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An iterative algorithm to solve periodic Riccati differential equations with an indefinite quadratic term *

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Abstract

An iterative algorithm to solve periodic Riccati differential equations (PRDE) with an indefinite quadratic term is proposed. In our algorithm, we replace the problem of solving a PRDE with an indefinite quadratic term by the problem of solving a sequence of PRDEs with a negative semidefinite quadratic term which can be solved by existing methods. The global convergence is guaranteed and a proof is given.

1. Introduction

We consider the following PRDE which arises from H_∞ control

$$-\dot{\Pi}(t) = A(t)^T\Pi(t) + \Pi(t)A(t) - \Pi(t)(B_2(t)B_2(t)^T - B_1(t)B_1(t)^T)\Pi(t) + C(t)^TC(t), \quad (1)$$

where $A : \mathbb{R}^+ \to \mathbb{R}^{n\times n}$, $B_1 : \mathbb{R}^+ \to \mathbb{R}^{n\times p}$, $B_2 : \mathbb{R}^+ \to \mathbb{R}^{n\times q}$, $C : \mathbb{R}^+ \to \mathbb{R}^{r\times n}$ are piecewise continuous, locally integrable, $T-$periodic functions and $\Pi : \mathbb{R}^+ \to \mathbb{R}^{n\times n}$ is the unique bounded symmetric positive semidefinite $T-$periodic stabilizing solution we seek. Our interest is in providing a new type of solution algorithm, built on recent developments for solving algebraic Riccati equations (AREs) with an indefinite quadratic term (see [1,2]).

Recall that in [1,2], the problem of solving an ARE with an indefinite quadratic term is replaced by the problem of solving a sequence of AREs with a negative semidefinite quadratic term and each of them can be solved by some existing algorithms (for example the Kleinman algorithm in [3]); then the solution of the original ARE can be approximated by the sum of the solutions of the AREs with a negative semidefinite term. Since AREs can be regarded as a special class of PRDEs, a natural question arising here is:"can we approximate the solution of an PRDE with a sign indefinite quadratic term by a sequence of PRDEs with a negative semidefinite quadratic term?" The answer is positive and an iterative algorithm to solve PRDE (1) will be given in this paper.

A key motivation of this paper comes from an increased interest in addressing periodic control problems for linear time-varying periodic systems. For linear time-invariant (LTI) control systems, to obtain an H_2 or an H_∞ controller, one needs to solve AREs and many algorithms are available. For linear time-varying periodic systems, in order to obtain an H_2 or an H_∞ controller, one needs to solve PRDEs which are typically more difficult to solve than AREs. Algorithms do exist, though their state of maturity is not equivalent to those for AREs. Roughly speaking, there are three categories of algorithms for solving PRDEs:

- *Hamiltonian methods.* This type of methods is based on the transformation from a PRDE into a Hamiltonian differential equation which is a linear differential equation; then the solution of the PRDE can be recovered from the solution of the Hamiltonian differential equation.
- *Structure exploiting methods.* In this type of methods, the linear continuous time-varying periodic system corresponding to the given PRDE is firstly approximated by a periodic discrete-time system; then the lifting in [15] is used to build a time-invariant descriptor system which is equivalent to the periodic discrete-time system. Finally, the structure of the lifted system matrix is exploited to obtain solutions of PRDEs.
- *Structure preserving methods.* Similar to the structure exploiting methods, a time-invariant descriptor system is obtained by lifting; then orthogonal matrix transformations are carried on this descriptor system to obtain solutions of PRDEs. In this type of methods, the structure of the lifted system matrix is not only exploited, but also preserved during computation.

There are certain disadvantages among existing methods to solve PRDEs. When the Hamiltonian methods are used to

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solve PRDEs, numerical instability may happen due to the inversion of possibly ill-conditioned matrices (see [16]). When the one-shot method (which is one of the structure exploiting methods) in [9] is used to solve a PRDE, two ODEs with unstable dynamics must be solved, and therefore this method is unreliable for systems with large periods (see [9]). A MATLAB package called PERIODIC SYSTEMS Toolbox (see [10]) has been developed to solve PRDEs with some numerically stable structure-exploiting and structure-preserving algorithms; however, it can only be used to solve H_2-type PRDEs (i.e. PRDEs with a negative semidefinite quadratic term) (see [10]), not for $H_\infty-$type PRDEs which are of the form (1).

