3$ is applied on the center of mass x of M.

The problem we want to solve is the following: given an arbitrary (not necessarily plane) T-periodic curve $\gamma \in C^3(\mathbb{R}, \mathbb{R}^3)$, we want to find a control force $f \in \mathcal{C}(\mathbb{R}, \mathbb{R}^3)$, applied to the point x, such that if $x(0) = \gamma(0)$, then $x(t) = \gamma(t)$, $\forall t \geq 0$ and $\|\zeta - e_3\|$ is sufficiently small, where $e_3 = (0, 0, 1)^T$. In other words, if at the initial time $x(0) = \gamma(0)$, then x follows all the curve γ and the rod remains close to the vertical without overturning.

To this goal, let $q = (x, \zeta) \in \mathbb{R}^3 \times S^2$ (where $S^2 = \{\zeta \in \mathbb{R}^3 : \|\zeta\| = 1\}$, be the vector of generalized coordinates, where x is the position of the center of mass of the moving base M and ζ the orientation versor of the rod. Let $L = T - U$ be the Lagrangian, where the kinetic energy is given by

$$T = 1/2(m + M)\|\dot{x}\|^2 + 1/2ml^2\|\dot{\zeta}\| + ml\langle\dot{\zeta}\, , \, \dot{x}\rangle$$

and the potential energy by

$$U = g\langle(M + m)x + lm\zeta\, , \, e_3\rangle\ .$$

The dynamic equations are derived through the Euler-Lagrange equation

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = f$$
$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\zeta}} - \frac{\partial L}{\partial \zeta} = 0\ .$$

Since $\zeta \in S^2$, $\dot{\zeta}$ is orthogonal to ζ, that is $\dot{\zeta} \in T_\zeta(S^2)$ the tangent space to S^2 at ζ, therefore we can suppose that $\dot{\zeta} = \zeta \times \omega$, with $\omega \in \mathbb{R}^3$ and $\langle\zeta\, , \, \omega\rangle = 0$.

The resulting dynamical system is

$$\begin{cases} (m + M)\ddot{x} + ml\ddot{\zeta} - (m + M)ge_3 = f \\ l\dot{\omega} = \zeta \times \ddot{x} + g(\zeta \times e_3) \end{cases} \tag{1}$$

Set $S^2_+ = \{\zeta = (\zeta_1, \zeta_2, \zeta_3) \in S^2 | \zeta_3 > 0\}$ and suppose that the following problem is solvable:

Problem: *There exists a $\zeta_0 \in S^2_+$ and $\omega_0 \in \mathbb{R}^3$ such that $\langle\zeta_0\, , \, \omega_0\rangle = 0$ and the following system*

$$\begin{cases} \dot{\zeta} = \zeta \times \omega \\ l\dot{\omega} = \zeta \times \ddot{\gamma} + g(\zeta \times e_3) \\ \omega(0) = \omega_0 \\ \zeta(0) = \zeta_0\ . \end{cases} \tag{2}$$

has a T-periodic solution, with the property

$$\zeta(t) \in S^2_+,\ \forall t \in \mathbb{R}\ ,$$

Remark that if (ζ, ω) is a solution of (2), then $\langle\zeta(t)\, , \, \omega(t)\rangle = 0$, $\forall t \geq 0$ since $\frac{d}{dt}\langle\zeta(t)\, , \, \omega(t)\rangle = 0$, $\forall t \geq 0$. Now if (ζ, ω) is a solution of (2), the control force given by

$$f = (m + M)\ddot{\gamma} + ml\ddot{\zeta} - (m + M)ge_3\ ,$$

drives point x along all the curve γ, that is the solution of system

$$\begin{cases} (m + M)\ddot{x} + ml\ddot{\zeta} - (m + M)ge_3 = f \\ l\dot{\omega} = \zeta \times \ddot{x} + g(\zeta \times e_3) \\ \dot{\zeta} = \zeta \times \omega \\ x(0) = \gamma(0), \\ \dot{x}(0) = \dot{\gamma}(0), \\ \langle\zeta(0)\, , \, \omega(0)\rangle = 0\ , \end{cases} \tag{3}$$

is such that $x(t) = \gamma(t)$, $\forall t \in [0, T]$ and the pendulum does not overturn since $\zeta(t) \in S^2_+$, $\forall t \geq 0$.

