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Improving Convergence of Iterative Feedback Tuning using Optimal External Perturbations

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Abstract— **Iterative Feedback Tuning constitutes an attractive control loop tuning method for processes in the absence of sufficient process insight. It is a purely data driven approach to optimization of the loop performance. The standard formulation ensures an unbiased estimate of the loop performance cost function gradient, which is used in a search algorithm.**

A slow rate of convergence of the tuning method is often experienced when tuning for disturbance rejection. This is due to a poor signal to noise ratio in the process data. A method is proposed for increasing the information content in data by introducing an optimal perturbation signal in the tuning algorithm. For minimum variance control design the optimal design of an external perturbation signal is derived in terms of the asymptotic accuracy of the Iterative Feedback Tuning method.

I. INTRODUCTION

Control design and tuning for disturbance rejection is one of the classic disciplines in control theory and control engineering science. Design of compensators for disturbance rejections is well documented e.g. [1], [2]. Given a particular control design, the tuning of the parameters in the controller can be conducted based on tuning rules or by minimization of some loop performance criterion. The performance criterion is typically a quadratic cost function with penalty on the process output and the control signals. Given a model of the system, the set of optimal control parameters which minimize the performance cost can be determined. In the absence of a sufficiently reliable model, the tuning can be performed based on data obtained from the loop, by a data driven optimization. Iterative Feedback Tuning is a method for optimizing controller parameters using closed loop data and this algorithm will form the basis for the results presented here. The basic algorithm was first presented in [3] and have been analyzed, extended and tested in a number of papers. [4] and [5] provide an extensive overview of the development of the method. This paper will propose a method for designing an optimal external pertubation signal which will inprove convergence of the tuning method.

The performance criterion used in the controller tuning is a function of the true system, the controller and external signals acting on the loop. The stochastic nature of the external signals implies that the performance cost is itself a random variable. For a linear system the minimum of the performance cost is explicitly defined when the number of data samples approach infinity.

Any change in the spectrum from an external signal to the process, Φ_r, will affect the spectrum of the process output Φ_y as well, hence also the minimum and the shape of the performance cost surface. By designing the spectrum of an external reference it is consequently possible to shape the performance cost function in order to improve the convergence properties of the search algorithm in the tuning method for the controller parameters. One has to be aware of, that shaping the cost function will also influence the location of the minimum in the controller parameter space. The cost function evaluated with external perturbation will be different from that of the original design problem. This is illustrated in figure 1 where two examples of a quadratic cost function are shown as function of two control parameters. Let the

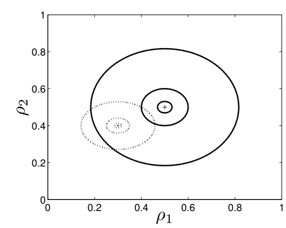

Fig. 1. Contours and minima for two cost functions with equal levels for the contour lines. ρ_1 and ρ_2 are parameter in the controller. The full lines and the cross refer to the original design criterion, J_0. The dotted lines and the dot in the center is the cost function when affected by an external perturbation, J_1.

original design J_0 refer to the disturbance rejection case where the reference signal to the loop is zero. J_1 is then the cost function for the case with external perturbation. Since the contour lines of J_1 are closer together than for J_0, the optimization with the perturbation is less sensitive to the stochastic element in the evaluation of the performance cost. The price is that the method converges towards a different minimum. Despite this unfortunate property, successful simulation studies are reported with respect to convergence using perturbation in Iterative Feedback Tuning for disturbance rejection [6]. The aim of this study is to design the optimal external perturbation signal spectrum when tuning a minimum variance controller with Iterative Feedback Tuning for the disturbance rejection problem.

The paper is organized as follows: Section II presents the basic Iterative Feedback Tuning algorithm for disturbance rejection together with an analysis of the accuracy of the

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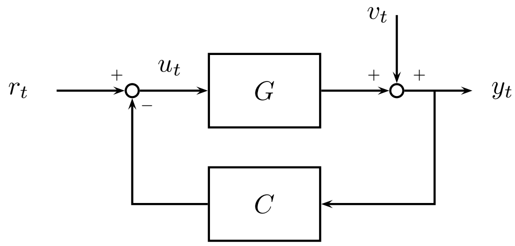


Fig. 2. A general feedback loop designed for disturbance rejection. The process, G, and the compensator in the feedback loop, C, is given as scalar linear transfer functions.

method. In section III the effect of having an external perturbation signal as the reference to the loop in the tuning method is presented. Section IV states the optimization problem for the parameterization of the perturbation signal. A simulation example in section V serves to illustrate the advantages of introducing an optimal external perturbation signal in the tuning algorithm for the disturbance rejection case.

