Proceedings of the
47th IEEE Conference on Decision and Control
Cancun, Mexico, Dec. 9-11, 2008

Limit Time Optimal Synthesis for a Two-Level Quantum System

Paolo Mason, Rebecca Salmoni, Ugo Boscain and Yacine Chitour

***Abstract*— For $\alpha \in (0, \pi/2)$, let $(\Sigma)_\alpha$ be the control system $\dot{x} = (F + uG)x$, where x belongs to the two-dimensional unit sphere S^2, $u \in [-1, 1]$ and F, G are 3×3 skew-symmetric matrices generating rotations with perpendicular axes and of respective norms $\cos(\alpha)$ and $\sin(\alpha)$. In this paper, we study the time optimal synthesis (TOS) from the north pole $(0, 0, 1)^T$ associated to $(\Sigma)_\alpha$, as the parameter α tends to zero; this problem is motivated by specific issues in the control of two-level quantum systems subject to weak external fields. The TOS is characterized by a "two-snakes" configuration on the whole S^2, except for a neighborhood U_α of the south pole $(0, 0, -1)^T$ of diameter at most $\mathcal{O}(\alpha)$. Inside U_α, the TOS depends on the relationship between $r(\alpha) := \pi/2\alpha - [\pi/2\alpha]$ and α. More precisely, we characterize three main relationships, by considering sequences $(\alpha_k)_{k\geq 0}$ satisfying (a) $r(\alpha_k) = \bar{r}$; (b) $r(\alpha_k) = C\alpha_k$ and (c) $r(\alpha_k) = 0$, where $\bar{r} \in (0, 1)$ and $C > 0$. In each case we describe the TOS and, in the case (a), we provide, after a suitable rescaling, the limit behavior of the corresponding TOS inside U_α, as α tends to zero.**

***Index Terms*— control-affine systems, optimal synthesis, control of quantum systems, minimum time, asymptotics**

I. INTRODUCTION

Let $\alpha \in (0, \pi/2)$. On the unit sphere $S^2 \subset \mathbf{R}^3$, consider the control system $(\Sigma)_\alpha$ defined by

$$(\Sigma)_\alpha \quad \dot{x} = (F + uG)x, \; x = (x_1, x_2, x_3)^T, \; \|x\|^2 = 1, \quad (1)$$

where $|u| \leq 1$ and F, G are two 3×3 skew-symmetric matrices representing two orthogonal rotations with axes of length respectively $\cos(\alpha)$ and $\sin(\alpha)$, $\alpha \in (0, \pi/2)$. With no loss of generality, we assume that

$$F := \begin{pmatrix} 0 & -\cos(\alpha) & 0 \\ \cos(\alpha) & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad (2)$$

$$G := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\sin(\alpha) \\ 0 & \sin(\alpha) & 0 \end{pmatrix} \quad (3)$$

In this paper, we aim at describing the time optimal synthesis (TOS for short) from the north pole $N := (0, 0, 1)^T$ for $(\Sigma)_\alpha$, i.e. for every $\bar{x} \in S^2$ we want to find the time optimal trajectory steering N to $\bar{x}$ in minimum time (see Figure 1).

In particular *we are interested in the qualitative shape of the time optimal synthesis in a neighborhood of the south pole $S = (0, 0, -1)^T$, in the limit $\alpha \to 0$*. The interest for that problem stems in quantum control issues. Indeed consider the population transfer problem for a two level quantum system driven by a single external field. This model describes the evolution of the z-component of the spin of a (spin 1/2) particle driven by a magnetic field, that is constant along the z-axis and controlled along the x-axis. Equivalently it describes the first two levels of a molecule driven by an external field without the rotating wave approximation [1], [2]. The dynamics of such a system is governed by the time dependent Schrödinger equation (in a system of units such that $\hbar = 1$):

$$i\frac{d\psi(t)}{dt} = (H_0 + \Omega(t)H_1)\psi(t). \quad (4)$$

Here $\psi(.) = (\psi_1(.), \psi_2(.))^T : [0, T] \to \mathbf{C}^2$, denotes the wave function and verifies $\sum_{j=1}^{2} |\psi_j(t)|^2 = 1$, i.e. $\psi(t)$ belongs to the sphere $S^3 \subset \mathbf{C}^2$. The free Hamiltonian H_0 and the controlled Hamiltonian H_1 are given by:

$$H_0 = \begin{pmatrix} -E & 0 \\ 0 & E \end{pmatrix}, \quad H_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad (5)$$

where $-E$, E, $(E > 0)$ are the two energy levels and the control $\Omega(.)$ is a real function describing the amplitude of the external field. Here $|\psi_1(t)|^2$ (resp. $|\psi_2(t)|^2$) represents the probability of measuring at time t the energy E (resp. $-E$). The control task consists of inducing a transition from the first eigenstate of H_0 (i.e., $|\psi_1|^2 = 1$) to any other *physical state*. We recall that two states ψ, ψ' are physically equivalent if they differ by a factor of phase. More precisely by physical state we mean a point of the two dimensional sphere (called *Bloch sphere* in this context) $S^2 = S^3/\sim$ where the equivalence relation $\sim$ is defined as follows: $\psi \sim \psi'$ (where $\psi, \psi' \in S^3$) if and only if $\psi = \exp(i\Phi)\psi'$, for some $\Phi \in [0, 2\pi[$. The projection $\Pi : S^3 \to S^2$ is called