Recently, the multi-shot method (see [8, 12]) has been developed to solve PRDEs. The multi-shot method is based on discretization techniques, which turn the continuous-time problem into an equivalent discrete-time problem for which satisfactory numerical methods already exist. As noted in [9], the multi-shot mothod has some important characteristics: (i) The ODEs with unstable dynamics in solving PRDEs with the multi-shot method are solved over small fractions of the period and all these ODEs can be solved independently. (ii) Only one ODE in the multi-shot method must be solved in sequence, in contrast to the one-shot method where two ODEs dependent on each other must be solved. (iii) The system's periodicity is exploited. All these characteristics make it likely that the multi-shot method can provide more reliable solutions of PRDEs. However, although the multi-shot method can be used to solve both H_2-type PRDEs and $H_\infty-$type PRDEs, it appears not to be incorporated into standard packages.

In our proposed algorithm, the problem of solving a PRDE with a sign indefinite quadratic term is equivalently replaced by the problem of solving a sequence of PRDEs with a negative semidefinite quadratic term; then the solution of the original PRDE can be approximated by the solutions of PRDEs with a negative semidefinite quadratic term which can be obtained by existing methods (for example use the standard package in [10]). To this end, it is worth pointing out that one can also use an iterative algorithm to solve H_2-type PRDEs (see [11]). In [11], the solution of an PRDE can be recursively approximated by the solutions of a sequence of periodic Lyapunov differential equations (PLDE). In fact, the iterative algorithm to solve H_2-type PRDEs in [11] is closely linked to the Kleinman algorithm in [3] since monotonic non-increasing sequences are constructed to approximate the solutions of H_2-type AREs (PRDEs) in both of them. Furthermore, a common idea between the algorithm in [11] and the Kleinman algorithm is to transform a nonlinear algebraic/differential equation into a sequence of linear algebraic/differential equations which are more easily solved.

To this end, it might be interesting to compare the storage and computational complexity between the multi-shot method and our algorithm. Suppose that we are given an H_∞ periodic Riccati equation and we have replaced the task of solving this H_∞ equation by k H_2 periodic Riccati equations. In such a situation, after some calculations, we obtain that the total storage in using our algorithm to solve this H_∞ Riccati equation is

$$(k+1)O(Nn^2)+O(Nnq)+O(Nnm)+O(Nnr), \qquad (2)$$

where N is the sampling number and $n,\ m,\ q,\ r$ are dimensions of periodic matrices in the H_∞ Riccati equation. If one uses the multi-method in [8] to solve this H_∞ equation, the total storage needed is

$$2O(N(2n))+O(Nn^2), \qquad (3)$$

where $2O(N(2n))$ is the storage for step 1 and step 3 of the multi-shoot method and $O(Nn^2)$ is the storage in step 4 of the multi-shoot method. Comparing (2) and (3), one cannot draw a conclusion as to which storage is bigger. Now we compute the computational complexity of using the multi-shoot method in [8] to solve this equation, then compute the computational complexity of using our proposed algorithm to solve this equation. After some calculations, we obtain that the total computational complexity of using the multiple shoot method to solve a periodic Riccati equation is

$$N(2\frac{1}{3}n^3+O(n^2))+O(Nn^3)+N\frac{4}{3}n^3. \qquad (4)$$

Now we calculate the computational complexity of using our proposed algorithm to solve an H_∞ periodic Riccati equation. As indicated above, we assume that we can reduce this H_∞ Riccati equation to k H_2 periodic Riccati equations; then from [17] we obtain that the computational complexity of using so called "fast algorithms" to solve these k H_2 periodic Riccati equations is

$$k\left(O\left((N-1)(2n+m)^3\right)+O\left((2n+m)^3\right)\right). \qquad (5)$$

Comparing (4) with (5), one cannot actually draw a conclusion as to which algorithm is more efficient.

