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An Alternative Approach to Designing Stabilizing Compensators for Saturating Linear Time-Invariant Plants

Sandip Roy, Yan Wan, Ali Saberi, and Babak Malek

Abstract— **We present a new methodology for designing low-gain linear time-invariant (LTI) controllers for semi-global stabilization of an LTI plant with actuator saturation, that is based on representation of a proper LTI feedback using a precompensator plus static-output-feedback architecture. We also mesh the new design methodology with time-scale notions to develop lower-order controllers for some plants.**

I. Introduction

Low-gain output feedback stabilization of linear time-invariant plants subject to actuator saturation has been achieved using the classical observer–followed–by–state–feedback controller architecture [1]–[3]. In this note, we discuss an alternative controller architecture for designing low-gain output feedback control of linear time-invariant (LTI) plants with saturating actuators. Specifically, we use a classical result of Ding and Pearson to show that a dynamic prefiltering together with static output feedback architecture can naturally yield a stabilizing low-gain controller under actuator saturation (Section 2). Subsequently, by using time-scale notions, we illustrate through a SISO example that lower-order controllers can be designed, in the case where the $j\omega$-axis eigenvalues of the plant are in fact at the origin (Section 3).

The reader may wonder what advantage the alternate architectures provide. Our particular motivation for developing the alternatives stems from our ongoing efforts on decentralized controller design, and in particular our effort to develop a low-gain methodology for decentralized plants [4]–[8]. In pursuing this goal, we have needed to use several novel controller structures, in particular ones that utilize pre-compensators and output derivatives together (see our works in [6]–[8]). This paper delineates the particular use of the new controller atchitectures in stabilization under saturation.

What this study of decentralized control makes clear is that freedoms in the structure of the controller facilitate design, because they can permit design that fit the structural limitations of the problem (in this case, decentralization). The alternatives to low-gain control that we propose here serve this purpose, because they naturally permit selection of a desirable controller architecture for the task at hand. While our primary motivation is in the decentralized controls arena, we believe that these alternate architectures may also be useful in such domains as adaptive control and plant inversion through lifting [7].

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All four authors are with the Department of Electrical Engineering, Washington State University, and can be reached at {sroy, ywan, saberi, bmalek}@eecs.wsu.edu.

II. Low-Gain Output Feedback Control through Precompensation

In this section, we demonstrate design of low-gain proper controllers for semi-global stabilization of LTI plants subject to actuator saturation, using a novel precompensator-based architecture. We also briefly discuss the connection of our design to the traditional observer-based design, and expose that the design is deeply related to a family of precompensator-based designs that also permit e.g. zero cancellation and relocation.

Formally, we demonstrate design of a proper output feedback compensator that achieves semi-global stabilization of the following plant $\mathcal{G}$:

$$\begin{aligned} \dot{\mathbf{x}} &= A\mathbf{x} + B\sigma(\mathbf{u}) \\ \mathbf{y} &= C\mathbf{x}, \end{aligned} \tag{1}$$

where $A \in R^{n\times n}$, $B \in R^{n\times m}$, $C \in R^{p\times n}$, $\sigma()$ is the standard saturation function, A has eigenvalues in the closed left-half-plane (CLHP), and the plant is observable and controllable[1]. Our design is fundamentally based on 1) positing a control architecture comprising a pre-compensator with a zero-free and uniform-rank structure together with a feedback of the output and its derivatives (see Figure 1), 2) designing the controller using this architecture, and 3) arguing that the designed controller admits a proper feedback implementation. This controller design directly builds on two early results: 1) Ding and Pearson's result [9] for pole-placement that is based on a dynamic pre-compensation + static feedback representation of a proper controller (Figure 1a and 1b); and 2) Lin and Saberi's effort [1] on stabilization under saturation using state feedback. For clarity, we cite the two results in the lemmas before we present our main result.

Lemma 1 is concerned with pre-compensator and feedback design for pole placement in a general LTI system, i.e. one of the form

$$\begin{aligned} \dot{\mathbf{x}} &= A\mathbf{x} + B\mathbf{u} \\ \mathbf{y} &= C\mathbf{x}, \end{aligned} \tag{2}$$

where $A \in R^{n\times n}$, $B \in R^{n\times m}$, and $C \in R^{p\times n}$.