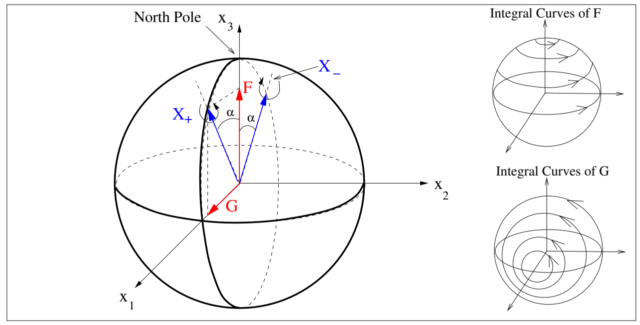

Fig. 1. Geometric interpretation of the system $(\Sigma)_\alpha$. The vector fields $X_+ := F + G$ and $X_- := F - G$ are two rotations of norm one making an angle α with the axis x_3.

P. Mason is with I.A.C-C.N.R., Viale del Policlinico 137, 00161 Rome, Italy, p.mason@iac.cnr.it.
R. Salmoni and Y. Chitour are with Laboratoire des signaux et systèmes, Université Paris-Sud, CNRS, Supélec, 91192 Gif-Sur-Yvette, France rebecca.salmoni@lss.supelec.fr, yacine.chitour@lss.supelec.fr.
U. Boscain is with Le2i, CNRS, Université de Bourgogne, BP 47870, 21078 Dijon Cedex, France, boscain@sissa.it.

978-1-4244-3124-3/08/$25.00 ©2008 IEEE

Hopf projection. A particularly interesting transition is of course from the first to the second eigenstates of H_0 (i.e., from $|\psi_1|^2 = 1$ to $|\psi_2|^2 = 1$).

In many applications, the external field should have bounded amplitude M, (i.e. $|\Omega(.)| \leq M$) and, in order to minimize the unavoidable effects of relaxation and decoherence [3], [4], the transfer should occur as quickly as possible. Therefore we end up addressing a minimum time control problem with one bounded control. Sometimes decoherence is also reduced by taking M small with respect to E: this guarantees that the energy injected by the control action into the system is close to the minimal one necessary to induce the transition.

As it was shown in [2], the projection of the minimum time control problem for the system (4) on the Bloch Sphere gives rise, after time renormalization, to the minimum time control problem for system (1) where **i)** the first and second eigenstates of H_0 project respectively onto the north pole N and the south pole S, **ii)** $\tan(\alpha) = M/E$ and $u(t) = \Omega(t)/M$. **iii)** H_0 projects on F and H_1 on G. The case $M << E$ corresponds now to the limit $\alpha \to 0$.

Nowadays two-level quantum systems are central in the implementation of the so-called quantum gates (the basic blocks of a quantum computer); see for instance [5].

The present paper is actually a continuation of [6], [2] in the sense that it answers questions raised in these papers.

In [6], the purpose was to provide a lower and an upper bound for $N(\alpha)$, the maximum number of switchings for time optimal trajectories for the left invariant control system

$$(S)_\alpha \quad \dot{g} = g(F + uG), \quad g \in SO(3), \quad |u| \leq 1, \qquad (6)$$

where F and G are defined in (2) and (3). Recall that, for such control systems, it is known (cf. for instance [6], [2]) that every time optimal trajectory is a finite concatenation of bang arcs (i.e. $u \equiv \pm 1$) or singular arcs ($u = 0$). A bang arc is an integral trajectory corresponding to the rotations

$$X_+ := F + G, \quad X_- := F - G, \qquad (7)$$

and is denoted by $e^{tX_\varepsilon}x$, $t \in [0, T]$, where $\varepsilon = \pm$, x is the starting point of the bang arc and T is its time duration. Moreover, a switching time – or simply a switching – along a time optimal trajectory is a time t_0 so that the control u is not constant in any open neighborhood of t_0.

To estimate $N(\alpha)$, a suitable Hopf map $\Pi : SO(3) \to S^2$ was introduced to project $(S)_\alpha$ onto $(\Sigma)_\alpha$. In particular, every time optimal trajectory of $(\Sigma)_\alpha$ is the projection by Π of a time optimal trajectory of $(S)_\alpha$. It results that, if a time optimal trajectory on S^2 has a certain number of switchings, then this number is lower than or equal to the maximum number of switchings for the optimal problem on $SO(3)$. The construction of time optimal trajectories of $(\Sigma)_\alpha$ was performed according to the general theory of time optimal synthesis on 2-D manifolds developed in [7], [8], [9], [10], [11], [12], [13], [14], and recently gathered in the book [15].

