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Multiobjective Controller Synthesis for Parameter Dependent Descriptor Systems via Dilated LMI Characterizations

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Abstract— **This paper proposes dilated LMI characterizations of admissibility, *D*-admissibility, H_∞ and H_2 norms for continuous-time descriptor systems. These dilated LMIs achieve less conservative results when dealing with robust admissibility/performance analysis of affine parameter-dependent descriptor systems. Furthermore, based on these conditions, the paper presents an iterative design procedure for multiobjective state feedback control of parameter dependent descriptor systems. The main idea underlying the proposed method is to linearize the products of controller parameter K and the auxiliary variable G by assigning a part of G. When initializing the proposed algorithm, the assignment takes into account an explicit characterization of G. The effectiveness of the proposed conditions and design method is shown through some numerical examples.**

I. Introduction

THE descriptor framework is very attractive for system modeling, as pointed out in [1], since it encompasses a wide class of systems. Descriptor models can preserve physical parameters in the coefficient matrices, and describe the dynamic part, static part, and even improper part of the system. Standard LMI characterizations for admissibility, *D*-admissibility (i.e. regularity, impulse immunity and all finite eigenvalues located in a prescribed convex region denoted by *D*), H_∞ and H_2 norms for descriptor systems are established in [2]~[10] (and references therein).

On the other hand, in the state-space case, the dilated (or extended) LMI characterizations enable us to use parameter-dependent Lyapunov functions for robust system analysis and synthesis ([11]~[16]) and independent Lyapunov functions for multiobjective control synthesis problems ([17]~[19]).

The first objective of this paper is to propose dilated LMI characterizations for continuous-time descriptor systems properties, such as admissibility, *D*-admissibility, H_∞ and H_2 norms. These conditions encompasses the dilated LMI characterizations in the state-space case. The connections with these results are given in this paper. Furthermore, we consider robust admissibility/performance analysis of a class of parameter-dependent descriptor systems whose coefficient matrices are affine functions of a parameter vector. Note that this class of descriptor systems is fairly general since it contains the class of state-space parameter-dependent models whose coefficient matrices are rational functions of a parameter vector (see for example [20]).

The second objective of this paper is to propose an iterative design method for multiobjective state feedback control of parameter-dependent descriptor systems. When using standard LMI characterizations for solving such problem, not only a common Lyapunov variable is enforced to convexify the synthesis problem but also only a constant Lyapunov variable is employed (otherwise gridding techniques of the parameter range are required). These restrictions inherently brings conservatism into the design. On the contrary, when using the dilated LMI conditions introduced in this paper, it is possible to achieve less conservative results since we can employ non-common parameter-dependent Lyapunov variables for each design specification. Nevertheless, in the second approach the products of the controller parameter K and the multipliers G have to be linearized. Thus, the key point of the procedure proposed in this paper, is to assign a part of G. In the initialization step, this assignment takes into account an explicit characterization of G in the descriptor case.

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II. Preliminaries

A. Basic facts on descriptor systems

Let us consider a descriptor system given by:

$$\begin{cases} E\dot{x} = Ax + Bw \\ z = Cx \end{cases} \qquad (1)$$

where $x \in \mathbb{R}^n$ is the descriptor variable, $w \in \mathbb{R}^q$ is the disturbance and $z \in \mathbb{R}^p$ is the controlled output. It is known that systems having direct transmission path from w to z can be transformed to (1) by augmenting the descriptor variable as pointed out in [1]. The matrix $E \in \mathbb{R}^{n\times n}$ may be singular and we denote its rank by $rank(E) = r \leq n$. If $\det(sE - A) \neq 0$ for some complex number s, then system (1) is said to be regular. A regular system of the form (1) is said to be impulse-free if:

$$\deg(\det(sE - A)) = rank(E)$$

Definition 1: [1] System (1) is admissible if it is regular and has neither impulsive modes nor unstable finite modes.

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Definition 2: [1] A regular descriptor system of the form (1) is finite dynamics stabilizable if there exists K such that $(E, A+BK)$ is admissible.

B. Standard LMI characterizations of admissibility, D-admissibility, H_2 and H_∞ norms

As defined above, the admissibility analysis of descriptor system (1) can be reduced to an LMI feasibility problem [3]. Let $E_0 \in \mathbb{R}^{n\times(n-r)}$ be any matrix of full-column rank such that $EE_0 = 0$.

Lemma 1: [3] System (1) is admissible if and only if there exist matrices $P > 0, P \in \mathbb{R}^{n\times n}$ and $Q \in \mathbb{R}^{(n-r)\times n}$ such that:

$$He\left\{A\left(PE^T + E_0Q\right)\right\} < 0 \quad (2)$$

Most of the simple regions of the complex plane for pole location that are useful in control application can be characterized in terms of LMI regions defined below.