Lemma 1: Consider a plant of the form (2) that is controllable and observable, with observability index v. Pre-compensation through addition of $v-1$ integrators to each

[1] In fact, the methods developed here trivially generalize to the case where the dynamics are stabilizable and detectable. We consider the observable and controllable case for the sake of clarity.

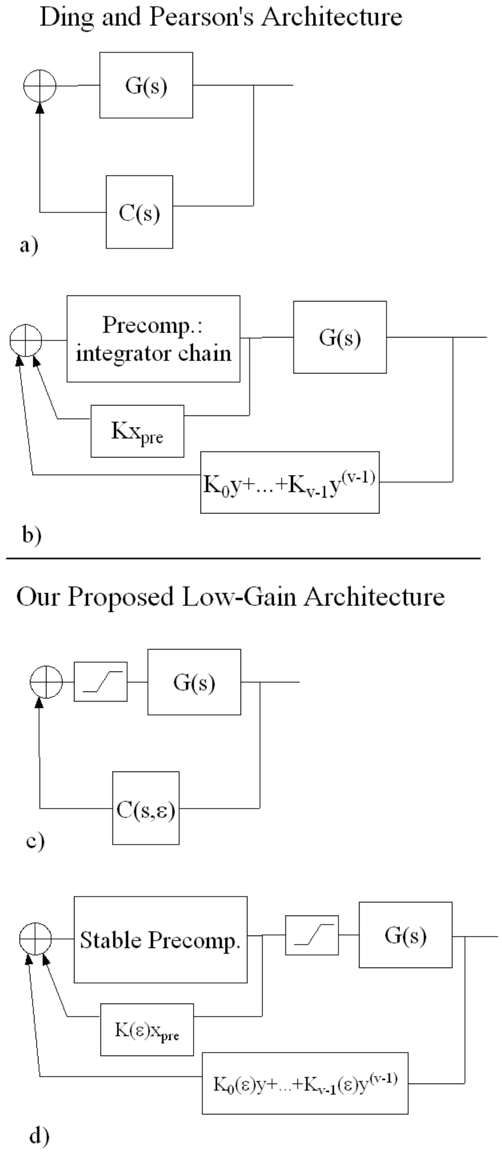

Fig. 1. Compensator artechitectures: a) and b) show the compensator artechitectures presented in Ding and Pearson [9], in particular, a precompensator-together-with-static feedback viewpoint. (b) is used to design a proper compensator of (a). c) and d) show the compensator artechitectures that stabilize a plant under input saturation.

plant input permits computation of the plant's state $\mathbf{x}$ *as a linear function of the plant's output* $\mathbf{y}$*, its derivatives up to* $\mathbf{y}^{(v)}$ *and the pre-compensator's state. A consequence of this computation capability is that it permits design of a proper feedback controller* $C(s)$ *that places the poles of the compensated plant at arbitrary locations (closed under conjugation).*

When the matrix C in the system (2) is not invertible, the classical method to obtain the state information from output is through observer design. This lemma of Ding and Pearson gives an alternative design for state estimation and feedback controller design, that is based on viewing the proper compensator $C(s)$ as a dynamic pre-compensation together with static feedback (Figure 1a and 1b). Specifically, the methodology of design is as follows: first, from the pre-compensator-based representation (Figure 1b), a computation of the plant state from the plant output and its derivatives together with the precompensator state can directly be obtained. Second, the classical state feedback methodology thus permits us to compute the static feedback in the pre-compensator-based representation, so as to place the closed-loop eigenvalues at desired locations. Third, the equivalence between the precompensator-based representation and a proper feedback controller is used to obtain a realization of the feedback control (Figure 1a). We kindly ask the reader to see [9], [10], both for the details of the state computation and the equivalence between the precompensator-based architecture and the proper feedback controller. In our development, we broadly replicate the design methodology of Ding and Pearson, but use a stable rather than neutral precompensator in order to obtain a controller that works under input saturation.

Lemma 2 is concerned with using linear state feedback control to semi-globally stabilize the plant:

$$\dot{\mathbf{x}} = A\mathbf{x} + B\sigma(\mathbf{u}), \tag{3}$$

where $A \in R^{n\times n}$, $B \in R^{n\times m}$, and $\sigma()$ is the standard saturation function. Please see Lin and Saberi's work [1] for the proof of the lemma.