II. ITERATIVE FEEDBACK TUNING FOR DISTURBANCE REJECTION

The algorithm for performing Iterative Feedback Tuning for disturbance rejection is illustrated in the following. The feedback loop in figure 2 depicts the signals and transfer functions which will be used in the algorithm for tuning the parameters $\boldsymbol{\rho}$ in C. The tuning is performed such that the effect of the noise, v_t, is rejected in an optimal sense. The objective in this paper is to minimize the minimum variance cost function:

$$J(\boldsymbol{\rho}_i) = \frac{1}{2N}\sum_{t=1}^{N}(y_t(\boldsymbol{\rho}_i) - y_t^d)^2 \tag{1}$$

where N number of data points in the discrete time horizon and y^d is the desired output response. For the disturbance rejection problem $r_t = 0$ and hence $y_t^d = 0$. The sensitivity of the cost function with respect to the controller parameters is

$$\frac{\partial J(\boldsymbol{\rho}_i)}{\partial \boldsymbol{\rho}} = \frac{1}{N}\sum_{t=1}^{N} y_t(\boldsymbol{\rho}_i)\frac{\partial y_t(\boldsymbol{\rho}_i)}{\partial \boldsymbol{\rho}} \tag{2}$$

where

$$\frac{\partial y_t}{\partial \boldsymbol{\rho}} = -\frac{\partial C(\boldsymbol{\rho}_i)}{\partial \boldsymbol{\rho}} G S^2(\boldsymbol{\rho}_i) v_t \tag{3}$$

when $S(\boldsymbol{\rho})$ is the sensitivity function. The minimization of the cost function is realized by iterating in the gradient scheme

$$\boldsymbol{\rho}_{i+1} = \boldsymbol{\rho}_i - \gamma R^{-1}\frac{\partial J(\boldsymbol{\rho}_i)}{\partial \boldsymbol{\rho}} \tag{4}$$

where R is a positive definite matrix. It could be chosen as the Hessian of the cost function with respect to the controller parameters, $\boldsymbol{\rho}$, or the identity matrix to achieve a Newton or a steepest decent algorithm respectively. If a model for the system is unknown, the gradient of the output and hence the cost function gradient can not be evaluated analytically. An estimate of the performance cost function gradient is

$$\widehat{\frac{\partial J(\boldsymbol{\rho}_i)}{\partial \boldsymbol{\rho}}} = \frac{1}{N}\sum_{t=1}^{N} y_t(\boldsymbol{\rho}_i)\widehat{\frac{\partial y_t(\boldsymbol{\rho}_i)}{\partial \boldsymbol{\rho}}} \tag{5}$$

where $\widehat{\frac{\partial y_t(\boldsymbol{\rho}_i)}{\partial \boldsymbol{\rho}}}$ is a estimate of (3). In the traditional Iterative Feedback Tuning framework the minimization of the cost function, (1), is based on data from two successive experiments [7].

- Collect data $\{y_t^1(\boldsymbol{\rho}_i)\}_{t=1,..,N}$ where $r_t^1 = 0$
- Collect data $\{y_t^2(\boldsymbol{\rho}_i)\}_{t=1,..,N}$ where $r_t^2 = -y_t^1$

This data is used to estimate the gradient of the output.

$$\widehat{\frac{\partial y_t}{\partial \boldsymbol{\rho}}} = \frac{\partial C(\boldsymbol{\rho}_i)}{\partial \boldsymbol{\rho}} y_t^2 \tag{6}$$

$$= \frac{\partial y_t}{\partial \boldsymbol{\rho}} + \frac{\partial C(\boldsymbol{\rho}_i)}{\partial \boldsymbol{\rho}} S(\boldsymbol{\rho}_i) v_t^2 \tag{7}$$

where (6) is the estimator for the gradient. When this expression is used to form the estimate for the performance cost function gradient (5), (7) imply that the estimate can be split into two terms: A analytic term, S_N, and a variance term, E_N. The latter term is due to the noise present in the second experiment.