Definition 3: A subset *D* of the complex plane is called an LMI region if there exist a symmetric matrix $R_1 \in \mathbb{R}^{d\times d}$ and a matrix $R_2 \in \mathbb{R}^{d\times d}$ such that

$$D = \left\{z \in \mathbb{C} : R_1 + R_2 z + R_2^T \bar{z} < 0\right\} \quad (3)$$

Definition 4: System (1) is *D*-admissible if it is regular, impulse free and its finite poles lie in *D*.

The following lemma recalls a necessary and sufficient condition for a system of the form (1) to be *D*-admissible.

Lemma 2:[6]System (1) is *D*-admissible if and only if there exist matrices $P > 0, P \in \mathbb{R}^{n\times n}$ and $Q \in \mathbb{R}^{(n-r)\times n}$ satisfying:

$$R_1 \otimes EPE^T + He\left\{R_2 \otimes APE^T\right\} + \Lambda \otimes \left(He\left\{AE_0Q\right\}\right) < 0 \quad (4)$$

Lemma 3: [11] For a given positive number γ_∞, the system (1) is admissible and

$$\left\|C\left(sE - A\right)^{-1} B\right\|_\infty < \gamma_\infty \quad (5)$$

holds if and only if there exist matrices $P > 0, P \in \mathbb{R}^{n\times n}$ and $Q \in \mathbb{R}^{(n-r)\times n}$ such that:

$$\begin{bmatrix} He\left\{A\left(PE^T + E_0Q\right)\right\} + BB^T & \left(PE^T + E_0Q\right)^T C^T \\ \bullet & -\gamma_\infty^2 I \end{bmatrix} < 0 \quad (6)$$

In order to ensure finiteness of the H_2 norm of system (1), we assume that the following condition holds ([2], [21])

$$KerC \supseteq KerE \quad (7)$$

Lemma 4: [2] For a given positive number γ_2, the descriptor system (1) with (7) is admissible and

$$\left\|C\left(sE - A\right)^{-1} B\right\|_2 < \gamma_2 \quad (8)$$

holds if and only if there exist matrices $P > 0, P \in \mathbb{R}^{n\times n}$ and $Q \in \mathbb{R}^{(n-r)\times n}$ such that:

$$\begin{aligned} & He\left\{A\left(PE^T + E_0Q\right)\right\} + BB^T < 0 \\ & trace\left\{CPC^T\right\} < \gamma_2^2 \end{aligned} \quad (9)$$

Other strict and non strict LMI versions of these lemmas exist in the literature (see, for instance, [7]).

III. DILATED LMI CHARACTERIZATIONS

Theorem 1: The descriptor system given by (1) is admissible if and only if there exist matrices $P_1 > 0, P_1 \in \mathbb{R}^{n\times n}$, $Q_1 \in \mathbb{R}^{(n-r)\times n}$, $Q_2 \in \mathbb{R}^{(n-r)\times n}$, $G_1 \in \mathbb{R}^{n\times n}$ and $G_2 \in \mathbb{R}^{n\times n}$ such that:

$$\begin{bmatrix} 0 & \left(P_1E^T + E_0Q_1\right)^T \\ \bullet & He\left\{E_0Q_2\right\} \end{bmatrix} + He\left\{\begin{bmatrix} A \\ -I \end{bmatrix}\begin{bmatrix} G_1 & G_2 \end{bmatrix}\right\} < 0 \quad (10)$$

Proof: System (1), with $w = 0$, is equivalent to the system given by

$$\begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix}\begin{pmatrix} \dot{x} \\ \dot{\zeta} \end{pmatrix} = \begin{bmatrix} 0 & A \\ I & -I \end{bmatrix}\begin{pmatrix} x \\ \zeta \end{pmatrix}, \zeta \in \mathbb{R}^n \quad (11)$$

According to Lemma 1, descriptor system (11) is admissible if and only if there exist matrices $P > 0, P \in \mathbb{R}^{2n\times 2n}$ and $Q \in \mathbb{R}^{(2n-r)\times 2n}$ such that (2) holds. Let us partition matrices P and Q such as:

$$P = \begin{bmatrix} P_1 & \bullet \\ P_2 & P_3 \end{bmatrix} \in \mathbb{R}^{2n\times 2n},\ P_1 > 0, P_1 \in \mathbb{R}^{n\times n}$$

and $$Q = \begin{bmatrix} Q_1 & Q_2 \\ Q_3 & Q_4 \end{bmatrix} \in \mathbb{R}^{(2n-r)\times(2n)},\ Q_1 \in \mathbb{R}^{(n-r)\times n} \quad (12)$$