Lemma 2: Consider a plant of the form (3) that satisfies two conditions: 1) all the eigenvalues of A *are located in the CLHP; 2) (A, B) is stabilizable. The plant can be semiglobally stabilized using linear static state feedback. That is, a parametrized family of compensators* $u = K(\epsilon)x$ *can be designed such that, for any specified ball of plant initial conditions* $\mathcal{W}$*, there exists* $\epsilon^*(\mathcal{W})$ *such that, for all* $0 < \epsilon \leq \epsilon^*(\mathcal{W})$*, the compensator* $K(\epsilon)$ *achieves local exponential stabilization and contains* $\mathcal{W}$ *in its domain of attraction.*

Now we are ready to present the main result. Specifically, the following theorem formalizes that a family of proper controllers can be designed for semi-global stabilization of $\mathcal{G}$, based on the precompensator-together-with-derivative-feedback architecture shown in Figure 1. The proof of the theorem makes clear the design methodology.

Theorem 1: The plant $\mathcal{G}$ (Equation 1) can be asymptotically semi-globally stabilized using proper feedback compensation of order mv, where v is the observability index of the plant. Specifically, a parametrized family of compensators $C(s,\epsilon)$ can be designed (Figure 1c) to achieve the following: for any specified ball of plant and compensator initial conditions $\mathcal{W}$, there exists $\epsilon^*(\mathcal{W})$ such that, for all $0 < \epsilon \leq \epsilon^*(\mathcal{W})$, $C(s,\epsilon)$ is locally exponentially stabilizing and contains $\mathcal{W}$ in its domain of attraction. The design can be achieved by developing a controller of the architecture shown in Figure 1d—i.e., comprising an m-input uniform-rank square-invertible zero-free precompensator P with input $\mathbf{u}_p$ together with a feedback of the form $\mathbf{u}_p = K_0(\epsilon)\mathbf{y} + K_1(\epsilon)\mathbf{y}^{(1)} + \ldots + K_{v-1}(\epsilon)\mathbf{y}^{(v-1)}$ (where $K_0(\epsilon), \ldots, K_{v-1}(\epsilon)$ are matrices of dimension $m \times p$)—and then constructing a proper implementation.

Proof:

We shall prove that, for the given ball of initial conditions, a family of proper compensaters $C(s,\epsilon)$ can be designed so that the actuator does not saturate, and further the closed-loop system without saturation is exponentially stable. We first note that, as long as the compensator permits a proper state-space implementation and the system operates in the linear regime, the additive contribution of the compensator's initial condition on the input can be made arbitrarily small through pre- and post-scaling of the compensator by a large gain and its inverse (see Figure 1). Thus, WLOG, we seek to verify that $||\mathbf{u}||_\infty < 1$ for the ball of initial states and assuming null compensator initial conditions. To do so, we will design a compensator of the architecture shown in Figure 1 that achieves the design goals, and then note a proper implementation.

To do this, let $\tilde{P}$ be any asymptotically stable LTI system of the following form:

$$\begin{bmatrix} \mathbf{y}_{\tilde{P}}^{(1)} \\ \mathbf{y}_{\tilde{P}}^{(2)} \\ \vdots \\ \mathbf{y}_{\tilde{P}}^{(v)} \end{bmatrix} = \begin{bmatrix} & I_m & & \\ & & \ddots & \\ & & & I_m \\ \tilde{Q}_0 & \dots & \dots & \tilde{Q}_{v-1} \end{bmatrix} \begin{bmatrix} \mathbf{y}_{\tilde{P}} \\ \mathbf{y}_{\tilde{P}}^{(1)} \\ \vdots \\ \mathbf{y}_{\tilde{P}}^{(v-1)} \end{bmatrix} + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ I_m \end{bmatrix} \mathbf{u}_{\tilde{P}}, \quad (4)$$

where $\mathbf{u}_{\tilde{P}} \in R^m$ and $\mathbf{y}_{\tilde{P}} \in R^m$ are the input and output to $\tilde{P}$. Notice that $\tilde{P}$ is square-invertible, zero-free, and uniform rank. Let us denote the ∞-norm gain of this plant as q.