$$\widehat{\frac{\partial J(\boldsymbol{\rho}_i)}{\partial \boldsymbol{\rho}}} = S_N(\boldsymbol{\rho}_i) + E_N(\boldsymbol{\rho}_i) \tag{8}$$

where

$$S_N(\boldsymbol{\rho}_i) = \frac{1}{N}\sum_{t=1}^{N} y_t^1(\boldsymbol{\rho}_i)\frac{\partial y_t(\boldsymbol{\rho}_i)}{\partial \boldsymbol{\rho}} \tag{9}$$

$$= \frac{1}{N}\sum_{t=1}^{N}(S(\boldsymbol{\rho}_i)v_t^1)\left(-\frac{\partial C(\boldsymbol{\rho}_i)}{\partial \boldsymbol{\rho}} G S(\boldsymbol{\rho}_i)^2 v_t^1\right)$$

$$E_N(\boldsymbol{\rho}_i) = \frac{1}{N}\sum_{t=1}^{N} y_t^1(\boldsymbol{\rho}_i)\left(\widehat{\frac{\partial y_t(\boldsymbol{\rho}_i)}{\partial \boldsymbol{\rho}}} - \frac{\partial y_t(\boldsymbol{\rho}_i)}{\partial \boldsymbol{\rho}}\right) \tag{10}$$

$$= \frac{1}{N}\sum_{t=1}^{N}(S(\boldsymbol{\rho}_i)v_t^1)\left(\frac{\partial C(\boldsymbol{\rho}_i)}{\partial \boldsymbol{\rho}} S(\boldsymbol{\rho}_i) v_t^2\right)$$

The expectation of the variance part is zero, since the noise in the first and second experiment is uncorrelated. The estimate of the cost function gradient produced by the Iterative Feedback Tuning method is therefore unbiased.

Given that the noise v is a zero mean, weakly stationary random signal, the key contribution in Iterative Feedback Tuning, is that it supplies an unbiased estimate of the cost function gradient, without requiring a plant model estimate, $\hat{G}$, [7]. Let the estimate, (5), be an unbiased and monotonically increasing function of $\boldsymbol{\rho}$. Using the estimate (5) in the gradient iteration (4) instead of the analytical expression (2), as a stochastic approximation method, will still make the algorithm converge to the expectation of the local minimizer provided that the sequence of γ_i in (4) fulfills condition (11) [8], [9].

$$\sum_{i=1}^{\infty}\gamma_i^2 < \infty, \quad \sum_{i=1}^{\infty}\gamma_i = \infty \tag{11}$$

This condition is fulfilled e.g. by having $\gamma_i = a/i$ where a is some positive constant.

A Gauss-Newton approximation of the Hessian to the performance cost function with respect to the controller parameters was suggested in [3]. This first order approximation can be estimated using the available signals from the tuning method

$$\hat{\mathbf{H}} = \frac{1}{N}\sum_{t=1}^{N} \widehat{\frac{\partial y_t}{\partial \boldsymbol{\rho}}} \left(\widehat{\frac{\partial y_t}{\partial \boldsymbol{\rho}}}\right)^T \tag{12}$$

A. Asymptotic accuracy of the tuning method

The variability of the gradient estimate will affect the asymptotic convergence rate of the tuning method. A quantitative analysis was performed by [10]. The result is as follows: With n being the iteration number and $\bar{\boldsymbol{\rho}}$ the optimal set of parameters, the sequence of random variables, $\sqrt{n}(\boldsymbol{\rho}_n - \bar{\boldsymbol{\rho}})$, converge *in distribution* to a normally distributed random variable with zero mean and covariance matrix $\boldsymbol{\Sigma}$.

$$\begin{aligned} &\sqrt{n}(\boldsymbol{\rho}_n - \bar{\boldsymbol{\rho}}) \xrightarrow{D} \mathcal{N}(0, \boldsymbol{\Sigma}) \\ &\boldsymbol{\Sigma} = a^2 \int_0^\infty e^{\boldsymbol{A}t}\boldsymbol{R}^{-1}\mathrm{Cov}\left[\widehat{\frac{\partial J(\bar{\boldsymbol{\rho}})}{\partial \boldsymbol{\rho}}}\right]\boldsymbol{R}^{-1}e^{\boldsymbol{A}^T t}dt \end{aligned} \tag{13}$$

The result in (13) is valid given the following set of conditions hold.