Let (ζ, ω) be a solution of our Problem. Differentiating the first equation of (2), we get

$$\ddot{\zeta} = \dot{\zeta} \times \omega + \zeta \times \dot{\omega} = (\zeta \times \omega) \times \omega + l^{-1}\zeta \times (\zeta \times (\ddot{\gamma} + ge_3))\ .$$

By the following identity valid for any $a, b, c \in \mathbb{R}^3$ $a \times (b \times c) = b\langle a\, , \, c\rangle - c\langle a\, , \, b\rangle$, we get, since $\langle\zeta\, , \, \omega\rangle = 0$ and $\|\zeta\| = 1$, that $\ddot{\zeta} = -\zeta\|\omega\|^2 + l^{-1}\zeta\langle\zeta\, , \, \ddot{\gamma} + ge_3\rangle - l^{-1}(\ddot{\gamma} + ge_3)$.

Being $\|\omega\| = \|\dot{\zeta}\|$, our Problem reduces to the following one:

Find $\zeta_0 \in S^2_+$ such that the following system

$$\begin{cases} \ddot{\zeta} = -\zeta\|\dot{\zeta}\|^2 + l^{-1}\zeta\langle\zeta\, , \, \ddot{\gamma} + ge_3\rangle - l^{-1}(\ddot{\gamma} + ge_3) \\ \zeta(0) = \zeta_0 \\ \dot{\zeta}(0) = \zeta_0 \times \omega_0 \end{cases} \tag{4}$$

has a T-periodic solution $\zeta \in \mathcal{C}^2(\mathbb{R}, S^2_+)$.

Set $\mathcal{B} = \{(z, w) \in \mathbb{R}^2 | \|(z, w)\| < 1\}$, then the map

$$\begin{array}{ccc} \mathcal{B} & \longrightarrow & S^2_+ \\ (z, w) & \rightsquigarrow & (z, w, \sqrt{1 - z^2 - w^2}) \end{array}$$

is a diffeomorphism, therefore our problem can be rewritten on $\mathcal{B}$, in terms of the variable (z, w), in the following way:

Find $(z_0, w_0, \dot z_0, \dot w_0) \in \mathcal{B} \times \mathbb{R}^2$ *such that the system*

$$\begin{cases} \begin{pmatrix} \ddot z \\ \ddot w \end{pmatrix} = -\begin{pmatrix} z \\ w \end{pmatrix}(\dot z^2 + \dot w^2 + \frac{(\dot z z + \dot w w)^2}{1-z^2-w^2}) + \\ +l^{-1}\begin{pmatrix} z \\ w \end{pmatrix}\langle\begin{pmatrix} z \\ w \\ \sqrt{1-z^2-w^2} \end{pmatrix}, \ddot\gamma + \begin{pmatrix} 0 \\ 0 \\ g \end{pmatrix}\rangle + \\ -l^{-1}\begin{pmatrix} \ddot\gamma_1 \\ \ddot\gamma_2 \end{pmatrix} \\ z(0) = z_0, \ w(0) = w_0 \\ \dot z(0) = \dot z_0, \ \dot w(0) = \dot w_0 \ , \end{cases} \tag{5}$$

has a T*-periodic solution.*

Now our purpose is to solve this problem applying the homotopy method presented in section II. To this end, let us consider the family of ordinary differential systems depending on the parameter $s \in \mathbb{R}$

$$\begin{pmatrix} \ddot z \\ \ddot w \end{pmatrix} = -\begin{pmatrix} z \\ w \end{pmatrix}(\dot z^2 + \dot w^2 + \frac{(\dot z z + \dot w w)^2}{1-z^2-w^2}) + \\ +l^{-1}\begin{pmatrix} z \\ w \end{pmatrix}\langle\begin{pmatrix} z \\ w \\ \sqrt{1-z^2-w^2} \end{pmatrix}, s\ddot\gamma + \begin{pmatrix} 0 \\ 0 \\ g \end{pmatrix}\rangle + \\ -sl^{-1}\begin{pmatrix} \ddot\gamma_1 \\ \ddot\gamma_2 \end{pmatrix} . \tag{6}$$

Remark that for $s = 1$, system (6) becomes system (5) and for $s = 0$, system (6) has an obvious T-periodic solution: the one identically zero, that is the pendulum is kept in the vertical unstable equilibrium in the point $(0, 0)$. Therefore setting $x = (z, w, \dot z, \dot w)$, system (6) falls into the class of systems $\dot x = F(t, s, x)$ considered in theorem 1. Therefore it is easy to see that all hypotheses of theorem 1 are verified, taking $n = 4$, $I = \mathbb{R}$, $\Omega = \mathcal{B} \times \mathbb{R}^2$ and the solution identically zero as $\tilde x$. Therefore we can apply theorem 1 and our problem will be solved if we can show that $\bar s > 1$ and the requested T-periodic solution will be $x(t, 1, \phi(1))$, as stated in the theorem. In this paper this condition is checked by numerical computation for a given curve γ, as shown in the simulations section of this paper. In paper [12] we have proved for the planar vertical pendulum that the problem can always be solved if the second derivative of γ is sufficiently small. A similar result holds for the spherical pendulum, however the proof is more involved and will be presented in future papers.