The question of studying $N(\alpha)$ was first addressed in [16] where, using the index theory developed by Agrachev, the authors proved that $N(\alpha) \leq [\pi/\alpha]$, where $[\cdot]$ stands for the integer part. That result was not only an indirect indication that $N(\alpha)$ would tend to infinity as α tends to zero, but it also provided a hint on the asymptotic of $N(\alpha)$ as α tends to zero. Notice that for $\alpha = 0$ the systems (1) and (6) are not controllable. With the techniques developed in [6], enough properties for the TOS associated to $(\Sigma)_\alpha$, $\alpha < \pi/4$, were identified in order to improve the upper bound of [16] and to actually show that, for α small

$$N(\alpha) \leq k_M + 5, \quad \text{where} \quad k_M := \left[\frac{\pi}{2\alpha}\right].$$

In [6], it is proved that, for $\alpha < \pi/4$, the extremals associated to $(\Sigma)_\alpha$ (i.e. the trajectories candidate for time optimality obtained after using the Pontryagin Maximum Principle, PMP for short), starting from the north pole N are bang-bang trajectories, i.e. finite concatenations of bang arcs of the type

$$e^{s_f X_{-\varepsilon'}} e^{v(s_i) X_{\varepsilon'}} \ldots e^{v(s_i) X_{-\varepsilon}} e^{s_i X_\varepsilon} N,$$

where the initial time duration s_i verifies $s_i \in (0, \pi]$, all the time durations of the interior bang arcs are equal to $v(s_i)$, where the function v is defined by

$$v(s_i) = \pi + 2 \arctan\left(\frac{\sin(s_i)}{\cos(s_i) + \cot^2(\alpha)}\right), \qquad (8)$$

and the final time duration s_f verifies $s_f \leq v(s_i)$. Of particular importance for the construction of the TOS, are the *switching curves*, i.e. the curves made by points where the control switches from $+1$ to -1 or viceversa and defined inductively by

$$C_1^\varepsilon(s) = e^{X_\varepsilon v(s)} e^{X_{-\varepsilon} s} N, \quad C_k^\varepsilon(s) = e^{X_\varepsilon v(s)} C_{k-1}^{-\varepsilon}(s), \quad (9)$$

where $\varepsilon = \pm 1$ and $k = 2, ..., k_M$. Since the PMP gives just a necessary condition for optimality, it is crucial to determine the time after which an extremal is no more optimal. In [6], it is shown that the number of bangs must be lower than or equal to $k_M + 1$ and the extremals cover the sphere S^2 according to the "two-snakes" configuration as depicted in Figure 2. The two "snakes" correspond to extremal trajectories starting respectively with control $+1$ and -1. For more details, see [6].

However, in [6], the construction of the TOS associated to $(\Sigma)_\alpha$ was not exhaustive. In particular, it was not shown that all the extremals are optimal up to $k_M - 1$ bangs arcs and the construction of the synthesis was not analytically complete in a neighborhood of the south pole S. There, the minimum time front develops singularities due to the compactness of S^2. In [6] only numerical simulations were provided, describing the evolution of the extremal front in a neighborhood of the south pole. As $\alpha \to 0$, these numerical simulations suggested the emergence of an interesting phenomenon (see Fig. 3): define the remainder

$$r(\alpha) := \pi/2\alpha - [\pi/2\alpha]. \qquad (10)$$

Then, there are three possible patterns of TOS in the neighborhood of the south pole S, each of them depending

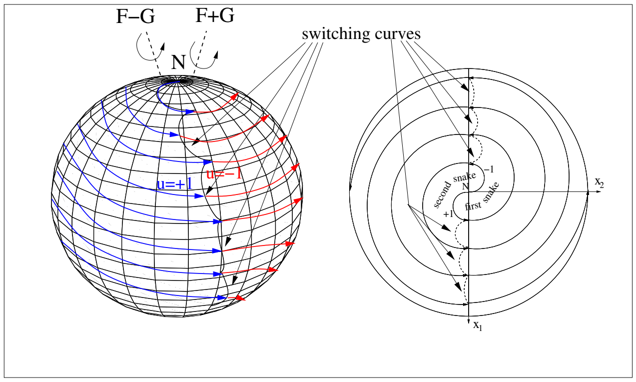

Fig. 2. The “two-snakes” configuration defined by the extremal flow. Notice that this set of trajectories covers the whole sphere, but in principle not all extremals are optimal and a point can be reached by more than one trajectory at the same or at different times.