1) The sequence $\boldsymbol{\rho}_n$ converges to a local isolated minimum $\bar{\boldsymbol{\rho}}$ of J
2) $\boldsymbol{H}(\bar{\boldsymbol{\rho}})$ is the true Hessian for $J(\boldsymbol{\rho})$ at $\bar{\boldsymbol{\rho}}$.
3) The sequence $\boldsymbol{\gamma}$ of steps in (4) is given by $\gamma_n = a/n$, where a is a positive constant.
4) There exists an index $\bar{n}$ and a matrix $\boldsymbol{R}$ such that $\boldsymbol{R}_n = \boldsymbol{R}$ for all $n > \bar{n}$.
5) The matrix $A = 1/2\boldsymbol{I} - a\boldsymbol{R}^{-1}\boldsymbol{H}(\bar{\boldsymbol{\rho}})$ is stable, i.e. the real part of all the eigenvalues is negative.
6) The covariance matrix $\mathrm{Cov}\left[\widehat{\frac{\partial \boldsymbol{J}(\bar{\boldsymbol{\rho}})}{\partial \boldsymbol{\rho}}}\right]$ is positive definite.

The result in (13) means that asymptotically the distribution for the deviation between the n'th iterate of the controller parameter and the true optimum is known, and that the method converges to the true local minimizer of the performance cost function. In [11] it is shown that the covariance expression for the distribution simplifies if $\boldsymbol{H}(\bar{\boldsymbol{\rho}})$, i.e. the true Hessian, is used as the matrix $\boldsymbol{R}$ in (4). Hence for a Newton-Raphson optimization

$$\boldsymbol{\Sigma} = \frac{a^2}{2a-1}\boldsymbol{R}^{-1}\mathrm{Cov}\left[\widehat{\frac{\partial J(\bar{\boldsymbol{\rho}})}{\partial \boldsymbol{\rho}}}\right][\boldsymbol{R}^{-1}]^T \tag{14}$$

As a measure of the quality of the controller for a given iteration, n, in the tuning algorithm [11] suggest the difference between the expected value of the performance cost with $C(\boldsymbol{\rho}_n)$ in the loop minus the theoretical minimum value. This quantity, ΔJ_n, will be referred to as the *control quality index.*

$$\Delta J_n \equiv \mathrm{E}[J(\boldsymbol{\rho}_n)] - J(\bar{\boldsymbol{\rho}}) \tag{15}$$

This index is by definition a positive measure. Expanding it in a Taylor series around the optimum up till second order gives the approximation:

$$\Delta J_n \approx \frac{1}{2}\mathrm{E}\left[\Delta\bar{\boldsymbol{\rho}}_n^T \boldsymbol{H}(\bar{\boldsymbol{\rho}})\Delta\bar{\boldsymbol{\rho}}_n\right] \tag{16}$$

where $\Delta\bar{\boldsymbol{\rho}}_n = \boldsymbol{\rho}_n - \bar{\boldsymbol{\rho}}$. The following asymptotic expression when $\mathbf{H}(\bar{\boldsymbol{\rho}})\mathbf{R}^{-1} = \mathbf{I}$ is given in [11].

$$\begin{aligned} &\lim_{n\to\infty} n\mathrm{E}[\Delta\bar{\boldsymbol{\rho}}_n^T \boldsymbol{H}(\bar{\boldsymbol{\rho}})\Delta\bar{\boldsymbol{\rho}}_n] = \\ &\qquad \frac{a^2}{2a-1}\mathrm{trace}\left\{\mathrm{Cov}\left[\widehat{\frac{\partial J(\bar{\boldsymbol{\rho}})}{\partial \boldsymbol{\rho}}}\right][\boldsymbol{R}^{-1}]^T\right\} \end{aligned} \tag{17}$$

From this analysis, it is seen that the covariance of the gradient estimate for the performance cost function influences both the asymptotic covariance of the distribution of $\Delta\bar{\boldsymbol{\rho}}_n$ and the quality measure of the controller with parameters $\boldsymbol{\rho}_n$. It is therefore of interest to decompose this covariance expression. The covariance of the gradient estimate in equation (8) can be divided into the following contributions.

$$\mathrm{Cov}\left[\widehat{\frac{\partial J(\boldsymbol{\rho}_i)}{\partial \boldsymbol{\rho}}}\right] = \mathrm{Cov}[S_N(\boldsymbol{\rho}_i)] + \mathrm{Cov}[E_N(\boldsymbol{\rho}_i)] \tag{18}$$

due to the independence of the signals v_t^1 and v_t^2. The asymptotic frequency-domain expressions of the two terms are [10]:

$$\begin{aligned} \lim_{N\to\infty} N\mathrm{Cov}[S_N(\boldsymbol{\rho}_i)] &= \frac{2}{2\pi}\int_{-\pi}^{\pi} |S(e^{j\omega},\boldsymbol{\rho}_i)|^4 \Phi_v^2(\omega)\times \\ &\mathcal{R}e\left\{G(e^{j\omega},\boldsymbol{\rho}_i)S(e^{j\omega},\boldsymbol{\rho}_i)\frac{\partial C(e^{j\omega},\boldsymbol{\rho}_i)}{\partial \boldsymbol{\rho}}\right\}\times \\ &\mathcal{R}e\left\{G(e^{j\omega},\boldsymbol{\rho}_i)S(e^{j\omega},\boldsymbol{\rho}_i)\frac{\partial C(e^{j\omega},\boldsymbol{\rho}_i)}{\partial \boldsymbol{\rho}}\right\}^T d\omega \end{aligned} \tag{19}$$

$$\begin{aligned} \lim_{N\to\infty} N\mathrm{Cov}[E_N(\boldsymbol{\rho}_i)] &= \frac{1}{2\pi}\int_{-\pi}^{\pi} |S(e^{j\omega},\boldsymbol{\rho}_i)|^4 \times \\ &\frac{\partial C(e^{j\omega},\boldsymbol{\rho}_i)}{\partial \boldsymbol{\rho}}\frac{\partial C^*(e^{j\omega},\boldsymbol{\rho}_i)}{\partial \boldsymbol{\rho}}\Phi_v^2(\omega)d\omega \end{aligned} \tag{20}$$

III. INTRODUCING EXTERNAL PERTURBATIONS IN THE TUNING

It is desired to improve the convergence rate and the asymptotic accuracy of the Iterative Feedback Tuning method. To achieve this, the signal to noise ratio in data used in the tuning method must be increased. An external perturbation signal will be used as reference in the first of the two experiments used in the tuning algorithm. The experiments are then defined as follows:

- Collect data $\{y_t^1(\boldsymbol{\rho}_i)\}_{t=1,..,N}$ where $r_t^1 = r_t^p$
- Collect data $\{y_t^2(\boldsymbol{\rho}_i)\}_{t=1,..,N}$ where $r_t^2 = -y_t^1$

where the external input r_t^p is characterized by the spectrum Φ_{r^p}.

The implication on the gradient estimate of the cost function from including this extra signal is

$$S_N(\boldsymbol{\rho}_i) = \frac{1}{N}\sum_{t=1}^{N}\left(S(\boldsymbol{\rho}_i)(Gr_t^p + v_t^1)\right) \times \left(-\frac{\partial C(\boldsymbol{\rho}_i)}{\partial \boldsymbol{\rho}} GS(\boldsymbol{\rho}_i)^2(Gr_t^p + v_t^1)\right) \quad (21)$$

$$E_N(\boldsymbol{\rho}_i) = \frac{1}{N}\sum_{t=1}^{N}\left(S(\boldsymbol{\rho}_i)(Gr_t^p + v_t^1)\right)\left(\frac{\partial C(\boldsymbol{\rho}_i)}{\partial \boldsymbol{\rho}} S(\boldsymbol{\rho}_i)v_t^2\right) \quad (22)$$

The implications of introducing an external perturbation signal in the experiments used by the Iterative Feedback Tuning method has been investigated in [12]. This external signal will affect both the asymptotic accuracy for the gradient estimate and the Hessian estimate (12). The qualitative effect from introducing external perturbation on the relevant functions in relation to the covariance of the cost function gradient estimate is.

- The asymptotic expressions for S_N and E_N are affine functions in the following variables. $S_N = f(\Phi_{r^p}^2, \Phi_v^2, \Phi_{r^p}\Phi_v)$ and $E_N = f(\Phi_v^2, \Phi_{r^p}\Phi_v)$, hence the asymptotic covariance estimate is also an affine function in $\Phi_{r^p}^2$, Φ_v^2 and $\Phi_{r^p}\Phi_v$.
- The Hessian estimate is an affine function in Φ_{r^p} and Φ_v only.