Based on our comparison results above, we can see that the benefit of our algorithm is not its efficiency to solve H_∞ periodic Riccati equations. However, it is expected that our algorithm has a high numerical reliability in solving H_∞ periodic Riccati equations and this point has been verified in the linear-invariant case of our algorithm. Recall example 3 in [2], which shows that we cannot obtain an accurate solution of this H_∞ algebraic Riccati equation by using certain standard algorithms; however, we can obtain an accurate solution of this equation by using our algorithm.

The paper is organized as follows: Section 2 gives some definitions and preliminary results. Section 3 presents our main result. Section 4 states the algorithm. Section 5 establishes our conclusion.

Notation: $\mathbb{R}^{n\times m}$ denotes the set of $n\times m$ real matrices; $\mathbb{R}^+$ denotes the set of nonnegative real numbers; $\mathbb{Z}$ denotes the set of integers with $\mathbb{Z}_{\geq a}$ denoting the set of integers greater or equal to $a\in\mathbb{R}$. Define function spaces as follows:

$$\begin{aligned}\mathscr{U} &= \left\{u:\mathbb{R}^+\to\mathbb{R}^m \;\middle|\; \int_{t_0}^{t_1}\|u(t)\|^2dt<\infty \quad \forall t_0,\, t_1\in\mathbb{R}^+\right\},\\ \mathscr{Y} &= \left\{y:\mathbb{R}^+\to\mathbb{R}^r \;\middle|\; \int_{t_0}^{t_1}\|y(t)\|^2dt<\infty \quad \forall t_0,\, t_1\in\mathbb{R}^+\right\}.\end{aligned}$$

2. Definitions and Preliminary Results

In this section, we will give some definitions and preliminary results.

To motivate the definitions in this section, we firstly define the following periodic control system Δ.

$$\Delta:\mathscr{U}\to\mathscr{Y}$$

given by the following equations:

$$\begin{aligned}x(0) &= x_0\\ \dot{x}(t) &= A(t)x(t)+B(t)u(t)\\ y(t) &= C(t)x(t)\end{aligned}$$

where $t\in\mathbb{R}^+$, $x_0\in\mathbb{R}^n$ is the initial state, $u(t)\in\mathbb{R}^m$ is the input value, $x(t)\in\mathbb{R}^n$ is the state value, and $y(t)\in\mathbb{R}^p$ is the output value. $A(t)\in\mathbb{R}^{n\times n}$, $B(t)\in\mathbb{R}^{n\times m}$, $C(t)\in\mathbb{R}^{r\times n}$ are piecewise continuous, locally integrable, $T-$periodic real matrices.

Definition 1 *[4] The system Δ is said to be* ***uniformly stabilizable*** *(respectively,* ***uniformly detectable****) if there exists $K:\mathbb{R}^+\to\mathbb{R}^{m\times n}$ (respectively, $L:\mathbb{R}^+\to\mathbb{R}^{n\times r}$) piecewise continuous and bounded on $\mathbb{R}$ such that the system $\dot{x}(t)=(A(t)-B(t)K(t))x(t)$ (respectively, $\dot{x}(t)=(A(t)-L(t)C(t))x(t)$) is exponentially stable.*

Definition 2 *Let $A,\ B_1,\ B_2,\ C$ be the matrix functions appearing in equation* (1)*. If there exists a bounded symmetric solution $\Pi(t)$ to PRDE* (1) *such that the system*

$$\dot{x}(t)=\left(A(t)+B_1(t)B_1(t)^T\Pi(t)-B_2(t)B_2(t)^T\Pi(t)\right)x(t)$$

is exponentially stable, then $\Pi(t)$ is called a ***stabilizing solution*** *of* (1).