II. THE HOMOTOPY METHOD

In this section we state and prove the main result of this paper: an existence theorem of periodic solutions for a family of ordinary differential systems $\dot x = F(t, s, x)$ depending on a parameter s. The procedure is the following. We suppose that for $s = 0$ the problem $\dot x = F(t, s, x)$ has a periodic solution $\tilde x$, then we find a family $\{x_s\}_{0 \le s \le \bar s}$ of solutions, which is a deformation of $\tilde x$ obtained by means of the Implicit Function Theorem applied to the Poincarè map related to the family $\dot x = F(t, s, x)$. In such way we find that there exists a periodic solution x_s for every s belonging to $[0, \bar s[$ which is the maximal interval of existence of a suitable differential system (11).

Theorem 1 (Main theorem): Let I be an open interval containing 0, Ω an open subset of $\mathbb{R}^n$,

$$\begin{array}{llll} F: & \mathbb{R} \times I \times \Omega & \longrightarrow & \mathbb{R}^n \\ & (t, s, x) & \rightsquigarrow & F(t, s, x) \end{array}$$

be a $\mathcal{C}^2$ map and let $\forall (s, y) \in I \times \Omega$, $\Phi(t, s, y)$ be the solution of

$$\begin{cases} \dot\Phi = \partial_x F(t, s, x(t, s, y))\Phi, \\ \Phi(0) = I \end{cases} \tag{7}$$

where $x(t, s, y)$ is the solution of

$$\begin{cases} \dot x = F(t, s, x) \ , \\ x(0) = y \ . \end{cases} \tag{8}$$

defined on its maximal interval of existence.

Suppose that the following hypotheses are verified:

a) $\forall (s, x) \in I \times \Omega$ the map $t \rightsquigarrow F(t, s, x)$ is T-periodic;

b) there exists a $\tilde x \in \mathcal{C}^1(\mathbb{R}, \Omega)$, T-periodic solution of

$$\dot{\tilde x}(t) = F(t, 0, \tilde x(t)), \ \forall t \in \mathbb{R} \ ; \tag{9}$$

c) it is

$$\det(I - \Phi(T, 0, \tilde x(0))) \neq 0 \ . \tag{10}$$

Let $\bar s$ be the supremum of $\hat s \ge 0$ such that there exists $\phi \in \mathcal{C}^1([0, \hat s], \mathbb{R}^n)$ with the following properties:

1) $\forall s \in [0, \hat s]$ the solution $t \rightsquigarrow x(t, s, \phi(s))$ is defined on $[0, T]$,

2) $\forall s \in [0, \hat s]$, det $(I - \Phi(T, s, \phi(s))) \neq 0$,

3) ϕ verifies the following differential system on $[0, \hat s]$:

$$\begin{cases} \dot\phi(s) = (I - \Phi(T, s, \phi(s)))^{-1}\Phi(T, s, \phi(s)) \cdot \\ \cdot \int_0^T \Phi^{-1}(p, s, \phi(s))\partial_s F(p, s, x(p, s, \phi(s)))dp \\ \phi(0) = \tilde x(0) \ . \end{cases} \tag{11}$$

Then $\bar s > 0$ and there exists a unique function $\phi \in \mathcal{C}^1([0, \bar s[, \mathbb{R}^n)$ solution of (11) on $[0, \bar s[$, such that $\{x(t, s, \phi(s))\}_{0 \le s < \bar s}$ is a family of T-periodic solutions of

$$\begin{cases} \dot x(t) = F(t, s, x(t)), \ \forall t \in \mathbb{R}, \\ x(0) = \phi(s) \ , \end{cases} \tag{12}$$

$\forall s \in [0, \bar s[$, this implies in particular that

$$x(t, 0, \phi(0)) = \tilde x(t), \ \forall t \in \mathbb{R} \ . \tag{13}$$

and

$$x(T, s, \phi(s)) = \phi(s), \ \forall s \in [0, \bar s[\ . \tag{14}$$

III. PROOF OF THE MAIN THEOREM

Remark that, in any case (13) is verified by uniqueness of the solution of system (11) and the definition of $x(t, s, y)$, in fact, being $\phi(0) = \tilde x(0)$, $\forall t \in \mathbb{R}$,

$$x(t, 0, \phi(0)) = x(t, 0, \tilde x(0)) = \tilde x(t) \ . \tag{15}$$

First of all we want to show that there exists $\hat s > 0$ and a function $\phi \in \mathcal{C}^1([0, \hat s], \mathbb{R}^n)$ such that properties 1), 2), 3) are verified. This will imply that $\bar s > 0$, by definition of $\bar s$.