Let us first consider pre-compensating the plant G using $\tilde{P}$, and then using feedback of the first v derivatives of the output along with the states of the precompensator (see Figure 1). That is, upon precompensation with $\tilde{P}$, we consider using a feedback controller of the form $\mathbf{u}_{\tilde{P}} = \sum_{i=0}^{v-1} K_i \mathbf{y}^{(i)} + \sum_{i=0}^{v-1} \tilde{K}_i \mathbf{y}_{\tilde{P}}^{(i)}$, where we have presciently used the notation K_i for the output-derivative feedbacks since these will turn out to be the gains in the conpensator diagrammed in Figure 1d, and where we suppress the dependence on ϵ in our notation for the sake of clarity. For convenience, let us define $K = \begin{bmatrix} K_0 & \dots & K_{v-1} \end{bmatrix}$, $\tilde{K} = \begin{bmatrix} \tilde{K}_0 & \dots & \tilde{K}_{v-1} \end{bmatrix}$, $\mathbf{y}(ext) = \begin{bmatrix} \mathbf{y} \\ \mathbf{y}^{(1)} \\ \vdots \\ \mathbf{y}^{(v-1)} \end{bmatrix}$, and $\mathbf{y}_{\tilde{P}}(ext) = \begin{bmatrix} \mathbf{y} \\ \mathbf{y}_{\tilde{P}}^{(1)} \\ \vdots \\ \mathbf{y}_{\tilde{P}}^{(v-1)} \end{bmatrix}$. In this notation, the controller becomes $\mathbf{u}_{\tilde{P}} = \begin{bmatrix} K & \tilde{K} \end{bmatrix} \begin{bmatrix} \mathbf{y}(ext) \\ \mathbf{y}_{\tilde{P}}(ext) \end{bmatrix}$.

We claim that such a controller can be designed, so that 1) the closed-loop system is asymptotically stable, 2) $||u_{\tilde{P}}||_\infty \leq \epsilon$ for the given ball of plant initial conditions and any $0 < \epsilon \leq \frac{0.9}{q}$, and 3) the controller gains K and $\tilde{K}$ are $\mathcal{O}(\epsilon)$. To see why, first note that, based on the fact that the relative degree of the precompensator equals the observability index, the state of the pre-compensated system $\widehat{\mathbf{x}} = \begin{bmatrix} \mathbf{x} \\ \mathbf{y}_{\tilde{P}}(ext) \end{bmatrix}$ is a linear function of $\begin{bmatrix} \mathbf{y}(ext) \\ \mathbf{y}_{\tilde{P}}(ext) \end{bmatrix}$. In particular, it is automatic that $\mathbf{x} = \begin{bmatrix} \mathbf{x} \\ \mathbf{y}_{\tilde{P}}(ext) \end{bmatrix} = Z \begin{bmatrix} \mathbf{y}(ext) \\ \mathbf{y}_{\tilde{P}}(ext) \end{bmatrix}$, where Z has the form $\begin{bmatrix} Z_1 & Z_2 \\ 0 & I \end{bmatrix}$, see Ding and Pearson's development [9] for the method of construction. Next, from Lemma 2, we see that a low-gain full state-feedback controller $\widehat{K}(\epsilon)$ of order ϵ can be developed for the precompensated plant, that stabilizes the plant and also makes the ∞-norm of the input less than ϵ for any $\epsilon > 0$, for the given ball of plant initial conditions. Thus, by applying the feedback $\widehat{K}(\epsilon) Z \begin{bmatrix} \mathbf{y}(ext) \\ \mathbf{y}_{\tilde{P}}(ext) \end{bmatrix}$, we can meet the three desired objectives.

It remains to be shown that the plant input $\mathbf{u}$ does not saturate upon application of this compensation. To do so, simply note that $||\mathbf{u}||_\infty \leq q||u_{\tilde{P}}||_\infty \leq 0.9$.

We can absorb the feedback of $\mathbf{y}_{\tilde{P}}(ext)$ into the pre-compensator, so that we obtain a control scheme comprising a precompensator P with dynamics

$$\begin{bmatrix} \mathbf{y}_P^{(1)} \\ \mathbf{y}_P^{(2)} \\ \vdots \\ \mathbf{y}_P^{(v)} \end{bmatrix} = \begin{bmatrix} & I_m & & \\ & & \ddots & \\ & & & I_m \\ \tilde{Q}_0 + \tilde{K}_0 & \dots & \dots & \tilde{Q}_{v-1} + \tilde{K}_{v-1} \end{bmatrix} \begin{bmatrix} \mathbf{y}_P \\ \mathbf{y}_P^{(1)} \\ \vdots \\ \mathbf{y}_P^{(v-1)} \end{bmatrix} + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ I_m \end{bmatrix} \mathbf{u}_P \quad (5)$$

together with feedback $\mathbf{u}_P = \sum_{i=0}^{v-1} K_i \mathbf{y}^{(i)}$.