(2) leads then to the following inequality

$$He\left\{\begin{bmatrix} 0 & A \\ I & -I \end{bmatrix}\left(\begin{bmatrix} P_1 & \bullet \\ P_2 & P_3 \end{bmatrix}\begin{bmatrix} E^T & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} E_0 & 0 \\ 0 & I \end{bmatrix}\begin{bmatrix} Q_1 & Q_2 \\ Q_3 & Q_4 \end{bmatrix}\right)\right\} < 0 \quad (13)$$

By denoting, $P_2E^T + Q_3 = G_1$ and $Q_4 = G_2$ it is easy to see that (13) is equivalent to the LMI (10). □

Remark 1: The result of Theorem 1 covers those appeared in the literature for admissibility of descriptor systems. In fact, by Theorem 1, (E, A) is admissible if and only if LMI (10) holds. Pre- and post-multiplying LMI (10) by $\begin{bmatrix} I & A \end{bmatrix}$ and $\begin{bmatrix} I & A \end{bmatrix}^T$ respectively leads to

$$He\left\{A\left(P_1E^T + E_0Q_1\right)\right\} + A\left(He\left\{E_0Q_2\right\}\right)A^T < 0 \quad (14)$$

By denoting $X = \left(P_1E^T + E_0\left(Q_1 + Q_2A^T\right)\right)^T$ (15)

(14) means that there exists matrix X such that $AX^T + XA^T < 0$ and $XE^T = EX^T \geq 0$. This is exactly the non strict LMI condition obtained in [7].

The result obtained for admissibility in Theorem 1 can be extended to the case of *D*-admissibility where the LMI region *D* is defined by (3).

Theorem 2: The descriptor system given by (1) is *D*-admissible if and only if there exist matrices

$P_1 > 0, P_1 \in \mathbb{R}^{n\times n}$, $Q_1 \in \mathbb{R}^{(n-r)\times n}$, $Q_2 \in \mathbb{R}^{(n-r)\times n}$, $G_1 \in \mathbb{R}^{n\times n}$, $G_2 \in \mathbb{R}^{n\times n}$ and $P_2 \in \mathbb{R}^{n\times n}$ such that:

$$R_1 \otimes \begin{bmatrix} E^T P_1 E & 0 \\ 0 & 0 \end{bmatrix} + He\left\{ R_2 \otimes \left(\begin{bmatrix} 0 & 0 \\ P_1 E^T & 0 \end{bmatrix} + \begin{bmatrix} A \\ -I \end{bmatrix} \begin{bmatrix} P_2 E^T & 0 \end{bmatrix} \right) \right\} + \Lambda \otimes He\left\{ \begin{bmatrix} 0 & 0 \\ E_0 Q_1 & E_0 Q_2 \end{bmatrix} + \begin{bmatrix} A \\ -I \end{bmatrix} \begin{bmatrix} G_1 & G_2 \end{bmatrix} \right\} < 0 \quad (16)$$

Proof: Follow the same lines as for Theorem 1.

Remark 2: The result on *D*-stability of state-space systems can be derived from Theorem 1 when considering $E = I$ and $E_0 = 0$.

Theorem 3: For a given positive number γ_∞, the system (1) is admissible and

$$\left\| C\left(sE - A\right)^{-1} B \right\|_\infty < \gamma_\infty \quad (17)$$

holds if and only if there exist matrices $P_1 > 0, P_1 \in \mathbb{R}^{n\times n}$, $Q_1 \in \mathbb{R}^{(n-r)\times n}$, $Q_2 \in \mathbb{R}^{(n-r)\times n}$, $G_1 \in \mathbb{R}^{n\times n}$ and $G_2 \in \mathbb{R}^{n\times n}$ such that:

$$\begin{bmatrix} 0 & \bullet & \bullet & \bullet \\ \phi & He\{E_0 Q_2\} & \bullet & \bullet \\ 0 & 0 & I & \bullet \\ 0 & 0 & 0 & -\gamma^2 I \end{bmatrix} + He\left\{ \begin{bmatrix} A & B \\ -I & 0 \\ 0 & -I \\ C & 0 \end{bmatrix} \begin{bmatrix} G_1 & G_2 & 0 & 0 \\ 0 & 0 & I & 0 \end{bmatrix} \right\} < 0 \quad (18)$$

where $\phi = P_1 E^T + E_0 Q_1$.

Remark 3: The above condition covers the condition stated in [17] for state-space systems. However, the last condition has an advantage due to the structure of the matrix

$$\begin{bmatrix} G_1^T & G_2^T & 0 & 0 \\ 0 & 0 & I & 0 \end{bmatrix}^T$$

which is not the case when using Finsler's lemma in the state-space case. Thus, in the case when $E = I$, that is, the descriptor system (1) reduces to a state-space system, the condition (18) can be more effective from the viewpoint of computational complexity.