In fact, since F is a $\mathcal{C}^2$ map, $\forall (y, s) \in I \times \Omega$ there exists and is unique the solution $x(t, s, y)$ of system (8) defined (as function of t) on its maximal interval of existence. For all $\epsilon > 0$, let $\Omega_\epsilon = \{x \in \Omega | d(x, \tilde x(0)) < \epsilon\}$ be the ϵ-neighborhood of $\tilde x(0)$ in Ω. Then by the theorems of continuous dependence of the solutions of differential systems, there exists an $\epsilon > 0$, $\sigma > 0$ such that $\forall s \in [0, \sigma[$, $\forall y \in \Omega_\epsilon$ the maximal interval of existence of the solution $x(t, s, y)$ contains $[0, T]$.

By the previous considerations, it is well defined the Poincaré map:

$$P: \begin{array}{lcl} [0,\sigma[\times\Omega_\epsilon & \to & \mathbb{R}^n \\ (s,y) & \rightsquigarrow & x(T,s,y) \end{array} \tag{16}$$

which is $\mathcal{C}^2$, since F is a $\mathcal{C}^2$ map. Let $\mathscr{P}: [0,\sigma[\times\Omega_\epsilon \to \mathbb{R}^n$ be the map defined by

$$\mathscr{P}(s,y) = P(s,y) - y = x(T,s,y) - y\ . \tag{17}$$

By the T-periodicity of $\tilde{x}$ and (15), it is

$$\mathscr{P}(0,\tilde{x}(0)) = x(T,0,\tilde{x}(0)) - \tilde{x}(0) = 0\ . \tag{18}$$

Moreover we want to show that

$$\det(\partial_y \mathscr{P}(0,\tilde{x}(0))) = \det(\partial_y P(0,\tilde{x}(0)) - I) \neq 0\ , \tag{19}$$

where I is the identity matrix. In fact $\forall i = 1,\ldots,n$

$$\partial_{y_i} P(s,y) = \partial_{y_i} x(T,s,y),\ \forall (s,y) \in [0,\sigma[\times\Omega_\epsilon\ .$$

By (8) and the regularity of F, $\partial_{y_i} x(T,s,y)$ is the solution of the following system

$$\begin{cases} \dot{\phi}_i = \partial_x F(t,s,x(t,s,y))\phi_i\ , \\ \phi_i(0) = e_i\ , \end{cases} \tag{20}$$

where e_i, $i = 1,\ldots,n$ is the i-th element of the canonical basis of $\mathbb{R}^n$ (that is the i-th column of the n-dimensional identity matrix). Therefore $\forall (s,y) \in [0,\sigma[\times\Omega_\epsilon$, $\partial_y P(s,y) = \Phi(T,s,y)$, where $\Phi(t,s,y)$ is the matrix solution of system

$$\begin{cases} \dot{\Phi} = \partial_x F(t,s,x(t,s,y))\Phi, \\ \Phi(0) = I\ . \end{cases} \tag{21}$$

Then $\partial_y \mathscr{P}(0,\tilde{x}(0)) = \Phi(T,0,\tilde{x}(0))$, which implies (19) by hypothesis (10).

Therefore, by the Implicit Function Theorem, there exists $\hat{s}, \omega_0 > 0$ and a unique map $y \in \mathcal{C}^1([0,\hat{s}], B(\tilde{x}(0),\omega_0))$ such that

$$\begin{cases} y(0) = \tilde{x}(0), \\ \mathscr{P}(s,y(s)) = 0, \forall s \in [0,\hat{s}]\ , \end{cases} \tag{22}$$

in other words it is

$$\begin{aligned} &\{(s,y(s)) \mid 0 \leq s \leq \hat{s}\} = \\ &= \{(s,y) \in [0,s_0[\times B(\tilde{x}(0),\omega_0) \mid \mathscr{P}(s,y) = 0\} \end{aligned}$$

and

$$\det \partial_y \mathscr{P}(s,y(s)) \neq 0,\ \forall s \in [0,\hat{s}]\ . \tag{23}$$