Finally, exactly analogously to the design method in [9]. we see automatically that the transfer function from $\mathbf{y}$ to $\mathbf{u}$ is in fact proper, and so the design admits a proper state-space implementation. Through appropriate scaling of the compensator, we thus see that saturation is avoided for the ball of plant and compensator initial conditions, while the dynamics without saturation are exponentially stable. Thus, semi-global stabilization has been verified. ■

We have given an alternative low-gain controller design for semi-global stabilization under saturation. It is worth stressing that the crux of the design is the ability to construct the plant's full state using only output derivatives, upon adequate dynamic precompensation. Let us now briefly conceptualize the connections of our design to observer-based designs and zero-cancellation ideas.

Remark: By choosing the $\tilde{Q}_i$ appropriately, we can set the gain q of the precompensator $\tilde{P}$ to an arbitrary value. Appropriate selection of the precompensator can potentially facilitate selection of more numerically-stable feedback gains, by permitting a larger input prior to the precompensator. We leave a careful analysis to future work.

A. Conceptual Connection with Observer-Based Designs

We have used a precompensator-plus-static-output-feedback representation for a class of feedback controllers to design low-gain stabilizers for LTI plants with actuator saturation, as an alternative to the traditional observer-based architecture for design. The proof of Theorem 1 makes clear the essence of our approach for design: by viewing the controller as comprising an arbitrary dynamic precompensator together with static feedback, we see that the full state of an LTI system can be obtained as a *static* mapping of output derivatives together with the precompensator state variables. This observation yields a design strategy where a dynamic precompensator's impulse response is designed followed by low-gain static state feedback, with the goal of ensuring that the output of their cascade is small (for the given ball of initial conditions). This is a different viewpoint from the traditional one in limited-actuation output-feedback design [1], [2], [5], where actuation capabilities are divided between observation and state-feedback tasks. We believe that this alternative viewpoint can inform design in such domains as decentralized control.

III. A Compensator that Exploits Time-Scale Structure

Our philosophy for low-gain control using a precomensation-plus-feedback architecture also permits construction of stabilizers that exploit time-scale structure in the plant. Specifically, we here demonstrate design of precompensators for semi-global stabilization of the plant $\mathcal{G}$, that are generally lower-order than those in Section 2 because they exploit time-scale separation in the plant. Conceptually, when stabilization under saturation is the goal, low-gain state feedback only need be provided for the plant dynamics associated with jw-axis eigenvalues (see e.g. [5] for use of this idea in observer-based designs). In the case where these eigenvalues are at the origin, the corresponding dynamics are in fact the slow dynamics of the system. Thus, through time-scale separation, we can design precompensation together with feedback so as to stabilize the *slow* dynamics under actuator saturation. The use of time-scale separation ideas to reduce the pre-compensator's order becomes rather intricate, and so we illustrate the design only for SISO plants for the sake of clarity. We shall use standard singular-perturbation notions to prove the result. Here is a formal statement:

Theorem 2: Consider a plant $\mathcal{G}$ (as specified in Equation 1) that is SISO, and has q poles at the origin. This plant can be semi-globally stabilized under actuator saturation using a proper dynamic compensator of order q.

Proof:

We shall prove that, for any given ball of plant and controller initial conditions, there exists a proper compensator of order q such that 1) the linear dynamics of the closed-loop is stable and 2) actuator saturation does not occur.

To this end, let us begin by denoting the the transfer function of the plant's linear dynamics by $G(s)$. We note that the transfer function can be written as $G(s) = \frac{b_m s^m+\ldots+b_0}{s^q(s^{n-q}+a_{n-q-1}s^{n-q-1}+\ldots+a_0)}$, where m is the number of plant zeros. We find it easiest to conceptualize the compensator as comprising a zero-free dynamic precompensator of order q, together with a feedback of the output y and its first $q-1$ derivatives, as shown in Figure 1. We choose the precompensator to be *any* stable system of this form. We denote the precompensator's transfer function by $C_p(s) = \frac{1}{(s^q+c_{q-1}s^{q-1}+\ldots+c_0)}$, and the feedback controller by $K(s) = k_{q-1}s^{q-1}+\ldots+k_0$. We note the entire compensator $K(s)C_p(s)$ has order q and is proper.