Theorem 4: For a given positive number γ_2, the system (1) with (7) is admissible and

$$\left\| C\left(sE - A\right)^{-1} B \right\|_2 < \gamma_2 \quad (19)$$

holds if and only if there exist matrices $P_1 > 0, P_1 \in \mathbb{R}^{n\times n}$, $Q_1 \in \mathbb{R}^{(n-r)\times n}$, $Q_2 \in \mathbb{R}^{(n-r)\times n}$, $G_1 \in \mathbb{R}^{n\times n}$, $G_2 \in \mathbb{R}^{n\times n}$ and $Z \in \mathbb{R}^{p\times p}$ such that:

$$\begin{bmatrix} 0 & \bullet & \bullet \\ \phi & He\{E_0 Q_2\} & \bullet \\ 0 & 0 & I \end{bmatrix} + He\left\{ \begin{bmatrix} A & B \\ -I & 0 \\ 0 & -I \end{bmatrix} \begin{bmatrix} G_1 & G_2 & 0 \\ 0 & 0 & I \end{bmatrix} \right\} < 0 \quad (20)$$

$$\begin{bmatrix} Z & CP \\ \bullet & P \end{bmatrix} > 0, \; trace(Z) < \gamma_2^2 \quad (21)$$

where $\phi = P_1 E^T + E_0 Q_1$.

IV. Robust admissibility/performance analysis

A. Parameter dependent-descriptor systems

In this section, we consider a parameter-dependant descriptor system whose coefficient matrices are affine functions of a time-invariant uncertain parameter vector $\theta = \begin{bmatrix} \theta_1 & \cdots & \theta_l \end{bmatrix}^T$:

$$\begin{cases} E\dot{x} = A(\theta)x + Bw \\ z = Cx \end{cases}, \; x \in \mathbb{R}^n \quad (22)$$

where $\theta_i \in \left[\underline{\theta}_i, \overline{\theta}_i\right]_{i=1\ldots l}$. θ is supposed to belong to the hyper-rectangle: $\Xi = \left\{ \left(\omega_1, \ldots, \omega_{2^l}\right) \setminus \omega_i \in \left\{ \underline{\theta}_i \;\; \overline{\theta}_i \right\} \right\}$ (23)

The matrix $E \in \mathbb{R}^{n\times n}$ may be singular and we denote its rank by $rank(E) = r \le n$ independently of θ.

This class of systems is fairly general since parameter-dependent descriptor systems with matrices E, B and C depending affinely on θ can be represented by (22). In fact, for example, the parameter-dependent system:

$$\begin{cases} \begin{bmatrix} 2+\theta_1 & 0 \\ 0 & 0 \end{bmatrix} \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{bmatrix} 1+2\theta_2 & 1 \\ 2 & -\theta_1 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{bmatrix} -\theta_1 \\ 1 \end{bmatrix} w \\ z = \begin{bmatrix} 3-\theta_1 & 0 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \end{cases}, \; \theta_i \in [-1,1]_{i=1,2}$$

can be rewritten as:

$$\begin{cases} \begin{bmatrix} 2 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{\zeta}_1 \\ \dot{\zeta}_2 \\ \dot{\zeta}_3 \end{pmatrix} = \begin{bmatrix} 1+2\theta_2 & 1 & 0 & -1 & 0 \\ 2 & -\theta_1 & 0 & 0 & 0 \\ \theta_1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -\theta_1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \zeta_1 \\ \zeta_2 \\ \zeta_3 \end{pmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ -1 \end{bmatrix} w \\ z = \begin{bmatrix} 3 & 0 & -1 & 0 & 0 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \zeta_1 \\ \zeta_2 \\ \zeta_3 \end{pmatrix} \end{cases}, \; \theta_i \in [-1,1]_{i=1,2}$$

Furthermore, note that state-space models whose coefficient matrices are rational functions of the parameter vector θ can also be represented by (22). For example, the state-space model given by: $\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{bmatrix} 1+\theta_1 & 2\theta_2^2 \\ 1 & \dfrac{-\theta_1}{2+\theta_2} \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, $\theta_i \in [-1,1]_{i=1,2}$

can be rewritten as:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{\zeta}_1 \\ \dot{\zeta}_2 \end{pmatrix} = \begin{bmatrix} 1+\theta_1 & 0 & 2\theta_2 & 0 \\ 1 & 0 & 0 & -\theta_1 \\ 0 & \theta_2 & -1 & 0 \\ 0 & -1 & 0 & 2+\theta_2 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \zeta_1 \\ \zeta_2 \end{pmatrix}, \; \theta_i \in [-1,1]_{i=1,2}$$