A. Unbiased gradient estimation with perturbation

From the general feedback loop, figure 2, it is seen that the closed loop transfer functions are given by

$$y_t = GS(\boldsymbol{\rho}_i)r_t^p + S(\boldsymbol{\rho}_i)v_t \quad (23)$$

It would be interesting to have a design of r^p which would not change the dynamics in the response of y or u with respect to the inputs, compared to the unperturbed case. If $r_t^p = \sqrt{\alpha}/G v_t$ would be realizable, the output in (23) will simplify to

$$y_t = GS(\boldsymbol{\rho}_i)\frac{\sqrt{\alpha}}{G}v_t + S(\boldsymbol{\rho}_i)v_t = (1+\sqrt{\alpha})S(\boldsymbol{\rho}_i)v_t$$

which is only a scaled expression of the output for the unperturbed case. This implies that the perturbation signal chosen here will not render the gradient estimate unbiased. Hence it is optimal in the sense that this design will contribute to a better signal to noise ratio without driving the optimization of the controller parameters to a biased optimum compared to the unperturbed case. In practical applications the actual random disturbance signal is unknown but the spectrum of the disturbance may be known. If the perturbation signal is generated using a signal with equal spectral properties as Φ_v the expected value of the gradient estimates will still be unbiased. If r_t^p and v_t are uncorrelated the spectrum of the output is:

$$\Phi_y = |G(e^{j\omega})|^2|S(e^{j\omega}, \boldsymbol{\rho}_i)|^2\Phi_{r^p} + |S(e^{j\omega}, \boldsymbol{\rho}_i)|^2\Phi_v \quad (24)$$

Following the optimal designs which has just be argued

$$\Phi_{r^p} = \frac{\alpha}{|G(e^{j\omega})|^2}\Phi_v \Rightarrow \Phi_y = (1+\alpha)|S(e^{j\omega}, \boldsymbol{\rho}_i)|^2\Phi_v$$

From these expressions it is seen that the only requirement is the noise spectrum and the magnitude function $|G(e^{j\omega})|^2$ in order to produce a spectrum of the output which is scaled with $(1+\alpha)$, compared to the unperturbed case. Insuring that the spectrum is scaled, is a less strict requirement than having the signal y scaled. E.g. let the true system model contain a time delay such that $G(q) = q^{-k}\bar{G}(q)$. Since $|G(e^{j\omega})|^2 = |\bar{G}(e^{j\omega})|^2$, a perturbation signal generated by $r_t^p = \sqrt{\alpha}/\bar{G} v_t$ would only scale up Φ_y by $(1+\alpha)$ but

$$y_t = GS(\boldsymbol{\rho}_i)\frac{\sqrt{\alpha}}{\bar{G}}v_t + S(\boldsymbol{\rho}_i)v_t = (1+\sqrt{\alpha}q^{-k})S(\boldsymbol{\rho}_i)v_t$$

which will change the dynamic response and hence render the gradient estimate of the cost function, (1), biased. This result gives some information for generation of the optimal perturbation signal for disturbance rejection tuning of the minimum variance controller. It is necessary to have an input signal with the same spectral properties as the random disturbance acting on the system. Furthermore this signal will have to be filtered through the inverse of the true plant dynamics.

In practice it is not possible to generate an optimal perturbation signal since the plant dynamics is unknown. On the other hand, the analysis in this section offers an optimal design strategy for the perturbation signal in case a plant estimate and noise model is available.

B. Influence of the perturbation power

If the perturbation signal spectrum is chosen as $\Phi_{r^p} = (\alpha/|G(e^{j\omega})|^2)\Phi_v$ then α is the only free parameter, and it will determine the power of the signal.

Using perturbations in the tuning algorithm will influence the covariance matrix of the performance cost function gradient estimate and hence the expected performance of the n'th iteration. Since the covariance matrix is proportional to the squared spectrum of the perturbation signal, it will be proportional to α^2. The true Hessian of the performance cost function, used in evaluation of Σ and ΔJ_n, is independent of the perturbation, since this Hessian is evaluated at the optimum for the unperturbed problem. In practice the true Hessian is not known and has to be estimated from the same perturbed data. The Hessian estimate is proportional to the perturbation spectrum and hence α. By substitution of the true Hessian with this perturbed Hessian estimate in the expressions for Σ and ΔJ_n, it will be expected that Σ will approach a constant value when $\alpha \to \infty$ while the control quality index will grow linearly.