With a little algebra, we find that the characteristic polynomial of the closed-loop system is $p(s) = s^q(s^{n-q} + a_{n-q-1}s^{n-q-1} + \ldots + a_0)(s^q + c_{q-1}s^{q-1} + \ldots + c_0) + (b_m s^m + \ldots + b_0)(k_{q-1}s^{q-1} + \ldots + k_0)$. We note that the polynomial $p_f(s) = (s^{n-q}+a_{n-q-1}s^{n-q-1}+\ldots+a_0)(s^q+c_{q-1}s^{q-1}+\ldots+c_0)$ has roots in the OLHP, by assumption, for the chosen stable pre-compensator $C_p(s)$.

Let us now consider a family of multiple derivative controllers, parameterized by a low-gain parameter $\epsilon > 0$. In particular, let us consider a controller with $k_i = \frac{a_0 c_0}{b_0}\gamma_i \epsilon^{q-i}$, where $s^q + \gamma_{q-1}s^{q-1} + \ldots + \gamma_0$ is a stable polynomial with roots $\lambda_1, \ldots, \lambda_q$, and ϵ is a low-gain parameter.

Next, we will verify that the characteristic polynomial has n roots that are within $\mathcal{O}(\epsilon)$ of the roots of $p_f(s)$, while the remaining q roots are within $\mathcal{O}(\epsilon^2)$ of $\epsilon\lambda_1, \ldots, \epsilon\lambda_q$. To prove this, notice first that $p(s) = s^q p_f(s) + (b_m s^m + \ldots + b_0)(\gamma_{q-1}\epsilon s^{q-1} + \ldots + \gamma_0\epsilon^q)\frac{a_0 c_0}{b_0}$. Noting that the entire second term in this expression is $\mathcal{O}(\epsilon)$, we see that the roots of $p(s)$ are $\mathcal{O}(\epsilon)$ perturbations of the roots of $s^q p_f(s)$. Thus, we see that n roots are within $\mathcal{O}(\epsilon)$ of the roots of $p_f(s)$, while the remaining are within $\mathcal{O}(\epsilon)$ of the origin.

To continue, let us consider the change of variables $\overline{s} = \frac{\epsilon}{s}$. Substituting into the closed-loop characteristic polynomial, we find that $p(\overline{s}) = (\frac{\epsilon}{\overline{s}})^q p_f(\overline{s}) + \epsilon^q(b_m\frac{\epsilon^m}{\overline{s}^m} + \ldots + b_0)(\frac{\gamma_{q-1}}{\overline{s}^{q-1}} + \ldots + \gamma_0)\frac{a_0 c_0}{b_0}$. Scaling the expression by $\frac{\overline{s}^{n+q}}{a_0 c_0}$, we obtain that the expression $p(\overline{s}) = 0$ is the following degree-$(n+q)$ polynomial equation in $\overline{s}$: $\epsilon^q(\gamma_0\overline{s}^{n+q} + \ldots + \gamma_{q-1}\overline{s}^{n+1}) + \epsilon^q\overline{s}^n + r(\overline{s}) = 0$, where $r(\overline{s})$ is a polynomial in $\overline{s}$ of degree no more that $\overline{s}^{n+q-1}$ with each term scaled by a coefficient of order ϵ^{q+1} or smaller. Thus, dividing by ϵ^q, we find that the solutions $\overline{s}$ to the equation are within $\mathcal{O}(\epsilon)$ of the solutions to $\gamma_0\overline{s}^{n+q} + \ldots + \gamma_{q-1}\overline{s}^{n+1} + \overline{s}^n = 0$. However, the roots of this equation are precisely $\frac{1}{\lambda_1}, \ldots, \frac{1}{\lambda_q}$, as well as 0 repeated n times. Noting that $s = \frac{\epsilon}{\overline{s}}$, we thus recover that q roots of the characteristic polynomial are within $\mathcal{O}(\epsilon^2)$ of $\epsilon\lambda_1, \ldots, \epsilon\lambda_q$. Thus, we have characterized all the poles of the closed-loop system. We notice that all the poles are guaranteed to be within the OLHP.