Definition 3 *Let $A,\ B_1,\ B_2,\ C$ be the real matrix functions appearing in* (1)*. Suppose there exists a bounded symmetric positive semidefinite $T-$periodic stabilizing solution $\Pi(t)$ to* (1)*. Let $P:\mathbb{R}^+\to\mathbb{R}^{n\times n}$. Let $\hat{A}_P:\mathbb{R}^+\to\mathbb{R}^{n\times n}$ be defined as*

$$\hat{A}_P(t) \;=\; A(t)+B_1(t)B_1(t)^TP(t)-B_2(t)B_2(t)^TP(t)$$

for all $t\in\mathbb{R}^+$, and let $\bar{A}_P:\mathbb{R}^+\to\ \mathbb{R}^{n\times n}$ be defined as

$$\bar{A}_P(t) \;=\; A(t)+B_1(t)B_1^T(t)P(t)-B_2(t)B_2(t)^T\Pi(t)$$

for all $t\in\mathbb{R}^+$.

We now set up some lemmas which are in parallel with the lemmas appearing in [1, 2, 13, 14].

The next lemma establishes some relations that will be very useful in the proof of the main theorem.

Lemma 4 *Let $A,\ B_1,\ B_2,\ C$ be the real matrix functions appearing in* (1)*, and let M be the set of smooth mappings from $\mathbb{R}^+$ to $\mathbb{R}^{n\times n}$ and $P,\ Z\in M$. Define*

$$\begin{aligned}F:\quad & M\longrightarrow M \qquad (6)\\ P(t) &\longmapsto \dot{P}(t)+P(t)A(t)+A(t)^TP(t)-P(t)(B_2(t)\\ &\quad B_2(t)^T-B_1(t)B_1(t)^T)P(t)+C(t)^TC(t).\end{aligned}$$

If $P(t)=P(t)^T$ and $Z(t)=Z(t)^T$ for all $t\in\mathbb{R}^+$, then

$$\begin{aligned}&F\left(P(t)+Z(t)\right)=F\left(P(t)\right)+\dot{Z}(t)+Z(t)\hat{A}_P(t)+\hat{A}_P(t)^TZ(t)\\ &\quad -Z(t)\left(B_2(t)B_2(t)^T-B_1(t)B_1(t)^T\right)Z(t) \qquad (7)\end{aligned}$$

for all $t\in\mathbb{R}^+$, where $\hat{A}_P(t)$ is defined in Defintion 3. Furthermore, if $P(t)=P(t)^T$ and $Z(t)=Z(t)^T$ for all $t\in\mathbb{R}^+$ and they satisfy

$$\begin{aligned}0=&\dot{Z}(t)+Z(t)\hat{A}_P(t)+\hat{A}_P(t)^TZ(t)-Z(t)B_2(t)B_2(t)^TZ(t)\\ &+F\left(P(t)\right) \qquad (8)\end{aligned}$$

for all $t\in\mathbb{R}^+$, then

$$F\left(P(t)+Z(t)\right)=Z(t)B_1(t)B_1(t)^TZ(t) \qquad (9)$$

for all $t\in\mathbb{R}^+$.

Proof: The results can be obtained by direct computations. □

The next two lemmas (Lemma 5-Lemma 6) are known general results. The first of these gives an existence result for the positive semidefinite $T-$periodic stabilizing solutions of a class of PRDEs.

Lemma 5 *[5, 6, 8] Consider the system Δ, and assume that (A,B) is uniformly stabilizable and (A,C) is uniformly detectable. Then, there exists a symmetric positive semidefinite $T-$periodic stabilizing solution $Z(t)$ satisfying the following PRDE*

$$\begin{aligned}-\dot{Z}(t)=&A(t)^TZ(t)+Z(t)A(t)-Z(t)B(t)B(t)^TZ(t)\\ &+C(t)^TC(t). \qquad (10)\end{aligned}$$

Furthermore, $Z(t)$ is the unique stabilizing solution of (10) *(i.e. there is no other stabilizing solution to* (10)*).*

Proof: See [5]. □

The next lemma gives a uniqueness result regarding the bounded stabilizing solution of (1).

Lemma 6 *Suppose there exists a bounded symmetric stabilizing solution* $\Pi(t)$ *to* (1)*; then this solution must be the unique stabilizing solution to* (1) *(i.e. there is no other stabilizing solution to* (1)*) and it is* $T-$*periodic. Furthermore, if* $\Pi(t) \geq 0$ *for all* $t \in \mathbb{R}^+$*, then the system* $\dot{x}(t) = \big(A(t) - B_2(t)B_2(t)^T\Pi(t)\big)x(t)$ *is exponentially stable.*

Proof: See [19]. □

The next lemma sets up some basic relationships between the stabilizing solution $\Pi(t)$ to equation (1) when it exists and the matrix functions P, Z satisfying equation (8).