In case the perturbation signal is *kept constant* between iterations, the covariance expression for the performance cost will change. Since the perturbation signal does not change between iterations it will be regarded as a deterministic signal. Hence the multiplication between signals driven by the perturbation signal r_p will not contribute to the covariance. That implies that terms in S_N with the squared spectrum of the perturbation signal will be zero, and that the covariance expression for E_N remains unchanged [12]. Having the same realization for the perturbation signal will give a covariance expression for the performance cost gradient estimate which is proportional to the perturbation signal spectrum and not the spectrum squared. Hence *a constant perturbation signal produces a covariance matrix* Σ *which approaches zero as the power of the perturbation signal is increased.*

IV. THE OPTIMIZATION PROBLEM

Let $\boldsymbol{\rho}_n(\Phi_{r^p})$ be the resulting controller after n iterations with Iterative Feedback Tuning where external perturbations with the spectrum Φ_{r^p} were used in the first experiment in the tuning algorithm. $J(\boldsymbol{\rho}_n(\Phi_{r^p}),0)$ then expresses the performance of the loop with this controller for the unperturbed operation and $J(\boldsymbol{\rho}_n(\Phi_{r^p}),\Phi_{r^p})$ is the value when the operation is perturbed by r^p. Let $\bar{\boldsymbol{\rho}}(0)$ be the optimal set of controller parameters for the original tuning problem and $\bar{\boldsymbol{\rho}}(\Phi_{r^p})$ be the optimal set for the perturbed operation. The design criterion for the perturbation spectrum will then minimize the control quality index for the perturbed tuning:

$$\Delta J_n(\Phi_{r^p}) \equiv \mathrm{E}[J(\boldsymbol{\rho}_n(\Phi_{r^p}),0)] - J(\bar{\boldsymbol{\rho}}(0),0) \tag{25}$$

Let the perturbation spectrum, Φ_{r^p}, be characterized by the parameters in a vector η. Demanding a upper bound, σ_{max}, on the perturbation signal power, r_0, leads to the following formulation of the perturbation signal optimization problem.

$$\begin{aligned} \min_{\eta} \quad & \Delta J_n(\Phi_{r^p}) \\ s.t. \quad & r_0 \leq \sigma_{max} \end{aligned} \tag{26}$$

This optimization aims at minimizing the expected distance between the performances of the optimal controller and the controller from the tuning algorithm which is affected by the perturbation spectrum.

In case the optimal perturbation design from section III-A can be realized, α, is the only free parameter in η. The covariance expression for Σ has shown that this parameter has to be as large as possible. Hence the optimization problem with the input power constraint on the perturbation signal is well posed. In reality the true model is unknown or only crudely approximated when Iterative Feedback Tuning is performed. In such case the optimization problem in equation (26) serves as a guide for how to choose the parameterization of the perturbation spectrum. A more general discussion of this optimization problem is presented in [12].

V. AN EXAMPLE

A simulation study is preformed in order to illustrate the ideas and advantages of introducing external perturbations in the Iterative Feedback Tuning method when tuning for disturbance rejection. For simplicity the control loop used is a discrete-time linear time-invariant transfer function model, and the controller has only two adjustable parameters. The random disturbance acting on the system is Gaussian white

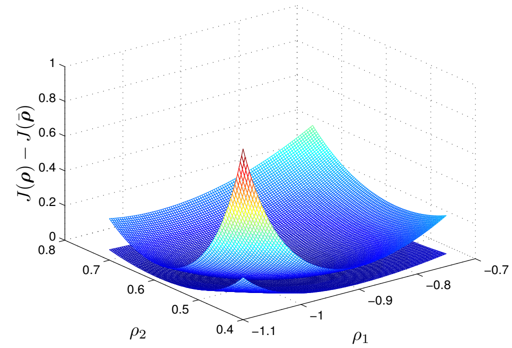


Fig. 3. Surface plot of the normalized performance cost function on a grid for the controller parameters. The lower surface is the performance cost when $\alpha = 0$ and the upper surface is for $\alpha = 1$. The same noise realization v_t has been used in each grid point and in the perturbation signal design in order to have a smooth surface.

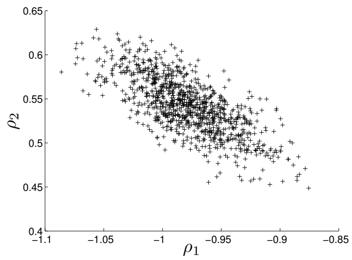
noise i.e. $e_t \in \mathcal{N}_{iid}(0,\sigma^2)$ where $\sigma = 1$. The nomenclature refers to the block diagram in figure 2 where $v_t = H(q)e_t$.

$$\begin{aligned} &\text{Plant model:} && G(q) = \frac{q^{-1} - 0.5q^{-2}}{1 - 0.3q^{-1} - 0.28q^{-2}} \\ &\text{Noise model:} && H(q) = \frac{1}{1 + 0.9q^{-1}} \\ &\text{Controller:} && C(q) = \rho_1 + \rho_2 q^{-1} \end{aligned} \tag{27}$$