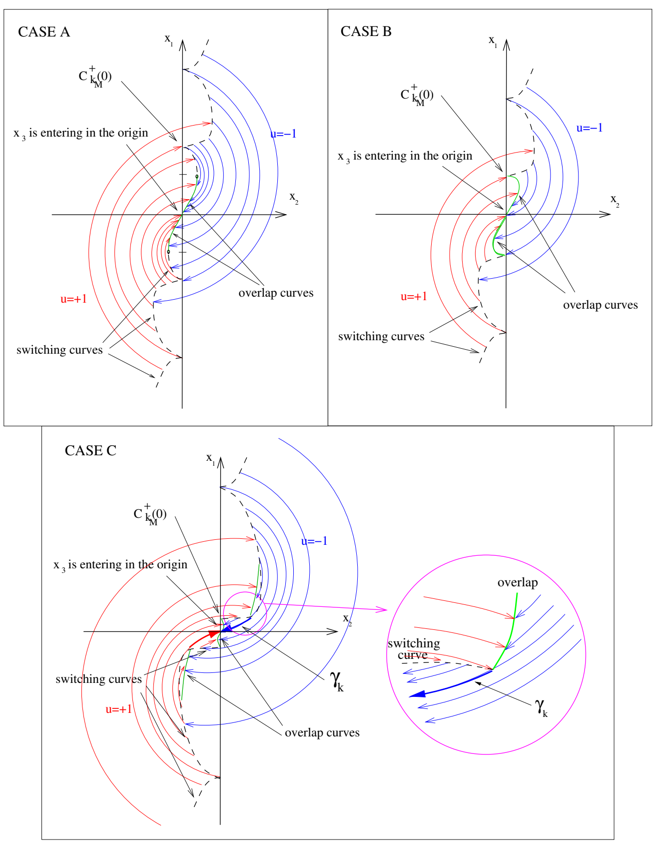

Fig. 3. Conjectured shapes of the synthesis in a neighborhood of the south pole. Switching curves are $\mathcal{C}^1$ curves made by points in which the control switches from $+1$ to -1 or viceversa. Overlap curves are $\mathcal{C}^1$ curves made by points reached optimally by more than one trajectory. The curve γ_k is a bang arc that is also an overlap curve since trajectories having a different history travel on it at the same time. The singularity appearing at the starting point of γ_k (called $(C,K)_1$ according to the taxonomy of [15]) is a singularity of the synthesis predicted by the general theory [15] and it is due to a nonlocal phenomenon.

on a relation between $r(\alpha)$ and α.

In [2], the TOS for (Σ_α) is studied in the context of quantum control as described previously. In that paper, the TOS, for $\alpha \geq \pi/4$ is completed and, in the case $\alpha < \pi/4$, further information are obtained for what concerns time optimal trajectories steering the north to the south pole (in fact the most interesting trajectories for the quantum mechanical problem). Such optimal trajectories belong to a set Ξ containing at most 8 trajectories, half of them starting with control $+1$ and the other half starting with control -1, and switching exactly at the same times. It is also proved that the cardinality of Ξ depends on the remainder $r(\alpha)$ defined in Eq. (10). For instance, for α and $r(\alpha)$ small enough, then Ξ contains exactly 8 trajectories (four of them are optimal) while if $r(\alpha)$ is close to 1, then Ξ contains only 4 trajectories (two of them are optimal).

In the next section we study the TOS associated to $(\Sigma)_\alpha$ as α tends to zero, focusing in particular on its behavior inside a neighborhood of the south pole. In this way we will be able to give a complete description of the TOS when α is small and therefore to answer the questions raised in [6], [2].

II. Limit time optimal synthesis

In this section we want to determine, roughly speaking, what could be a possible limit for the TOS associated to $(\Sigma)_\alpha$ as α tends to zero and then to state a convergence result (in some suitable sense) of the TOS associated to $(\Sigma)_\alpha$ to that limit. Note that we will just state the main results; details are given in [17]. To proceed, we embark on the study of a geometric object $\mathcal{F}(\alpha, T)$ called the *extremal front at time* T along $(\Sigma)_\alpha$ and defined as the set of points reached at time T by extremal trajectories starting from N. The PMP says that the extremal front $\mathcal{F}(\alpha, T)$ contains the *minimum time front* $OF(\alpha, T)$, i.e. the set of points reached at time T by time optimal trajectories starting from N. When $\mathcal{F}(\alpha, T) = OF(\alpha, T)$, we say that $\mathcal{F}(\alpha, T)$ is *optimal*.

First it is possible to see that, in the case in which k_M is odd (being the other case analogous), the extremal front $\mathcal{F}(\alpha, k_M\pi)$ is made up of the union of two curves $\mathcal{E}^\varepsilon(\alpha,\cdot) : (0,\pi] \to S^2$, $\varepsilon = \pm$, with $\mathcal{E}^\varepsilon(\alpha,\cdot) = \Pi_{x_3}\mathcal{E}^{-\varepsilon}(\alpha,\cdot)$, where Π_{x_3} is the orthogonal symmetry with respect to the x_3-axis. Moreover, for α small enough, $\mathcal{E}^\varepsilon(\alpha,\cdot)$ admits a convergent power series of the type $\sum_{l\geq 0} f_l^\varepsilon(s, r(\alpha))\alpha^l$, where the $f_l^\varepsilon(s,r)$ are real-analytic functions of $(s,r) \in \mathbf{R}^2$, 2π-periodic in s with

$$f_0^+(s,r) = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}, \quad f_1^+(s,r) = \begin{pmatrix} -2r\mathbf{c}_s \\ 2r\mathbf{s}_s \\ 0 \end{pmatrix},$$

$$f_2^+(s,r) = \begin{pmatrix} \frac{\pi}{2}(4r + \mathbf{c}_s)\mathbf{s}_s^2 \\ \frac{\pi}{4}(3 + 8r\mathbf{c}_s + \mathbf{c}_{2s})\mathbf{s}_s \\ 2r^2 \end{pmatrix}$$

$$f_l^-(s,r) = \Pi_{x_3} f_l^+(s,r)\,, \tag{11}$$

where $\mathbf{c}_s$, $\mathbf{s}_s$ stand for $\cos s$, $\sin s$, respectively.