Now consider the response for a ball of initial conditions $\mathcal{W}$. As in the proof of Theorem 1, we notice that the initial state of the precompensator is of no concern in terms of causing saturation, since the precompensator can be pre- and post-scaled by an arbitrary positive constant. Thus, WLOG, let us consider selecting among the family of compensators, to avoid saturation for a given ball of plant initial conditions and assuming zero precompensator initial conditions. Through consideration of the closed-loop dynamics associated with the slow eigenvalues $(\epsilon\lambda_1, \ldots, \epsilon\lambda_q)$, we recover immediately (see the proof of Lemma 1 in [1]) that, for any specified ball of initial conditions, $||y^{(i)}(t)||_\infty$, is at most of order $\frac{1}{\epsilon^{q-1-i}}$ for $i = 1, \ldots, q-1$. Thus, from the expression for the feedback controller, we find that the maximum value of the precompensator input $\overline{u}$ is $\mathcal{O}(\epsilon)$, say $v_1\epsilon + \mathcal{O}(\epsilon^2)$ for the given ball of plant initial conditions. Furthermore, the stable precompensator imparts a finite gain, say v_2, so the maximum value of $u(t)$ is $v_1 v_2 \epsilon + \mathcal{O}(\epsilon^2)$. Thus, for any given ball of initial conditions, we can choose ϵ small enough so that actuator saturation does not occur. Since actuator saturation is avoided and the closed-loop poles are in the OLHP, stability is proved.

■

Conceptually, the reduction in the controller order permitted by Theorem 2 is founded on focusing the control effort on only the slow dynamics of the system. That is, the controller is designed only to place the eigenvalues at the origin at desired locations (that are linear with respect to the low-gain parameter ϵ); simply using small gains enforces that the remaining eigenvalues remain far in the OLHP. Thus, one only needs to add precompensation to permit estimation of the part of the state associated with the slow dynamics. In this way, stability can be guaranteed and saturation avoided, without requiring as much precompensation as would be needed to estimate the whole state.

We notice that the time-scale-based design is aligned with the broad philosophy of our alternative low-gain design, in the sense that it provides freedom in compensator design. In particular, as with the design in Section 2, we notice that *any* stable precompensator can be used for the time-scale-exploiting design, and further design of feedback component in the architecture only requires knowlege of the DC gain of the plant.

IV. EXAMPLE

In this example, we demonstrate the design of a low-gain controller that semi-globally stabilizes the following plant:

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -3 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \sigma(\mathbf{u}) \qquad (6)$$

$$\mathbf{y} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \mathbf{x}. \qquad (7)$$

Specifically, we show that the compensator of the form $\mathbf{u}(t)^{(v)} = \sum_0^{v-1} \mathbf{A}_i \mathbf{u}(t)^{(i)} + \sum_0^{(v-1)} \mathbf{B}_i \mathbf{y}(t)^{(i)}$, where v is the observability index of the plant, can stabilize the plant under saturation. As developed in the article, the design is achieved by first designing a pre-compensator together with output feedback control law, and then implementing the controller in the proper feedback representation above. We shall use the notation from the above development in our illustration.

Let us begin with the precompensator-plus-output-feedback design. To begin, we notice that the observability index of this system is 2. As per the proof of Theorem 1, let us thus choose $\tilde{P}$ to be

$$\begin{bmatrix} \dot{\mathbf{y}}_{\tilde{P}} \\ \ddot{\mathbf{y}}_{\tilde{P}} \end{bmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{I} & -\mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{y}_{\tilde{P}} \\ \dot{\mathbf{y}}_{\tilde{P}} \end{bmatrix} + \begin{bmatrix} \mathbf{0} \\ \mathbf{I} \end{bmatrix} \mathbf{u}_{\tilde{P}} \qquad (8)$$

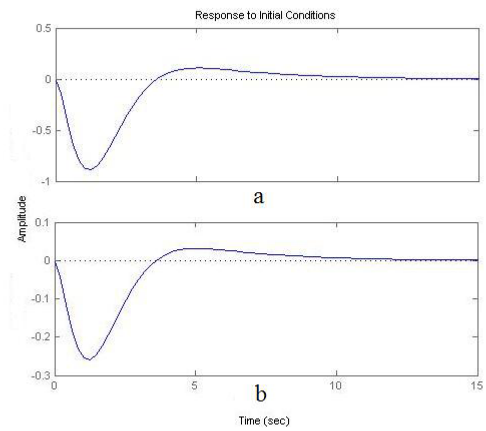

Fig. 2. The inputs a) u_1 and b) u_2 are shown, for $\epsilon = 0.5$. Each plant state variable is initialized at 0.99.