B. Dilated LMI conditions for robust admissibility and H_∞ performance analysis

Corollary 1: The descriptor system (22) is robustly admissible if there exist matrices $P_1^i > 0, P_1^i \in \mathbb{R}^{n\times n}$, $Q_1^i \in \mathbb{R}^{(n-r)\times n}$, $Q_2^i \in \mathbb{R}^{(n-r)\times n}$, $G_1 \in \mathbb{R}^{n\times n}$ and $G_2 \in \mathbb{R}^{n\times n}$ such that: $\forall i \in \left\{1, \cdots, 2^l\right\}$

$$\begin{bmatrix} 0 & \left(P_1^i E^T + E_0 Q_1^i\right)^T \\ \bullet & He\left\{E_0 Q_2^i\right\} \end{bmatrix} + He\left\{\begin{bmatrix} A(\omega_i) \\ -I \end{bmatrix}\begin{bmatrix} G_1 & G_2 \end{bmatrix}\right\} < 0 \quad (24)$$

The following corollary assesses the robust H_∞ performance analysis of system (22).

Corollary 2: For a given positive number γ_∞, the system (22) is robustly admissible and

$$\left\| C\left(sE - A(\theta)\right)^{-1} B \right\|_\infty < \gamma_\infty \quad (25)$$

holds for all $\theta \in \Xi$, if there exist matrices $P_1^i > 0, P_1^i \in \mathbb{R}^{n\times n}$, $Q_1^i \in \mathbb{R}^{(n-r)\times n}$, $Q_2^i \in \mathbb{R}^{(n-r)\times n}$, $G_1 \in \mathbb{R}^{n\times n}$ and $G_2 \in \mathbb{R}^{n\times n}$ such that: $\forall i \in \left\{1, \cdots, 2^l\right\}$

$$\begin{bmatrix} 0 & \bullet & \bullet & \bullet \\ \phi_i & He\left\{E_0 Q_2^i\right\} & \bullet & \bullet \\ 0 & 0 & I & \bullet \\ 0 & 0 & 0 & -\gamma^2 I \end{bmatrix} + He\left\{\begin{bmatrix} A_i & B_i \\ -I & 0 \\ 0 & -I \\ C_i & 0 \end{bmatrix}\begin{bmatrix} G_1 & G_2 & 0 & 0 \\ 0 & 0 & I & 0 \end{bmatrix}\right\} < 0 \quad (26)$$

where $\phi_i = P_1^i E + E_0 Q_1^i$.

Corollary 3: For a given positive number γ_2, the system (22) is robustly admissible and

$$\left\| C\left(sE - A(\theta)\right)^{-1} B \right\|_2 < \gamma_2 \quad (27)$$

holds for all $\theta \in \Xi$, if there exist matrices $P_1^i > 0, P_1^i \in \mathbb{R}^{n\times n}$, $Q_1^i \in \mathbb{R}^{(n-r)\times n}$, $Q_2^i \in \mathbb{R}^{(n-r)\times n}$, $G_1 \in \mathbb{R}^{n\times n}$ and $G_2 \in \mathbb{R}^{n\times n}$ such that: $\forall i \in \left\{1, \cdots, 2^l\right\}$

$$\begin{bmatrix} 0 & \bullet & \bullet \\ \phi_i & He\left\{E_0 Q_2^i\right\} & \bullet \\ 0 & 0 & I \end{bmatrix} + He\left\{\begin{bmatrix} A(\omega_i) & B \\ -I & 0 \\ 0 & -I \end{bmatrix}\begin{bmatrix} G_1 & G_2 & 0 \\ 0 & 0 & I \end{bmatrix}\right\} < 0 \quad (28)$$

$$\begin{bmatrix} Z & CP_1^i \\ \bullet & P_1^i \end{bmatrix} > 0, \; trace(Z) < \gamma_2^2$$

where $\phi_i = P_1^i E + E_0 Q_1^i$.

In the same manner we can present a corollary to Theorem 2 associated to the parameter-dependent descriptor system (22).

V. Iterative Design for Multiobjective State-Feedback Control of Parameter-Dependent Descriptor Systems

A. An iterative design procedure

Let us consider the following parameter-dependent descriptor system described by:

$$\begin{cases} E\dot{x} = A(\theta)x + B_2 w_2 + B_\infty w_\infty + Bu \\ z_2 = C_2 w_2 \\ z_\infty = C_\infty w_\infty \end{cases}, \theta \in \Xi \quad (29)$$

where Ξ is introduced by (23).