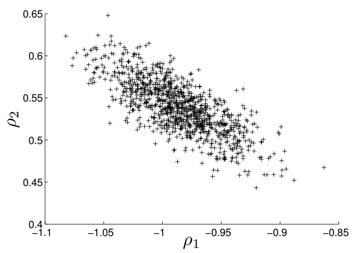
The external perturbation signal is given by $r_t^p = \sqrt{\alpha} H/G \epsilon_t$ where the plant model G and the noise model H is asumed to be known and $\epsilon_t \in \mathcal{N}_{iid}(0,1)$. An external perturbation will increase the value of the performance cost when applied. Figure 3 shows two normalized cost functions as surfaces on a grid of controller parameter values. These surface plots are smooth functions since the same noise realization has been used for each grid. Another constant signal with the same noise properties has been used for the perturbation signal design. The cost function is of course only a smooth function when the number of samples, $N \to \infty$, which is not practically realizable. In this simulation $N = 512$. The two surfaces have the same minimum in figure 3, and for this idealized case

$$J(\boldsymbol{\rho},\Phi_{r^p}) = (\sqrt{\alpha}+1)^2 J(\boldsymbol{\rho},0)$$

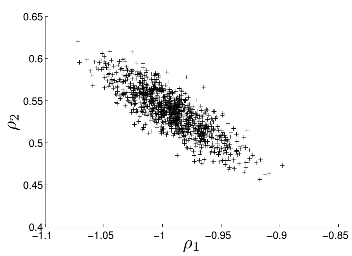
This property means that the perturbation gives the desired change in the curvature of the performance cost function to yield a faster convergence. In order to illustrate this result further a series of Monte Carlo experiments are performed using the Iterative Feedback Tuning with perturbations. Initially the controller parameters has the optimal value, but due to the stochastic nature of the data the tuning will move the parameters away from this value for repeated iterations. In 1000 experiments, 10 iterations have been performed from the same optimal starting point, and the values of the resulting set of parameters has been saved. $N = 1000$ data points has been collected and used in each iteration of the tuning. For four different values of α in the perturbation signal, the results are presented in figure 4. The variance and



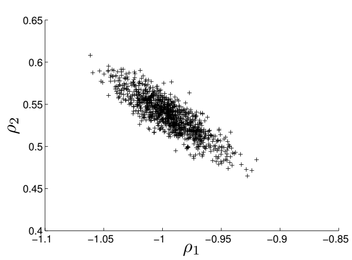
(a) $\alpha = 0$



(b) $\alpha = 1$



(c) $\alpha = 5$



(d) $\alpha = 10$

Fig. 4. The final control parameters from 1000 Monte Carlo experiments each with 10 iterations in the Iterative Feedback Tuning method. All iterations are initiated at the optimal value for the controller parameters. The value of α in scaling the perturbation signal has been changed in four steps from zero to 10.

the cross-covariance of the resulting controller parameters are reported in table I. Form the results of the Monte Carlo simulations in figure 4 and table I it is obvious that

TABLE I

THE VARIANCE AND THE CROSS-COVARIANCE FOR THE RESULTING SET OF CONTROLLER PARAMETERS FROM THE MONTE CARLO SIMULATIONS FOR DIFFERENT VALUES OF α IN THE PERTURBATION SIGNAL.

Variance	$\sigma^2_{\rho_1}$	$\sigma^2_{\rho_2}$	σ_{ρ_1,ρ_2}
$\alpha = 0$	0.00124	0.00103	-0.000817
$\alpha = 1$	0.00116	0.00102	-0.000820
$\alpha = 5$	0.000757	0.000743	-0.000623
$\alpha = 10$	0.000522	0.000531	-0.000451

increasing the value of α in the perturbation signal, produces an optimization problem with a statistically better defined optimum.

VI. CONCLUSIONS

A general framework for the use of external perturbation signals in Iterative Feedback Tuning of minimum variance control for the disturbance rejection problem has been presented. The effect on the asymptotic accuracy is disused in relation to the spectral properties of the external signal. It is shown that the optimal design of this perturbation depends on both the true system and the spectrum of the process noise. For practical applications where the plant model is unkown this design serves as a gide for designing the spectrum of an external pertubation in order to improve the tuning convergence. Through simulations the optimal design is shown not introduce bias. Increasing the input power in this signal will lead to a faster convergence of the tuning method due to a steeper curvature of the performance cost function.

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