Lemma 7 *Let* $A,\ B_1,\ B_2,\ C$ *be the matrix functions appearing in* (1)*,* $P(t) = P(t)^T \in \mathbb{R}^{n\times n}$ *for all* $t \in \mathbb{R}^+$ *and* $Z(t) = Z(t)^T \in \mathbb{R}^{n\times n}$ *for all* $t \in \mathbb{R}^+$ *satisfying equation* (8)*, and a bounded stabilizing* $\Pi(t) = \Pi(t)^T \in \mathbb{R}^{n\times n}$ *for all* $t \in \mathbb{R}^+$ *satisfying equation* (1)*, and let* $\bar{A}_P$ *be the function defined in Definition 3. Then*

(i) $\Pi(t) \geq (P(t)+Z(t))$ *for all* $t \in \mathbb{R}^+$ *if* $\bar{A}_P(t)$ *is exponentially stable,*

(ii) $\bar{A}_{P+Z}(t)$ *is exponentially stable if* $\Pi(t) \geq (P(t)+Z(t))$ *for all* $t \in \mathbb{R}^+$.

Proof: The proof can be given by using Lemma 4 and some existing results and it is omitted due to space limitations. □

3. Main Result

In this section, we set up the main theorem by constructing two positive semidefinite matrix series $P_k(t)$ and $Z_k(t)$, and we also prove that the series $P_k(t)$ is monotonically non-decreasing and converges to the unique bounded symmetric positive semidefinite $T-$periodic stabilizing solution $\Pi(t)$ of PRDE (1) if such a solution exists.

Theorem 8 *Let* $A,\ B_1,\ B_2,\ C$ *be the real matrix functions appearing in* (1)*. Suppose that* (C,A) *is uniformly detectable and* (A,B_2) *is uniformly stabilizable, and define* $F: M \longrightarrow M$ *as in* (6)*. Suppose there exists a bounded symmetric positive semidefinite* $T-$*periodic stabilizing solution* $\Pi(t)$ *of PRDE* (1)*.*
Then

(I) two square matrix function series $Z_k(t)$ *and* $P_k(t)$ *can be defined for all* $k \in \mathbb{Z}_{\geq 0}$ *recursively as follows:*

$$P_0(t) = 0 \quad \forall t \in \mathbb{R}^+, \tag{11}$$

$$A_k(t) = A(t) + B_1(t)B_1(t)^T P_k(t) - B_2(t)B_2(t)^T P_k(t) \quad \forall t \in \mathbb{R}^+, \tag{12}$$

$Z_k(t) \geq 0$ *is the unique* $T-$*periodic stabilizing solution of*

$$-\dot{Z}_k(t) = Z_k(t)A_k(t) + A_k(t)^T Z_k(t) - Z_k(t)B_2(t)B_2(t)^T Z_k(t) + F(P_k(t)), \tag{13}$$

and then

$$P_{k+1}(t) = P_k(t) + Z_k(t) \quad \forall t \in \mathbb{R}^+; \tag{14}$$

(II) the two series $P_k(t)$ *and* $Z_k(t)$ *in part (I) have the following properties:*

1) $(A(t) + B_1(t)B_1(t)^T P_k(t), B_2(t))$ *is uniformly stabilizable* $\forall k \in \mathbb{Z}_{\geq 0}$*,*

2) $F(P_{k+1}(t)) = Z_k(t)B_1(t)B_1(t)^T Z_k(t)\ \forall k \in \mathbb{Z}_{\geq 0}\ \forall t \in \mathbb{R}^+$*,*

3) $A(t) + B_1(t)B_1(t)^T P_k(t) - B_2(t)B_2(t)^T P_{k+1}(t)$ *is exponentially stable* $\forall k \in \mathbb{Z}_{\geq 0}$*,*