Assumption (7) is supposed to be verified and we suppose also that system (29) is finite dynamics stabilizable for all $\theta \in \Xi$ (see Definition 2).

Since θ is in a compact and connected region, we can take $\forall i \in \{1, \ldots, l\} : \theta_i \in [-1, 1]$ without loss of generality. We denote, in the sequel, $A_0 = A(0)$ and, for regular systems (29), we denote $T_2^\theta, T_\infty^\theta$ the transfer functions from w_2 to z_2 and from w_∞ to z_∞.

Assume γ_∞ is a given positive scalar. Then the problem is to find a state feedback $u = K(\theta)x$, depending affinely on θ, which minimizes $\gamma_2 = \left\|T_2^\theta\right\|_2$ under $\left\|T_\infty^\theta\right\|_\infty < \gamma_\infty$. This problem can be formulated using dilated LMI conditions as below.

Problem 1: Find symmetric matrices Z, $P_2^i > 0$, $P_\infty^i > 0$, and matrices Q_{12}^i, Q_{22}^i, $Q_{1\infty}^i$, $Q_{2\infty}^i$, G_{12}, G_{22}, $G_{1\infty}$, $G_{2\infty}$ solution of the following optimization problem:

Minimize γ_2

Subject to $\gamma_2^2 \geq trace(Z)$

$$\begin{bmatrix} 0 & \bullet & \bullet \\ P_2^i E^T + E_0 Q_{12}^i & He\left\{E_0 Q_{22}^i\right\} & \bullet \\ 0 & 0 & I \end{bmatrix} + He\left\{\begin{bmatrix} A(\omega_i) + BK(\omega_i) & B_2 \\ -I & 0 \\ 0 & -I \end{bmatrix}\begin{bmatrix} G_{12} & G_{22} & 0 \\ 0 & 0 & I \end{bmatrix}\right\} < 0 \quad (30)$$

$$\begin{bmatrix} Z & C_2 P_2^i \\ \bullet & P_2^i \end{bmatrix} > 0 \quad (31)$$

$$\begin{bmatrix} 0 & \bullet & \bullet & \bullet \\ P_\infty^i E^T + E_0 Q_{1\infty}^i & He\left\{E_0 Q_{2\infty}^i\right\} & \bullet & \bullet \\ 0 & 0 & I & \bullet \\ 0 & 0 & 0 & -\gamma_\infty^2 I \end{bmatrix} + He\left\{\begin{bmatrix} A(\omega_i) + BK(\omega_i) & B_\infty \\ -I & 0 \\ 0 & -I \\ C_\infty & 0 \end{bmatrix}\begin{bmatrix} G_{1\infty} & G_{2\infty} & 0 & 0 \\ 0 & 0 & I & 0 \end{bmatrix}\right\} < 0 \quad (32)$$

$$P_\infty^i > 0 \quad (33)$$

Note that a pole-clustering constraint can be taken into account at this step but is not considered here for brevity reasons.

Theorem 5: For any arbitrary prescribed positive number $\beta > 0$, a regular descriptor system given by (1) is admissible if and only if there exist matrices $P_1 > 0, P_1 \in \mathbb{R}^{n\times n}$, $Q_1 \in \mathbb{R}^{(n-r)\times n}$, $Q_2 \in \mathbb{R}^{(n-r)\times n}$ and $G_1 \in \mathbb{R}^{n\times n}$ such that:

$$\begin{bmatrix} 0 & \left(P_1 E^T + E_0 Q_1\right)^T \\ \bullet & He\left\{E_0 Q_2\right\} \end{bmatrix} + He\left\{\begin{bmatrix} A \\ -I \end{bmatrix} G_1 \begin{bmatrix} I & \beta(NM)^T \end{bmatrix}\right\} < 0 \quad (34)$$

where (N, M) is a Kronecker-Weierstrass transformation couple of (E, A).

Remark 4: The condition (34) obviously covers the condition stated in [18] for state-space systems. Indeed, in that case we have $N = M = I$, $E_0 = 0$.

Condition (34) is used in the initialization step of the iterative design procedure proposed next for solving Problem 1.

Algorithm1: Fix $\beta > 0$ and $\beta_\infty > 0$. Let $K(\theta)^{(j)}$ be the parameter-dependent state feedback designed at the *j*-th iteration.

Algorithm1.1: H$_\infty$ minimization

(a1) Find a state feedback gain K_0 such that $(E, A_0 + BK_0)$ is admissible.

(a2) Compute $G^{(0)} = NM$, where N, M are associated to the pair $(E, A_0 + BK_0)$ (we will mention after how to compute $G^{(0)}$).