As a trivial consequence, we deduce that for $r \in [0,1]$, $s \in \mathbf{R}$ and α small enough, we have

$$\mathcal{E}^\varepsilon(\alpha, s) = f_0^\varepsilon(s, r(\alpha)) + f_1^\varepsilon(s, r(\alpha))\alpha + f_2^\varepsilon(s, r(\alpha))\alpha^2 + \mathcal{O}(\alpha^3) \quad (12)$$

and

$$\frac{\partial}{\partial s}\mathcal{E}^\varepsilon(\alpha, s) = \frac{\partial}{\partial s} f_1^\varepsilon(s, r(\alpha))\alpha + \frac{\partial}{\partial s} f_2^\varepsilon(s, r(\alpha))\alpha^2 + \mathcal{O}(\alpha^3), \quad (13)$$

where $|\mathcal{O}(\alpha^3)| \leq \bar{C}|\alpha|^3$ with $\bar{C} > 0$ constant independent of (r, s, α).

The first crucial result is the following.

Proposition 1: $\mathcal{F}(\alpha, T)$ is optimal for $T \leq (k_M - 1)\pi$ and α small enough. Moreover $\mathcal{F}(\alpha, (k_M - 1)\pi)$ is a circle of radius $2(1 + r(\alpha))\alpha$ up to order α^2.

As a consequence of the optimality of $\mathcal{F}(\alpha, (k_M - 1)\pi)$, we get that all the extremals of the "two-snakes" configuration depicted in Figure 2 are optimal up to time $(k_M - 1)\pi$. In other words, if U_α is the connected component of $S^2 \setminus \mathcal{F}(\alpha, (k_M - 1)\pi)$ containing the south pole, we obtain the optimal synthesis on $S^2 \setminus U_\alpha$. Notice that U_α is a neighborhood of the south pole of size proportional to α.

In that way we answer the question stated in [6] about optimality of extremals of the two snake configuration in the case α small and it is, of course, of interest for applications to the two-level quantum system.

The expressions (12)–(13) are central tools to understand the possible asymptotic behaviors of the TOS associated to $(\Sigma)_\alpha$, as α tends to zero.

For this purpose we observe that the expressions of f_1^+ and f_2^+ in (11) depend explicitly on the remainder $r(\alpha)$. This fact suggests the need to impose particular relationships between α and $r(\alpha)$ in order to define any asymptotic behavior. In other words we must let α goes to zero only along certain subsequences $(\alpha_k)_{k\geq 0}$ where a specific relationship holds between α_k and $r(\alpha_k)$. By analyzing Eq. (11) it is possible to determine such relationships and to prove that the conjectures made in [6] about the qualitative shape of the synthesis near the south pole were true (see Figure 3). In particular there are exactly three qualitatively different asymptotic behaviours of the synthesis as α goes to zero, described by the following cases.

First, we consider the case in which α is arbitrarily small, with $r(\alpha) \in (0,1)$ uniformly far from 0 and 1. To simplify further the discussion, it is reasonable to consider the following.

(C1) For $\bar{r} \in (0,1)$, let α tend to zero along the subsequence $\alpha_k := \frac{\pi}{2(k+\bar{r})}$, so that $r(\alpha_k) = \bar{r}$.

In this case $\mathcal{E}^\varepsilon(\alpha, \cdot)$ is approximated, up to order α^2, by the expression $S + f_1^\varepsilon(\cdot, \bar{r})$. As a consequence $\mathcal{F}(\alpha, k_M\pi)$ is approximately a circle of radius $2\bar{r}\alpha$ centered at the south pole. We are then able to give a qualitative description of the optimal synthesis, as stated below in Theorem 1. We then deduce that, if α is small enough and $r(\alpha)$ is far enough from 0 and 1, the synthesis in a neighborhood of the south pole is topologically equivalent to the limit synthesis obtained, as k tends to infinity, along the sequence α_k above. That synthesis turns out to be exactly the one described in Figure 3 (case B), as predicted in [6].

It remains then to consider the cases in which $r(\alpha)$ can be arbitrarily close to 0 or 1. For this purpose we first consider the case in which $r(\alpha)/\alpha$ remains bounded above and below by positive constants as α tends to zero. From Eq. (11) it is clear that this is equivalent to say that $f_1^\varepsilon(\cdot, r)\alpha$ is comparable to $f_2^\varepsilon(\cdot, r)\alpha^2$. For simplicity we consider the following.

(C2) For $C > 0$, let α tend to zero along a subsequence $(\alpha_k)_{k\geq 0}$ such that $r(\alpha_k) = C\alpha_k$.

In this case $\mathcal{E}^\varepsilon(\alpha, \cdot)$ is well approximated by $S + (f_1^\varepsilon(\cdot, C) + f_2^\varepsilon(\cdot, 0))\alpha^2$. If $C > \pi/4$, the synthesis is equivalent to that of the previous case. On the other hand if $C < \pi/4$ the synthesis is more complicated and it turns out to be exactly the one described in Figure 3 (case C), as predicted in [6].