4) $\Pi(t) \geq P_{k+1}(t) \geq P_k(t) \geq 0\ \forall k \in \mathbb{Z}_{\geq 0}\ \forall t \in \mathbb{R}^+$*;*

(III) the limit

$$P_\infty(t) := \lim_{k\to\infty} P_k(t)$$

exists for all $t \in \mathbb{R}^+$ *with* $P_\infty(t) \geq 0$ *for all* $t \in \mathbb{R}^+$*. Furthermore,* $P_\infty(t) = \Pi(t)$ *is the unique* $T-$*periodic stabilizing solution of PRDE* (1)*, which is also positive semidefinite.*

Proof: We construct the series for $Z_k(t)$ and $P_k(t)$ to show results (I) and (II) together by an inductive argument. Firstly we show that (I) and (II) are true when $k = 0$. Then, given $k \in \mathbb{Z}_{\geq 0}$ where (I) and (II) are satisfied, we will show that (I) and (II) are also satisfied for $k+1$.

Case $k = 0$

Since $P_0(t) = 0$ via (11), $(II1)$ is trivially satisfied by assumption. Since equation (13) reduces to

$$-\dot{Z}_0(t) = Z_0(t)A(t) + A(t)^T Z_0(t) - Z_0(t)B_2(t)B_2(t)^T Z_0(t) + C(t)^T C(t), \tag{15}$$

then by Lemma 5 that there exists a unique $T-$periodic positive semidefinite and stabilizing solution $Z_0(t)$ for equation (15); hence $Z_0(t) \geq 0$ for all $t \in \mathbb{R}^+$. Since $P_1(t) = P_0(t) + Z_0(t)$ for all $t \in \mathbb{R}^+$ via (14), then we have $F(P_1(t)) = Z_0(t)B_1(t)B_1(t)^T Z_0(t)$ for all $t \in \mathbb{R}^+$ by Lemma 4 and $(II2)$ is satisfied. Then by Lemma 5 $(A(t) - B_2(t)B_2(t)^T Z_0(t))$ is exponentially stable (since $Z_0(t)$ is the stabilizing solution of (15)), hence $(II3)$ is satisfied on noting that $P_0(t) = 0$ and $P_1(t) = Z_0(t)$ for all $t \in \mathbb{R}^+$. We can show $(II4)$ is satisfied by the following steps:

1. Since $Z_0(t) \geq 0$ for all $t \in \mathbb{R}^+$ and $P_1(t) = P_0(t) + Z_0(t)$ for all $t \in \mathbb{R}^+$, then $P_1(t) \geq P_0(t)$ for all $t \in \mathbb{R}^+$;
2. Since $P_0(t) = 0$ for all $t \in \mathbb{R}^+$, $(A(t) + B_1(t)B_1(t)^T P_0(t) - B_2(t)B_2(t)^T \Pi(t)) = (A(t) - B_2(t)B_2(t)^T \Pi(t))$ is exponentially stable (see Lemma 6);
3. Since $(A(t) + B_1(t)B_1(t)^T P_0(t) - B_2(t)B_2(t)^T \Pi(t))$ is exponentially stable, then $\Pi(t) \geq (P_0(t) + Z_0(t)) = P_1(t)$ for all $t \in \mathbb{R}^+$ by Lemma 7.

Inductive step for $k \in \mathbb{Z}_{\geq 0}$

We now consider an arbitrary $q \in \mathbb{Z}_{\geq 0}$, suppose that (I) and (II) are satisfied for $k = q \in \mathbb{Z}_{\geq 0}$, and show that (I) and (II) are also satisfied for $k = q+1$. Since $F(P_{q+1}(t)) = Z_q(t)B_1(t)B_1(t)^T Z_q(t) \geq 0$ for all $t \in \mathbb{R}^+$ by inductive assumption $(II2)$, sufficient conditions for the existence of a unique positive semidefinite $T-$periodic stabilizing solution $Z_{q+1}(t)$ to (13) are (see Lemma 5):

(α) (A_{q+1}, B_2) is uniformly stabilizable;

(β) $(B_1^T Z_q, A_{q+1})$ is uniformly stabilizable.