(a3) Minimize γ_∞ under the constraints (32)-(33) with $G_{2\infty} = \beta_\infty G_{1\infty} G^{(0)}$. Set $K(\theta)^{(0)} = K(\theta)$ where $K(\theta)$ is the solution and $j = 1$.

(a4) Minimize γ_∞ under the constraints (32)-(33) with $K(\theta) = K(\theta)^{(j-1)}$. Compute $G^{(j)} = (G_{1\infty})^{-1} G_{2\infty}$ where $(G_{1\infty}, G_{2\infty})$ is the solution.

(a5) Minimize γ_∞ under the constraints (32)-(33) with $G_{2\infty} = \beta_\infty G_{1\infty} G^{(j)}$. Set $K(\theta)^{(j)} = K(\theta)$ where $K(\theta)$ is the solution.

(a6) If the desired γ_∞ is reached or a stopping criterion is satisfied, exit. Otherwise, set $j = j + 1$ and go to Step (a4).

Algorithm1.2: H$_2$ minimization under H$_\infty$ constraint

(b1) Set $K(\theta)^{(0)} = K(\theta)$ where $K(\theta)$ is the solution of the *Initialization* part of the algorithm. Set $j = 1$.

(b2) Solve the optimization problem (30)-(33) with $K(\theta) = K(\theta)^{(j-1)}$ and $G_{12} = G_{1\infty} = G_1$, $G_{22} = G_{2\infty} = G_2$. Compute $G^{(j)} = (G_1)^{-1} G_2$ where (G_1, G_2) is the solution.

(b3) Solve the optimization problem (30)-(33) with $G_{22} = G_{2\infty} = \beta G_1 G^{(j)}$ and $G_{12} = G_{1\infty} = G_1$. Set $K(\theta)^{(j)} = K(\theta)$ where $K(\theta)$ is the solution.

(b4) If the desired γ_2 is reached or a stopping criterion is satisfied, exit. Otherwise, set $j = j + 1$ and go to Step (b2).

Note that $G^{(0)}$ is computed for the closed-loop pair $(E, A_0 + BK_0)$ which is admissible. In that case, a singular value decomposition form can be obtained:

$$RES = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix},\ R(A_0 + BK_0)S = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}$$

where matrices R, S are nonsingular. We then derive matrices M, N as follows:

$$M = \begin{bmatrix} I & -A_2 A_4^{-1} \\ 0 & A_4^{-1} \end{bmatrix} R,\ N = S \begin{bmatrix} I & 0 \\ -A_4^{-1} A_3 & I \end{bmatrix}.$$

Remark 5: The existence of a solution in $K(\theta)^{(0)}$ is not guaranteed. However, the initialization method proposed here seems to be effective when tested on a large number of examples. In the contrary, assigning arbitrarily $G^{(0)}$ do not lead to a solution in most cases.

Remark 6: Local convergence of each part of the algorithm is guaranteed by construction.

B. Numerical examples

Example: Consider the uncertain descriptor system (29) with the following data:

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\ B_2 = \begin{bmatrix} 0 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}\ B_\infty = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}\ B = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}\ C_2 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}^T\ C_\infty = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}^T$$

$$A(\theta) = \begin{bmatrix} \theta_1 & -5 & \theta_1 \\ 1 + 2\theta_2 & 2\theta_1 & 5 \\ 10 & \theta_1 & 1 + \theta_2 \end{bmatrix}, \theta_i \in [-1, 1]_{i=1,2}$$

The problem is to find a state feedback controller $u = K(\theta)x$ which minimizes $\gamma_2 = \left\| T_2^\theta \right\|_2$ under the constraint $\left\| T_\infty^\theta \right\|_\infty < 2$. The initial controller is selected as:

$$K_0 = [\text{-26.0564 -163.2829 -1.0187}]$$

which is obtained by solving a feasible LMI problem such as $(E, A_0 + BK_0)$ is admissible. After 30 iterations of Algorithm1.1, with $\beta_\infty = 1$, we obtain a controller:

$$K(\theta) = K^0 + \theta_1 K^1 + \theta_2 K^2$$

with: $K^0 = [\text{-10.1636 -2.7238 -1.6683}]$,

$K^1 = [\text{-0.4706 -1.6511 0.0060}]$ and $K^2 = [\text{-0.0963 0.0276 -1.0791}]$

Figure 1, shows the convergence of Algorithm 1.1.