If α and $r(\alpha)$ tend to zero with $r(\alpha)/\alpha$ tending to infinity (resp. to zero) it is possible to see that the synthesis is qualitatively equivalent to the one of case (C1) (resp. (C2)).

The third interesting case is the following.

(C3) Let α tend to zero along the subsequence $\alpha_k := \frac{\pi}{2k}$, so that $r(\alpha_k) = 0$.

In this case the extremal front at time $k_M\pi$ contains the south pole and the corresponding optimal front reduces to that point. The optimal synthesis is then described starting from the extremal front $\mathcal{F}(\alpha, (k_M - 1)\pi) = OF(\alpha, (k_M - 1)\pi)$, and it corresponds to the one described in Figure 3 (case A), as predicted in [6].

Similarly, one can see that in the case in which α is small and $r(\alpha)$ is close to 1, the optimal synthesis is qualitatively equivalent either to that of Case (C1) or to that of Case (C3), and this concludes the description of the possible asymptotic behaviors as α tends to 0.

Remark 2.1: It is interesting to notice that numerical simulations show that for α decreasing to zero continuously, the qualitative shape of the optimal synthesis described in Figure 3 alternates cyclically in the order BCABCA....

Let us describe the case (C1) in more details. Since $\mathcal{F}(\alpha, k_M\pi)$ is approximated, up to $\mathcal{O}(\alpha^2)$, by a circle of center S and radius $2\bar{r}\alpha$, we are able to show that it is optimal, so that all the extremals of the "two-snakes" configuration depicted in Figure 2 are optimal up to time $k_M\pi$. In other words, if V_α is the connected component of $S^2 \setminus \mathcal{F}(\alpha, k_M\pi)$ containing the south pole, we obtain the optimal synthesis on $S^2 \setminus V_\alpha$.

As α tends to zero, V_α collapses on S. Hence one must rescale the problem by a factor $1/\alpha$, in order to describe the TOS inside V_α. Also notice that since we are in a neighborhood of the south pole we can project the problem on the plane (x_1, x_2). We are now in a position to define a possible limit behavior for the TOS inside V_α. Let M_α be the linear mapping from $\mathbf{R}^3$ onto $\mathbf{R}^2$ defined as the composition of the projection $(x_1, x_2, x_3) \mapsto (x_1, x_2)$ followed by the dilation by $1/\alpha$. Denote by $(\widetilde{\Sigma})_\alpha$ (resp. $\widetilde{OF}(\alpha, k_M\pi)$) the

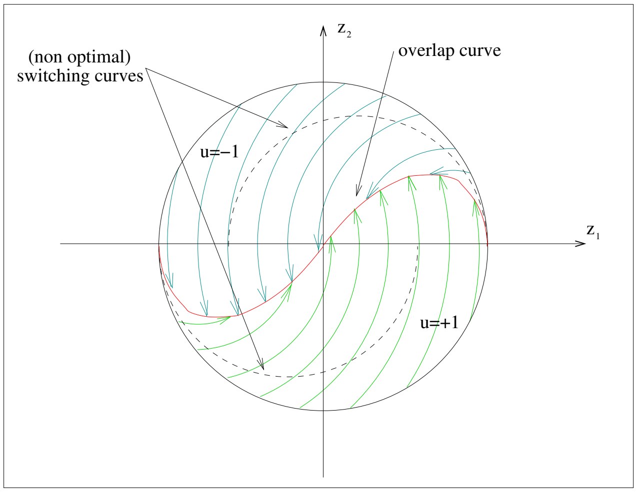

Fig. 4. Optimal synthesis for the linear pendulum

image by M_α of $(\Sigma)_\alpha$ (resp. $OF(\alpha, k_M\pi)$). Then, $(\widetilde{\Sigma})_\alpha$ is a perturbation by $\mathcal{O}(\alpha^2)$ of the forced linear pendulum

$$(Pen): \quad \begin{cases} \dot{z}_1 = -z_2, \\ \dot{z}_2 = z_1 + u, \qquad (z_1, z_2) \in \mathbf{R}^2, \quad |u| \le 1 \end{cases} \tag{14}$$

while $\widetilde{OF}(\alpha, k_M\pi)$ is a perturbation by $\mathcal{O}(\alpha^2)$ of $C(0, 2\bar{r})$, the planar circle of center $(0,0)$ and radius $2\bar{r}$. As a consequence, the candidate limit TOS inside V_α is the one associated to the problem of reaching in minimum time every point of the ball $B(0, 2\bar{r})$ starting from $C(0, 2\bar{r})$, along the dynamics of the standard linearized pendulum. To prove such a result, we first study the above mentioned optimal control problem and show that the corresponding TOS is characterized by an overlap curve γ^o_{pen}, which is the set of points $z \in \mathbf{R}^2$ with $z_1 z_2 \ge 0$ and belonging to the locus.

$$z_1^4 + z_2^4 + 2z_1^2 z_2^2 - 4\bar{r}^2 z_1^2 + (4 - 4\bar{r}^2) z_2^2 = 0\,.$$