Differentiating (22) on $[0,\hat{s}]$ we get that $y(s)$ verifies the following differential system:

$$\begin{cases} \dot{y}(s) = (I - \partial_y \mathscr{P}(s,y(s))^{-1}\partial_s \mathscr{P}(s,y(s)), \forall s \in [0,\hat{s}] \\ y(0) = \tilde{x}(0)\ . \end{cases} \tag{24}$$

Remark that $\partial_s \mathscr{P}(s,y) = \partial_s P(s,y) = \partial_s x(T,s,y)$ and by the regularity of F, $\partial_s x(T,s,y) = Z(T,s,y)$ where $Z(t,s,y)$ is the solution of the following system:

$$\begin{cases} \dot{Z} = \partial_x F(t,s,x(t,s,y))Z + \partial_s F(t,s,x(t,s,y)) \\ Z(0) = 0\ , \end{cases}$$

that is

$$\begin{aligned} &\partial_s P(s,y(s)) = \Phi(T,s,y(s))\cdot \\ &\cdot \int_0^T \Phi^{-1}(p,s,y(s))\partial_s F(p,s,x(p,s,y(s)))dp\ . \end{aligned}$$

In other words $y(s)$ is the solution of

$$\begin{cases} \dot{y}(s) = (I - \Phi(T,s,y(s))^{-1}\Phi(T,s,y(s))\cdot \\ \cdot \int_0^T \Phi^{-1}(p,s,y(s))\partial_s F(p,s,x(p,s,y(s)))dp\ , \\ y(0) = \tilde{x}(0)\ , \forall s \in [0,\hat{s}], \end{cases}$$

which is the same Cauchy problem as (11), then setting

$$\phi(s) = y(s),\ \forall s \in [0,\hat{s}]\ , \tag{25}$$

we have shown that there exists $\hat{s} > 0$ and a function $\phi \in \mathcal{C}^1([0,\hat{s}],\mathbb{R}^n)$ such that properties 1), 2) and 3) are verified.

Therefore $\bar{s} > 0$ and by uniqueness of solution of system (11) (remark that F is $\mathcal{C}^2$) there exists a unique $\phi \in \mathcal{C}^1([0,\bar{s}[,\mathbb{R}^n)$ such that (11) is verified on $[0,\bar{s}[$,

the solution $t \rightsquigarrow x(t,s,\phi(s))$ is defined on $[0,T]$ and

$$\det\ (I - \Phi(T,s,\phi(s))) \neq 0,\ \forall s \in [0,\bar{s}[\ . \tag{26}$$

Moreover we have also shown that there exists $\hat{s} \leq \bar{s}$ such that

$$\mathcal{P}(s,\phi(s)) = 0,\ \forall s \in [0,\hat{s}]\ ,$$

that is

$$x(T,s,\phi(s)) = \phi(s),\ \forall s \in [0,\hat{s}]\ , \tag{27}$$

in other words $\{x(t,s,\phi(s))\}_{0\leq s\leq\hat{s}}$ is a family of T-periodic solutions. Let s_M be the supremum of $\hat{s}$ such that property (27) is verified. By definition, $s_M \leq \hat{s}$. The theorem will be proved if we show that $s_M = \bar{s}$. By contradiction suppose that $s_M < \bar{s}$. Let $\{s_n\}_{n\in\mathbb{N}} \subset [0,s_M[$ such that $\lim_{n\to+\infty} s_n = s_M$, by definition of s_M, $\{x(t,s_n,\phi(s_n))\}_{n\in\mathbb{N}}$ is a sequence of T-periodic solution of system (12). This sequence converges uniformly to $x(t,s_M,\phi(s_M))$ on $[0,T]$ by the theorems on the continuous dependence of the solution on the initial data and the parameter s. Moreover $x(t,s_M,\phi(s_M))$ is T-periodic since

$$\begin{aligned} &\phi(s_M) = \lim_{n\to+\infty} \phi(s_n) = \\ &= \lim_{n\to+\infty} x(T,s_n,\phi(s_n)) = x(T,s_M,\phi((s_M)))\ , \end{aligned}$$

and $\det(I - \Phi(T,s_M,\phi(s_M))) \neq 0$ by (26), then applying the same reasoning as before with $\phi(s_M)$ instead of $\tilde{x}(0)$ and with $x(t,s_M,\phi(s_M))$ instead of $\tilde{x}(t)$, there exists an $\epsilon > 0$ such that