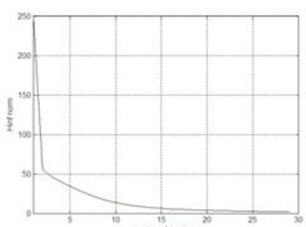


Figure 1. Achieved H$_\infty$ norm

After 67 iterations of Algorithm1.2, with $\beta = 1$, we obtain a controller: $K(\theta) = K^0 + \theta_1 K^1 + \theta_2 K^2$

with: $K^0 = [\text{-14.6171 -15.6301 -5.7937}]$, $K^1 = [\text{0.8375 -3.5119 0.9226}]$ and $K^2 = [\text{-1.6012 0.0114 -1.8101}]$

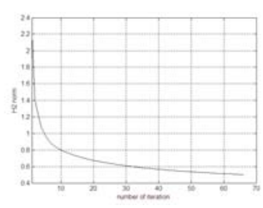


Figure 2. Achieved H$_2$ norm with $\left\| T_\infty^\theta \right\|_\infty < 2$

which achieves an H$_2$ norm of $\left\| T_2^\theta \right\|_2 = 0.5$. Figure 2, shows the convergence of Algorithm 1.2.

Example: Consider the uncertain descriptor system (29) with the following data:

$$E=\begin{bmatrix}3&1&0&0\\0&2&0&0\\0&0&1&0\\0&0&0&0\end{bmatrix}\quad B_2=\begin{bmatrix}1&2\\0&1\\-1&0\\1&1\end{bmatrix}\quad B_\infty=\begin{bmatrix}0&1\\0&1\\0&-2\\1&0\end{bmatrix}\quad B=\begin{bmatrix}1\\0\\-1\\1\end{bmatrix}\quad C_2=\begin{bmatrix}0&1\\1&0\\-1&-1\\1&0\end{bmatrix}^T\quad C_\infty=\begin{bmatrix}1&0\\0&1\\1&0\\0&1\end{bmatrix}^T$$

$$A(\theta)=\begin{bmatrix}-2+2\theta_1&-1&1&-\theta_1\\2\theta_1&-1+5\theta_2&1&-1\\0&1+\theta_1&-4&0\\1&-1&0&1+\theta_2\end{bmatrix},\theta_i\in[-1,1]_{i=1,2}$$

The problem is to find a state feedback controller $u=K(\theta)x$ which minimizes $\gamma_2=\left\|T_2^\theta\right\|_2$ under the constraint $\left\|T_\infty^\theta\right\|_\infty<3$. The initial controller is selected as:

$$K_0=[\text{-33.6870}\quad 14.1261\quad \text{-69.5059}\quad \text{-1.0116}]$$

which is obtained by solving a feasible LMI problem such as (E,A_0+BK_0) is admissible. After 22 iterations of Algorithm1.1, with $\beta_\infty=1$, we obtain a controller:

$$K(\theta)=K^0+\theta_1K^1+\theta_2K^2$$

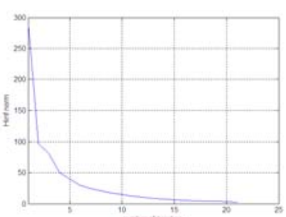
with:
$$K^0=[\text{-1.0328 } 5.6882\ 0.6194 \text{ -1.6903}],$$
$$K^1=[0.9638\ 0.1486 \text{ -0.0097 } 0.0473],$$
$$K^2=[\text{-0.3441 } 1.9644\ \text{ -0.0644 -0.7566}]$$

After 25 iterations of Algorithm1.2, with $\beta=1$, we obtain a controller: $K(\theta)=K^0+\theta_1K^1+\theta_2K^2$

with:
$$K^0=[\text{-0.7577 } 23.6750\ \ 1.2607 \text{ -3.8215}]$$
$$K^1=[2.6168\ \ 0.2002\ \ 0.007\ \ 0.0294]$$
$$K^2=[\text{-1.9497 } 7.3928 \text{ -0.1618 } 0.8156]$$

which achieves an H_2 norm of $\left\|T_2^\theta\right\|_2=1.86$.

The convergence of Algorithms 1.1 and 1.2 is shown in the following figures:



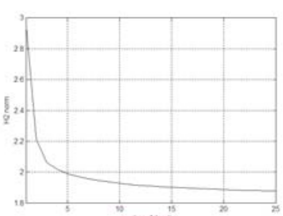


Figure 3. Achieved H_∞ norm Figure 4. Achieved H_2 norm

VI. Conclusion

In this paper, we proposed dilated LMI characterizations of admissibility, *D*-admissibility, H_∞ and H_2 norms for continuous-time descriptor systems. As in the state-space case, these dilated LMIs achieved less conservative results when dealing with robust admissibility/performance analysis of affine parameter-dependent descriptor systems. Furthermore, based on these dilated LMI conditions, we presented an iterative design procedure for multiobjective state feedback control of parameter-dependent descriptor systems. The underlying linearization method can be adapted to the state-space multiobjective feedback design problem when using a descriptor representation of such systems. A forthcoming paper will test this alternative design method and compare it with existing ones.

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