The optimal synthesis inside $C(0, 2\bar{r})$ is then described by the following feedback, defined on $B(0, 2\bar{r}) \setminus \gamma^o_{pen}$: "above" γ^o_{pen}, the control u is constantly equal to -1 and "below" γ^o_{pen}, it is constantly equal to 1 (see Fig. 4). Finally, the asymptotic result we have in this case is:

Theorem 1: For $\bar{r} \in (0,1)$, let $(\alpha_k)_{k\ge1}$ be the sequence defined by $\alpha_k := \frac{\pi}{2(k+\bar{r})}$ for $k \ge 1$. Consider γ^o_{pen}, the overlap curve of the TOS for the optimal control problem consisting of starting from $C(0, 2\bar{r})$, the planar circle of center $(0,0)$ and radius $2\bar{r}$, and reaching in minimum time every point of $B(0, 2\bar{r})$ along the control system (14). Then, for k large enough, the TOS associated to $(\widetilde{\Sigma})_{\alpha_k}$ inside $\widetilde{OF}(\alpha_k, k_M\pi)$ is characterized by an overlap curve $\gamma^o_{\alpha_k}$ so that the optimal feedback takes the value -1 "above" $\gamma^o_{\alpha_k}$, and the value 1 "below" $\gamma^o_{\alpha_k}$. Moreover, $\gamma^o_{\alpha_k}$ converges to γ^o_{pen} in the C^0 topology, uniformly with respect to $\bar{r}$ in any compact interval of $(0,1)$, as k goes to infinity.

Remark 2.2: Notice that the sequence $(\alpha_k)_{k\ge1}$ defined above has been chosen in order to simplify the previous statement. Indeed the same result could be restated in a more general way by taking an arbitrary sequence $(\tilde{\alpha}_k)_{k\ge1}$ converging to zero and such that $r(\tilde{\alpha}_k)$ converges to $\bar{r}$, or letting the remainder vary on a compact subinterval of $(0,1)$.

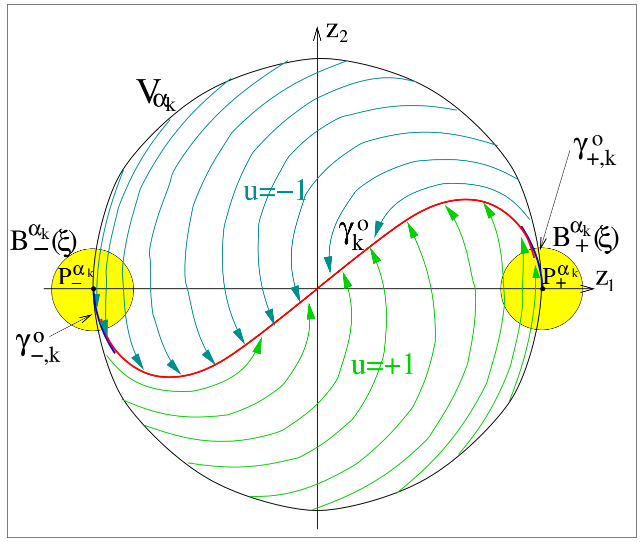

Fig. 5. Propositions 2 and 3

The complete proof of Theorem 1 is given in [17]. Here we just outline its main steps.

We start by defining V_α as the image with respect to M_α of the neighborhood of the south pole enclosed by $OF(\alpha, k_M\pi)$. First of all, by studying its asymptotic expression, it is possible to prove that the switching curve $C^\varepsilon_{k_M}$ is nowhere locally optimal, provided that α is small enough and $r(\alpha) \ge \bar{r} > 0$. This means that the shape of the synthesis inside V_α is completely determined if we identify the cut locus.

Let now $P^\alpha_\varepsilon := M_\alpha(C^\varepsilon_{k_M}(0))$ for $\varepsilon = \pm$ and let $B^\alpha_\varepsilon(\xi)$ be the ball of center P^α_ε and radius ξ. We look separately at the shape of the synthesis far from P^α_ε and inside neighborhoods $B^\alpha_\varepsilon(\xi)$ of P^α_ε, $\varepsilon = \pm$ (see Figure 5). Then Theorem 1 is obtained as a consequence of the following results:

Proposition 2: Let $\bar{r} \in (0,1)$ and $\alpha_k := \frac{\pi}{2(k+\bar{r})}$. Then for any $\xi > 0$ there exist a positive integer $\bar{k}$ and a compact interval $I \subset (0,\pi)$ such that it is possible to find a curve γ^o_k, defined on I for $k \ge \bar{k}$, verifying the following: γ^o_k divides $V_{\alpha_k} \setminus (B^{\alpha_k}_+(\xi) \cup B^{\alpha_k}_-(\xi))$ in two connected components, such that above γ^o_k the optimal feedback associated to the synthesis for $\alpha = \alpha_k$ takes the value -1, and below γ^o_k it is equal to 1, and in particular γ^o_k is an overlap curve for $\alpha = \alpha_k$. Moreover, γ^o_k converges to γ^o_{pen} in the C^0 topology of I.