The eigenvalues of this system are located at $-1/2 \pm \sqrt{3}/2i$, hence it is clearly asymptotically stable.

Now let us pre-compensate the plant (Equation 6) using $\tilde{P}$ and design feedback of the form

$$\mathbf{u}_{\tilde{P}} = \tilde{K}_0(\epsilon)\mathbf{y}_{\tilde{P}} + \tilde{K}_1(\epsilon)\dot{\mathbf{y}}_{\tilde{P}} + K_0(\epsilon)\mathbf{y} + K_1(\epsilon)\dot{\mathbf{y}} \qquad (9)$$

To design the feedback, we first recover the state of the entire system including the pre-compensator from the outputs $\mathbf{y}_{\tilde{P}}, \dot{\mathbf{y}}_{\tilde{P}}, \mathbf{y}, \dot{\mathbf{y}}$ through linear transformation, and then apply the low-gain state feedback design (see [1]) to shift eigenvalues of the whole system left by $-\epsilon$. Doing so, we straightfowardly obtain $\tilde{K}_0(\epsilon)$, $\tilde{K}_1(\epsilon)$, $K_0(\epsilon)$, and $K_1(\epsilon)$, see [11] for the expressions.

Hence, the control scheme can be viewed as comprising a precompensator P:

$$\begin{bmatrix} \dot{\mathbf{y}}_P \\ \ddot{\mathbf{y}}_P \end{bmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{I} + \tilde{K}_0(\epsilon) & -\mathbf{I} + \tilde{K}_1(\epsilon) \end{bmatrix} \begin{bmatrix} \mathbf{y}_P \\ \dot{\mathbf{y}}_P \end{bmatrix} + \begin{bmatrix} \mathbf{0} \\ \mathbf{I} \end{bmatrix} \mathbf{u}_P \quad (10)$$

and the feedback flow $\mathbf{u}_P = K_0(\epsilon)\mathbf{y} + K_1(\epsilon)\dot{\mathbf{y}}$.

Finally, the above procedure leads to the proper feedback compensator design

$$\begin{aligned} \mathbf{u}^{(2)}(t) = & (-\mathbf{I} + \tilde{K}_0(\epsilon))\mathbf{u}(t) + (-\mathbf{I} + \tilde{K}_1(\epsilon))\mathbf{u}^{(1)}(t) \\ & + K_0(\epsilon)\mathbf{y}(t) + K_1(\epsilon)\mathbf{y}^{(1)}(t) \end{aligned} \qquad (11)$$

Now let us show how ϵ can be chosen. WLOG, let us assume that the precompensator initial conditions to nil, with the understanding that scaling of the precompensator (with appropriately revised proper implementation) permits design with non-zero compensator initial conditions. In particular, consider the case where initial conditions of the plant are in a ball $\mathcal{W}$ with infinity-norm radius 1, i.e., where each initial condition has a magnitude less than or equal to 1. We find that $\epsilon^*(\mathcal{W}) \approx 0.5$ through an exhaustive search. Thus ϵ can be chosen between 0 and 0.5. Trajectories of the two inputs are shown for an initial condition at the edge of the ball, for two different values of ϵ.

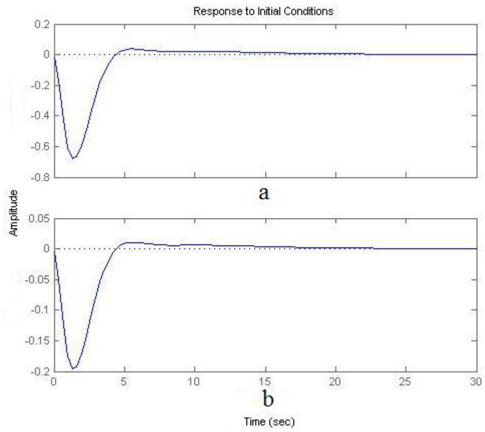

Fig. 3. The inputs a) u_1 and b) u_2 are shown, for $\epsilon = 0.25$. Each plant state variable is initialized at 0.99.

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