$$x(t,s,\phi(s)) = \phi(s),\ \forall s \in [s_M, s_M + \epsilon]\ ,$$

which contradicts the definition of s_M. $\square$

IV. On the computation of $\phi(s)$

This section discusses some numerical issues related to the dynamic inversion method presented in this paper. From the proof of the main theorem, we can deduce that $\phi(s)$ is computed by

$$\begin{cases} \dot{\phi}(s) = (I - \Phi(T,s,\phi(s)))^{-1} w(T,s,\phi(s)) \\ \phi(0) = \tilde{x}(0) \end{cases} \tag{28}$$

where $w(t,s,y)$ and $\Phi(t,s,y)$ are the solutions of the following system

$$\begin{cases} \dot{w}(t) = \partial_x F(p,s,x(p,s,y))w(t) + \partial_s F(p,s,x(p,s,y)) \\ \dot{\Phi} = \partial_x F(t,s,x(t,s,y))\Phi, \\ \dot{x} = F(t,s,x)\ , \\ w(0) = 0,\ \Phi(0) = I,\ x(0) = y\ . \end{cases} \tag{29}$$

Each evaluation of $\dot{\phi}(s)$ requires the numerical computation of (29). The method is therefore computationally intensive because involves the solution of a large number of differential equations.

There are two main numerical problems related to the solution of (28) in systems in which the internal dynamics are hyperbolic, such as the spherical pendulum. First, computation of $\dot{\phi}(s)$ needs inverting matrix $(I - \Phi(T,s,\phi(s)))$, which is ill-conditioned because $\Phi(T,s,\phi(s))$ has stable and unstable eigenvalues which decrease and respectively grow exponentially with respect to period time T, (the ill-conditioning becomes more relevant for larger values of T). Second, the solution of (28) is very sensitive to numerical errors and a very small error in computation of function ϕ can produce strong deviations in the resulting internal dynamics.

We present here a computational method that helps facing the two problems outlined above. Roughly speaking the first problem is addressed by avoiding the direct computation of the inverse of $(I - \Phi(T,s,\phi(s)))$, the second problem is taken into account by adding an error feedback term to (28) that keeps the numerical error small.

I) Set $t_0 : 0 < t_0 < T$ and define $\chi(t,s,y) = (I - \Phi(t,s,y))^{-1}$, $\forall t \in [t_0, +\infty[$, remark that $t_0 > 0$ because $\chi(0,s,y)$ is not defined, being $\Phi(0,s,y) = I$. Differentiating $\chi(t,s,y)$ with respect to t and using the fact that for an invertible matrix function $A(t)$, $\frac{dA^{-1}}{dt}(t) = -A^{-1}(t)\frac{dA}{dt}(t)A^{-1}(t)$ we obtain

$$\begin{aligned} \dot{\chi}(t,s,y) = -\chi(t,s,y)\big(&-\partial_x F(t,s,x(t,s,y))\cdot \\ &\cdot\Phi(t,s,y)\big)\chi(t,s,y) = \\ = -\chi(t,s,y)&\big(\partial_x F(t,s,x(t,s,y))\cdot \\ \cdot(-I + &(I - \Phi(t,s,y))\big)\chi(t,s,y) = \\ = \chi(t,s,y)\partial_x F&(t,s,x(t,s,y))(\chi(t,s,y) - I)\ . \end{aligned}$$

Set $\forall t \in [t_0, +\infty[$, $z(t,s,y) = (I - \Phi(t,s,y))^{-1}w(t,s,y) = \chi(t,s,y)w(t,s,y)$, differentiating with respect to t , by (30), we get:

$$\begin{aligned} \dot{z}(t,s,y)(t) = \dot{\chi}(t,s,y)&w(t,s,y) + \\ +\chi(t,s,y)\big(\partial_x F(t,s,&x(t,s,y))w(t,s,y) + \\ +\partial_s F(t,s,x(t,s,y))\big) = \chi&(t,s,y)\big(\partial_x F(t,s,x(t,s,y))\cdot \\ \cdot z(t,s,y) + \partial_s F&(t,s,x(t,s,y))\big)\ . \end{aligned}$$

By the previous considerations we deduce that:

$\dot{\phi}(s) = z(T)$, where $z(t)$ is the first component of the solution (z,χ,x) of the following system

$$\begin{cases} \dot{z}(t) = \chi(t)(\partial_x F(t,s,\xi(t))z(t) + \\ \quad +\partial_s F(t,s,\xi(t)) \text{ on } [t_0,T]\ , \\ \dot{\chi}(t) = \chi(t)\partial_x F(t,s,\xi(t))(\chi(t) - I) \text{ on } [t_0,T]\ , \\ \dot{\xi}(t) = F(t,s,\xi(t))\ , \\ z(t_0) = (I - \Phi(t_0,s,\phi(s)))^{-1}w(t_0), \\ \chi(t_0) = (I - \Phi(t_0,s,\phi(s)))^{-1}, \\ \xi(t_0) = x(t_0,s,\phi(s))\ . \end{cases} \tag{30}$$

To determine $\Phi(t_0,s,\phi(s)))^{-1}$, $w(t_0)$ and $x(t_0,s,\phi(s))$, we integrate system (29) in the interval $[0,t_0]$, with t_0 sufficiently small such that matrix $(I - \Phi(t_0,s,\phi(s)))$ is still well-conditioned and, for larger values of t (that is on $[t_0,T]$), the integration is continued using (30) where matrix $(I - \Phi(t,s,\phi(s)))$ is ill-conditioned.

II) As stated in (14), the property $\phi(s) = x(T,s,\phi(s))$ holds, $\forall s \in [0,\bar{s}[$ but, the error function $e(s) = \phi(s) - x(T,s,\phi(s))$ can become significant because the numerical error in the computation of $\phi(s)$ induces a large error on the final state $x(T,s,\phi(s))$ because of the instability of the system internal dynamics.

This problem can be taken into account by substituting system (28) with the following one

$$\begin{cases} \dot{\phi}(s) = \tilde{z}(T) - \chi(T)Ke(s) \\ \phi(0) = \tilde{x}_0\ , \end{cases} \tag{31}$$

where $\tilde{z}(T)$ is the result of the numerical computation of $z(T)$ obtained with the method presented in I) and K is a positive definite gain matrix.

Let us define $\eta(s) = \tilde{z}(T) - z(T)$ the numerical error term, then differentiating $e(s)$ with respect to s, by (31), we get

$$\dot{e} = \big(I - \Phi(T,s,\phi(s))\big)\big(\eta(s) - \chi(T)Ke(s)\big)$$

that is e verifies the following system

$$\begin{cases} \dot{e} = -Ke(s) + (I - \Phi(T,s,\phi(s)))\eta(s) \\ e(0) = 0\ , \end{cases}$$

therefore for a given T, the numerical error can be made arbitrarily small by taking the gain matrix K with the minimum eigenvalue sufficiently large.

V. Simulation results

We set the length of the pendulum rod $l = 1$ and consider the following eight-shaped 2π-periodic trajectory in $\mathbb{R}^3$ (see Figure 2)

$$\gamma(t) = \begin{pmatrix} 4\sin t \\ \sin(2t) \\ \sin t \end{pmatrix}\ .$$

We found a 2π-periodic solution of the internal dynamics (5) using the homotopy method outlined in section II. The initial state $(z(0), w(0), \dot{z}(0), \dot{z}(0))$ is given by $\phi(1)$ and has been computed numerically solving differential equation (11), since $\bar{s} > 1$ as found in the computation. Note that the right term of this equation is computed by solving differential equation (7), therefore this approach is based on the solution of two "nested" differential equations.

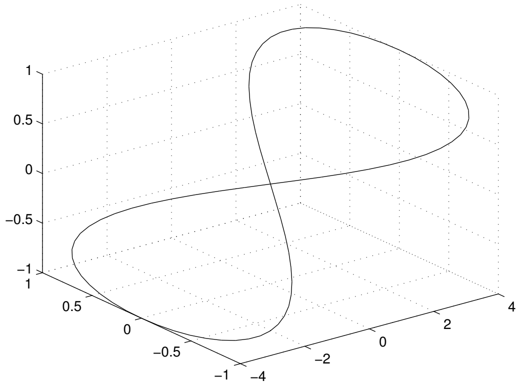

Fig. 2. Eight-shaped curve in $\mathbb{R}^3$.

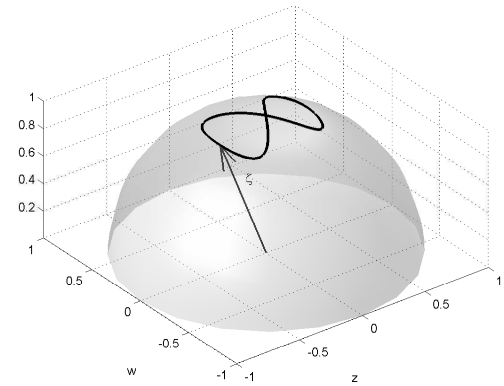

Fig. 4. Function $\zeta(t) \in S^2_+$.