Since $A(t) + B_1(t)B_1(t)^T P_{q+1}(t) = A_{q+1}(t) + B_2(t)B_2(t)^T P_{q+1}(t)$ and $A(t) + B_1(t)B_1(t)^T P_q(t) - B_2(t)B_2(t)^T P_{q+1}(t) = A_{q+1}(t) - B_1(t)B_1(t)^T Z_q(t)$ for all $t \in \mathbb{R}^+$, conditions (α) and (β) are clearly equivalent to the following two conditions respectively:

$(\alpha 1)$ $(A + B_1 B_1^T P_{q+1}, B_2)$ is uniformly stabilizable;

$(\beta 1)$ $(B_1^T Z_q, A + B_1 B_1^T P_q - B_2 B_2^T P_{q+1})$ is uniformly detectable.

We will now show the existence of $Z_{q+1}(t)$ is guaranteed via the following two points:

1. Since result $(II4)$ holds by inductive assumptions, we have $\Pi(t) \geq P_{q+1}(t)$ for all $t \in \mathbb{R}^+$, and thus $(A(t) + B_1(t)B_1(t)^T P_{q+1}(t) - B_2(t)B_2(t)^T \Pi(t))$ is exponentially stable by Lemma 7 Part (ii). Hence $(A + B_1 B_1^T P_{q+1}, B_2)$ is uniformly stabilizable and thus condition $(\alpha 1)$ and result $(II1)$ for $k = q+1$ are satisfied;
2. Since $(A + B_1 B_1^T P_q - B_2 B_2^T P_{q+1})$ is exponentially stable by inductive assumption $(II3)$, condition $(\beta 1)$ is also satisfied.

Since conditions $(\alpha 1)$ and $(\beta 1)$ hold, there exists a unique positive semidefinite $T-$periodic stabilizing solution $Z_{q+1}(t)$ for equation (13) with $k = q+1$. Since $P_{q+2}(t) = P_{q+1}(t) + Z_{q+1}(t)$ for all $t \in \mathbb{R}^+$, $(II2)$ is trivially satisfied for $k = q+1$ via Lemma 4. Since $Z_{q+1}(t)$ is the stabilizing solution to (13), it follows that $(A_{q+1}(t) - B_2(t)B_2(t)^T Z_{q+1}(t)) = (A(t) + B_1(t)B_1(t)^T P_{q+1}(t) - B_2(t)B_2(t)^T P_{q+2}(t))$ is exponentially stable, hence $(II3)$ is satisfied when $k = q+1$. Since $P_{q+2}(t) = P_{q+1}(t) + Z_{q+1}(t)$ and $Z_{q+1}(t) \geq 0$ for all $t \in \mathbb{R}^+$, $P_{q+2}(t) \geq P_{q+1}(t)$ for all $t \in \mathbb{R}^+$. Also, since $\Pi(t) \geq P_{q+1}(t) = P_q(t) + Z_q(t)$ for all $t \in \mathbb{R}^+$ by inductive assumption $(II4)$ for $q \in \mathbb{Z}_{\geq 0}$, it follows that $(A(t) + B_1(t)B_1(t)^T P_{q+1}(t) - B_2(t)B_2(t)^T \Pi(t))$ is exponentially stable via Lemma 7 Part (ii) and this in turn gives $\Pi(t) \geq P_{q+2}(t)$ for all $t \in \mathbb{R}^+$ via Lemma 7 Part (i). Hence $(II4)$ is satisfied for $k = q+1$.

Inductive Conclusion

The case for $k = 0$ and the inductive step establish that (I) and (II) are true $\forall k \in \mathbb{Z}_{\geq 0}$, so the proof for (I) and (II) is completed.

