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Coprime Factor Model Reduction for Continuous-time Uncertain Systems

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Abstract— **The paper considers the problem of coprime factor model reduction for a class of continuous-time uncertain systems with structured norm bounded uncertainty. The proposed method is applicable to the uncertain systems which may be robustly unstable, overcoming the robust stability restriction in the balanced truncation approach. A systematic approach is presented to construct a contractive coprime factor for the underlying uncertain system, based on the use of LMIs. This enables the balanced truncation to be applied to the contractive coprime factor to obtain the reduced uncertain system. Error bound on the L_2-induced norm of the resulting coprime factor is derived.**

I. INTRODUCTION

In recent years, there has been growing interests in model reduction problems for uncertain systems. The balanced truncation method for discrete-time uncertain systems can be traced back to [1] within the framework of linear fractional transformations (LFTs), which is further developed in [2] for multidimensional and uncertain systems. The recent paper [3] studies the balanced truncation for continuous-time uncertain systems using LFT representations. Similar method is applied to model reduction of linear parameter dependent (LPD) systems in [4]. The model reduction problems are also addressed in [5], [6] for linear time-varying systems and in [7], [8], [9] for linear parameter-varying systems. The reader is referred to [10], [11] for closely related problems, such as approximation, truncation and simplification of uncertain systems.

It is shown in [1], [2], [3], [4] that the balanced truncation methods can guarantee robust stability of the reduced systems and yield bounds on model reduction error from an input-output perspective. However, the original uncertain systems are required to satisfy certain *robust stability* conditions to proceed with the balanced truncation approaches. This requirement prevents its application to those uncertain systems which may be robustly unstable. One of the common solutions, for the nominal linear time invariant (LTI) systems, to overcome this difficulty is to use the coprime factor approach; see for example [12], [13], [14], [15]. This approach is then extended to discrete-time uncertain systems in [16] to obtain reduced-order uncertain systems with guaranteed error bounds on the derived coprime factors. In [17], coprime factorization for LPD systems is considered. This motivates us to seek for a coprime factor model reduction method for continuous-time uncertain systems.

In this paper, we focus on the coprime factor model reduction problems for continuous-time uncertain systems. The uncertain systems under consideration are described in terms of LFTs with structured norm bounded uncertainty. A systematic approach based on the use of linear matrix inequalities (LMIs) is proposed for coprime factorization and contractive coprime factorization of the underlying uncertain systems. This enables us to apply the balanced truncation method in [3] to the resulting coprime factors to obtain the reduced-order uncertain systems. It is also shown that, besides the guaranteed error bounds, the reduced-order coprime factors retain the property of contractiveness.

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II. BACKGROUND

A. Notation

Let $\mathbf{L}_2^m[0,\infty)$ be the space of square integrable functions in $\mathbf{R}^m$, and $\mathcal{L}(\mathbf{L}_2^m)$ denote the space of all linear bounded operators mapping from $\mathbf{L}_2^m$ to $\mathbf{L}_2^m$. The gain of an operator Δ in $\mathcal{L}(\mathbf{L}_2^m)$ is given by $\|\Delta\| = \sup_{z\in\mathbf{L}_2^m[0,\infty), z\neq 0} \frac{\|\Delta z\|}{\|z\|}$, and the adjoint operator of Δ is denoted as Δ^* if Δ is linear, and if $\Delta = \Delta^*$, $\Delta < 0$ means that $x^*\Delta x < 0$ for any $x \neq 0$ in $\mathbf{R}^m$. We also use M^* to denote the complex conjugate transpose of a complex matrix M. The state-space realization of a transfer matrix is denoted by $G(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D\end{array}\right] := C(sI-A)^{-1}B + D$.

B. Problem Formulation

We consider the uncertainty structure

$$\mathbf{\Delta^c} = \{\mathrm{diag}(\Delta_1, \cdots, \Delta_k) : \Delta_i \in \mathcal{L}(\mathbf{L}_2^{h_i}), \Delta_i \text{ causal}, \|\Delta_i\| \leq 1\},$$

and the uncertain systems of the following form:

$$\mathcal{G}_\Delta : \begin{cases} \dot{x} = Ax + E\xi + Bu, \\ z = Kx + Gu, \\ y = Cx + D\xi, \\ \xi = \Delta z, \quad \Delta \in \mathbf{\Delta^c}, \end{cases} \tag{1}$$

where $x(t) \in \mathbf{R}^n$ is the *state*, $u(t) \in \mathbf{R}^m$ is the *control input*, $z(t) \in \mathbf{R}^h$ is the *uncertainty output*, $y(t) \in \mathbf{R}^l$ is the *measured output* and $\xi(t) \in \mathbf{R}^h$ is the *uncertainty input*; here $h = h_1 + \cdots + h_k$.