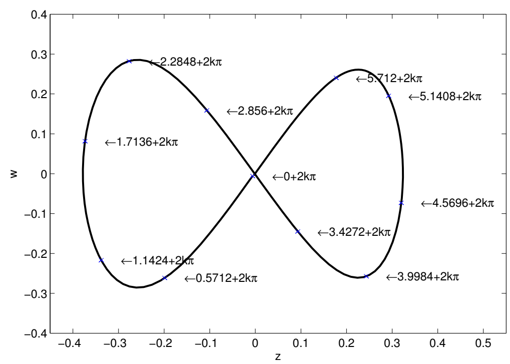

Fig. 3. Internal dynamics projected on plane (z, w).

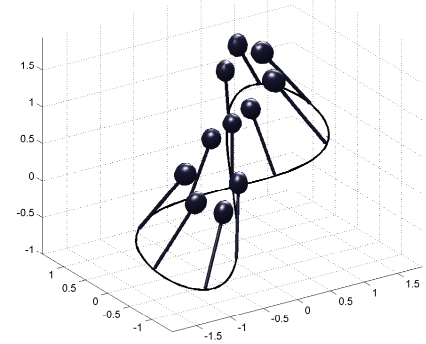

Fig. 5. Pendulum motion.

The resulting 2π-periodic trajectories for internal dynamics projected on plane (z, w) are shown in Figure 3. Figure 4 represents the attitude versor ζ and Figure 5 represents the pendulum motion along the curve.

VI. Conclusions

We have presented a method for non causal dynamic inversion for nonminimum phase nonlinear systems and discussed a numerical implementation. The approach has been applied for solving the exact output tracking problem for the spherical pendulum. In paper [12] we have proved for the planar vertical pendulum that the problem can always be solved if the second derivative of γ is sufficiently small. A similar result holds for the spherical pendulum, however the proof is more involved and will be presented in future papers.

References

[1] A. S. Shiriaev, H. Ludvigsen, O. Egeland, Swinging up the spherical pendulum via stabilization of its first integrals, Automatica (40) (2004) 73–85.

[2] B. A.M., D. E. Chang, L. N.E., J. Marsden, Controlled lagrangians and the stabilization of mechanical systems. ii. potential shaping, Automatic Control, IEEE Transactions on 46 (10) (Oct 2001) 1556–1571.

[3] N. A. Chaturvedi, N. H. McClamroch, Asymptotic stabilization of the hanging equilibrium manifold of the 3d pendulum, Int. J. Robust Nonlinear Control 17 (2007) 1435–1454.

[4] L. Guangyu, M. Iven, N. Dragan, Way point tracking of a spherical inverted pendulum, Control and Automation, IEEE International Conference on (May 30 2007-June 1 2007) 654–659.

[5] G. Liu, D. Nesic, I. Mareels, Non-linear stable inversion-based output tracking control for a spherical inverted pendulum, International Journal of Control 81 (1) (2008) 116–133.

[6] S. Devasia, D. Chen, B. Paden, Nonlinear inversion-based output tracking, IEEE Transactions on Automatic Control AC-41 (7) (1996) 930–942.

[7] P. T. Kabamba, R. W. Longman, J. Sun, A homotopy approach to the feedback stabilization of linear systems,, Journal of Guidance, Control, and Dynamics 10 (5) (1987) 422–432.

[8] J. Dunyak, J. L. Junkins, L. T. Watson, Robust nonlinear least squares estimation using the chow-yorke homotopy method, Journal of Guidance, Control, and Dynamics (1984) 752–755.

[9] V. V. Dikoussar, Continuation methods in boundary value problems (1994) 65–70.

[10] K. Reif, K.Weinzierl, A. Z. R. Unbehauen, A homotopy approach for nonlinear control synthesis, Automatic Control, IEEE Transactions on 43 (9) (1998) 1311–1318.

[11] L. Consolini, M. Tosques, On the vtol aircraft exact tracking with bounded internal dynamics via a poincarè map approach, IEEE Trans. On Automatic Control 52 (9) (2007) 1757–1762.

[12] L. Consolini, M. Tosques, On the existence of small periodic solutions for the 2-dimensional inverted pendulum on a cart, SIAM Journal on Applied Mathematics 68 (2) (2007) 486–502.

[13] L. Consolini, M. Tosques, A morphing method for exact tracking control of nonminimum phase non linear systems, in: Proc. of the IEEE Conference on Decision and Control, 2007, pp. 5240–5245.