(III) Since the sequence $P_k(t)$ is monotone (i.e. $P_{k+1}(t) \geq P_k(t)$) for all $t \in \mathbb{R}^+$ and bounded above by $\Pi(t)$ (i.e. $\Pi(t) \geq P_k(t)$ for all $t \in \mathbb{R}^+$), the sequence converges to a limit $P_\infty(t)$ (see pp. 33-34 in [7] for the details), and convergence of the sequence $P_k(t)$ to $P_\infty(t)$ implies convergence of $Z_k(t)$ to 0 since

$$Z_\infty(t) := \lim_{k \to \infty} Z_k(t) = \lim_{k \to \infty} (P_{k+1}(t) - P_k(t)) = 0.$$

Then it is clear from $(II4)$ that $P_\infty(t) \geq 0$ for all $t \in \mathbb{R}^+$, from $(II2)$ that $P_\infty(t)$ must satisfy $F(P_\infty(t)) = 0$ for all $t \in \mathbb{R}^+$, and from $(II3)$ that $(A(t) + B_1(t)B_1(t)^T P_\infty(t) - B_2(t)B_2(t)^T P_\infty(t))$ must be exponentially stable. Thus $P_\infty(t) \geq 0$ is a stabilizing solution to $F(P_\infty(t)) = 0$. Since $\Pi(t) \geq 0$ is a stabilizing solution to $F(\Pi(t)) = 0$ and the stabilizing solution of PRDE (1) is always unique (see Lemma 6), it is clear that $P_\infty(t) = \Pi(t)$ for all $t \in \mathbb{R}^+$. □

The following corollary gives a condition under which there does *not* exist a stabilizing solution $\Pi(t) \geq 0$ to $F(\Pi(t)) = 0$. This is useful for terminating the recursion in finite iterations.

Corollary 9 *Let A, B_1, B_2, C be the functions appearing in* (1). *Suppose (C, A) is uniformly detectable and (A, B_2) is uniformly stabilizable, and let $\{P_k(t)\}$ and $F : M \longrightarrow M$ be defined as in Theorem 8. If $\exists k \in \mathbb{Z}_{\geq 0}$ such that $(A + B_1 B_1^T P_k, B_2)$ is not uniformly stabilizable, then there does not exist a stabilizing solution $\Pi(t) \geq 0$ to $F(\Pi(t)) = 0$.*

Proof: Restatement of Theorem 8, implication $(II1)$. □

4. Algorithm

Let A, B_1, B_2, C be the real matrix functions appearing in (1). Suppose (C, A) is uniformly detectable and (A, B_2) is uniformly stabilizable, an iterative algorithm for finding the bounded symmetric positive semidefinite $T-$periodic stabilizing solution of equation (1) is given as follows:

1. Let $P_0(t) = 0$ and $k = 0$.

2. $A_k(t) = A(t) + B_1(t)B_1(t)^T P_k(t) - B_2(t)B_2(t)^T P_k(t)$.

3. Construct (for example use the MATLAB package in [10]) the unique real symmetric $T-$periodic stabilizing solution $Z_k(t) \geq 0$ which satisfies

$$\begin{aligned} -\dot{Z}_k(t) =& Z_k(t)A_k(t) + A_k(t)^T Z_k(t) - Z_k(t)B_2(t)B_2(t)^T Z_k(t) \\ &+ F(P_k(t)), \end{aligned} \tag{16}$$

where $F(P_k(t))$ is given in (6).

4. Set $P_{k+1}(t) = P_k(t) + Z_k(t)$.

5. If $\sup_{t\in\mathbb{R}^+} \|Z_k(t)\| < \varepsilon$ where ε is a prescribed tolerance, then set $\Pi(t) = P_{k+1}(t)$ and exit. Otherwise, go to step 6.

6. If $(A + B_1 B_1^T P_{k+1}, B_2)$ is uniformly stabilizable, then increment k by 1 and go back to step 2. Otherwise, exit as there does not exist a real bounded symmetric stabilizing solution $\Pi(t) \geq 0$ satisfying (1).

5. Conclusion

In this paper, an iterative algorithm to compute the stabilizing solution of a PRDE with a sign indefinite quadratic term is given. By using our proposed algorithm, we can reduce a PRDE with a sign indefinite quadratic term to a sequence of PRDEs with a negative semidefinite quadratic term which can be solved by existing methods; then the stabilizing solution of the original PRDE can be approximated by the sum of the solutions of these PRDEs with a negative semidefinite quadratic term. Our algorithm has a quadratic rate of convergence and a natural game theoretic interpretation; these last two observations are not established here and will be published elsewhere. In a longer version of this work, we will also provide simulation data.

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