Proposition 3: Consider the notations defined above. Then there exist $\xi > 0$, τ_ε, $\varepsilon = \pm$, with $0 < \tau_- < \tau_+ < \pi$ and a positive integer $\bar{k}$ such that, for every $k \ge \bar{k}$, it is possible to find two curves $\gamma^o_{-,k}$ and $\gamma^o_{+,k}$, defined respectively on $[0, \tau_-]$ and $[\tau_+, \pi]$, verifying the following: $\gamma^o_{\varepsilon,k}$ divides $V_{\alpha_k} \cap B^{\alpha_k}_\varepsilon(\xi)$ in two connected components, such that above $\gamma^o_{\varepsilon,k}$ the optimal feedback associated to the synthesis for $\alpha = \alpha_k$ takes the value -1, and below $\gamma^o_{\varepsilon,k}$ it is equal to 1, and in particular the $\gamma^o_{\varepsilon,k}$ are overlap curves for $\alpha = \alpha_k$. Moreover, $\gamma^o_{-,k}$ and $\gamma^o_{+,k}$ converge to γ^o_{pen} in the C^0 topology, respectively, of $[0, \tau_-]$ and $[\tau_+, \pi]$.

The proofs of the previous results rely on different implicit function arguments that, for reasons of space, we will not specify here.

Results analogous to Theorem 1 can be proved in the cases (C2) and (C3). Details are given in [17].

III. Conclusion

In this paper, we built the time optimal synthesis for a two-level quantum system driven by an external field, in the case $M << E$, where M is the bound on the fields and $-E$ and E are the two energy levels. We answered several questions stated in [6], [2], regarding the locus where extremals lose optimality and the shape of the synthesis at the south pole and in particular we showed that there are three main patterns which cyclically alternate as $M/E \to 0$. For these three cases it is possible to characterize a concept of "asymptotic" optimal synthesis in the "noncontrollability" limit $M/E \to 0$ (see in particular Theorem 1).

References

[1] R. Allen and J. H. Eberly, *Optical Resonance and Two-Level Atoms*. New York: Wiley, 1975.

[2] U. Boscain and P. Mason, "Time minimal trajectories for a spin $1/2$ particle in a magnetic field," *J. Math. Phys.*, vol. 47, no. 6, pp. 062 101, 29, 2006.

[3] R. Alicki and K. Lendi, *Quantum dynamical semigroups and applications*, ser. Lecture Notes in Physics. Berlin: Springer-Verlag, 1987, vol. 286.

[4] G. Lindblad, "On the generators of quantum dynamical semigroups," *Comm. Math. Phys.*, vol. 48, no. 2, pp. 119–130, 1976.

[5] M. A. Nielsen and I. L. Chuang, *Quantum computation and quantum information*. Cambridge: Cambridge University Press, 2000.

[6] U. Boscain and Y. Chitour, "Time-optimal synthesis for left-invariant control systems on $\mathrm{SO}(3)$," *SIAM J. Control Optim.*, vol. 44, no. 1, pp. 111–139 (electronic), 2005.

[7] U. Boscain and B. Piccoli, "Extremal synthesis for generic planar systems," *J. Dynam. Control Systems*, vol. 7, no. 2, pp. 209–258, 2001.

[8] ——, "On automaton recognizability of abnormal extremals," *SIAM J. Control Optim.*, vol. 40, no. 5, pp. 1333–1357 (electronic), 2002.

[9] A. Bressan and B. Piccoli, "Structural stability for time-optimal planar syntheses," *Dynam. Contin. Discrete Impuls. Systems*, vol. 3, no. 3, pp. 335–371, 1997.

[10] ——, "A generic classification of time-optimal planar stabilizing feedbacks," *SIAM J. Control Optim.*, vol. 36, no. 1, pp. 12–32 (electronic), 1998.

[11] B. Piccoli, "Regular time-optimal syntheses for smooth planar systems," *Rend. Sem. Mat. Univ. Padova*, vol. 95, pp. 59–79, 1996.

[12] ——, "Classification of generic singularities for the planar time-optimal synthesis," *SIAM J. Control Optim.*, vol. 34, no. 6, pp. 1914–1946, 1996.

[13] H. J. Sussmann, "The structure of time-optimal trajectories for single-input systems in the plane: the general real analytic case," *SIAM J. Control Optim.*, vol. 25, no. 4, pp. 868–904, 1987.

[14] ——, "Regular synthesis for time-optimal control of single-input real analytic systems in the plane," *SIAM J. Control Optim.*, vol. 25, no. 5, pp. 1145–1162, 1987.

[15] U. Boscain and B. Piccoli, *Optimal syntheses for control systems on 2-D manifolds*, ser. Mathématiques & Applications (Berlin) [Mathematics & Applications]. Berlin: Springer-Verlag, 2004, vol. 43.

[16] A. A. Agrachëv and R. V. Gamkrelidze, "Symplectic geometry for optimal control," in *Nonlinear controllability and optimal control*, ser. Monogr. Textbooks Pure Appl. Math. New York: Dekker, 1990, vol. 133, pp. 263–277.

[17] P. Mason, R. Salmoni, U. Boscain, and Y. Chitour, "Limit time optimal synthesis for a control-affine system on S^2," *SIAM J. Control Optim.*, vol. 47, no. 1, pp. 111–143, 2008.