Let the nominal system be denoted by

$$M = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22}\end{bmatrix} = \left[\begin{array}{c|cc} A & E & B \\ \hline K & \mathbf{0}_h & G \\ C & D & \mathbf{0}_{l\times m}\end{array}\right].$$

Then, the uncertain system (1) can be represented as an LFT,

$$\mathcal{G}_\Delta = \mathcal{F}_u(M, \Delta) := M_{22} + M_{21}\Delta(I - M_{11}\Delta)^{-1}M_{12},$$

provided that $I - M_{11}\Delta$ is non-singular.

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Define the following operators

$$\begin{bmatrix} \mathcal{A}_\Delta & \mathcal{B}_\Delta \\ \mathcal{C}_\Delta & \mathcal{D}_\Delta \end{bmatrix} = \begin{bmatrix} A+E\Delta K & B+E\Delta G \\ C+D\Delta K & D\Delta G \end{bmatrix}. \quad (2)$$

The uncertain system (1) can also be rewritten as

$$\mathcal{G}_\Delta : \begin{cases} \dot{x} = \mathcal{A}_\Delta x + \mathcal{B}_\Delta u, \\ y = \mathcal{C}_\Delta x + \mathcal{D}_\Delta u, \quad \Delta \in \boldsymbol{\Delta^c}. \end{cases}$$

In what follows, robust stability, stabilizability and detectability of the uncertain system (1) are defined.

Definition 1 (Robust Stability [18]): The uncertain system (1) is *robustly stable* if $(I - M_{11}\Delta)^{-1}$ exists in $\mathcal{L}(\mathbf{L}_2^h)$ and is causal, for all $\Delta \in \boldsymbol{\Delta^c}$.

Definition 2: The uncertain system (1) is said to be *robustly stabilizable* if there exists a static state feedback law $u = Fx$ such that the corresponding closed-loop uncertain system is robustly stable. Similarly, the system (1) is said to be *robustly detectable* if the dual of the system (1) is robustly stabilizable.

The following lemma states a necessary and sufficient condition for robust stability, which is given in terms of the positive commutant set corresponding to $\boldsymbol{\Delta^c}$ defined as

$$\boldsymbol{P_\Theta} = \{\mathrm{diag}(\theta_1 I_{h_1}, \cdots, \theta_k I_{h_k}) : \theta_i > 0\}. \quad (3)$$

Lemma 3: (see [18]) The system (1) is robustly stable if and only if there exist $\Theta \in \boldsymbol{P_\Theta}$ and $X > 0$, such that

$$A^*X + XA + K^*\Theta K + XE\Theta^{-1}E^*X < 0. \quad (4)$$

In this paper, we aim to seek for a model reduction scheme for the uncertain system (1), possibly robustly unstable, to obtain a reduced system $\mathcal{G}_{r\Delta} = \mathcal{F}_u(M_r, \Delta)$ with M_r of order $d < n$.

C. Balanced Truncation

We briefly review the balanced truncation model reduction technique for the uncertain system (1) recently presented in [3]. It is assumed in this section that the uncertain system (1) is robustly stable. First controllability and observability Gramians for the uncertain system (1) are defined as follows.

Definition 4: Matrices $S > 0$, $P > 0$ are said to be a *generalized controllability Gramian* and *generalized observability Gramian* for the uncertain system (1), respectively, if the following linear, operator inequalities hold,

$$\mathcal{A}_\Delta S + S\mathcal{A}_\Delta^* + \mathcal{B}_\Delta \mathcal{B}_\Delta^* < 0, \quad (5)$$

$$\mathcal{A}_\Delta^* P + P\mathcal{A}_\Delta + \mathcal{C}_\Delta^* \mathcal{C}_\Delta < 0 \quad \forall \Delta \in \boldsymbol{\Delta^c}. \quad (6)$$

Here, $\mathcal{A}_\Delta, \mathcal{B}_\Delta, \mathcal{C}_\Delta$ are as defined in (2).

The following results from [3] provide a numerical approach to solve for generalized Gramians of the uncertain system (1), as defined in Definition 4.

Proposition 5: If there exist matrices $S > 0$ and $\bar{\Lambda}_c \in \boldsymbol{P_\Theta}$ solving the following LMI:

$$\begin{bmatrix} SA^* + AS + E\bar{\Lambda}_c E^* & SK^* & B \\ \star & -\bar{\Lambda}_c & G \\ \star & \star & -I_m \end{bmatrix} < 0, \quad (7)$$

S is a generalized controllability Gramian for the uncertain system (1).

Proposition 6: If there exist matrices $P > 0$ and $\Lambda_o \in \boldsymbol{P_\Theta}$ solving the following LMI:

$$\begin{bmatrix} A^*P + PA + K^*\Lambda_o K & PE & C^* \\ \star & -\Lambda_o & D^* \\ \star & \star & -I_l \end{bmatrix} < 0, \quad (8)$$

P is a generalized observability Gramian for the uncertain system (1).

Theorem 7: The following statements are equivalent:

(i) The uncertain system (1) is robustly stable.

(ii) The LMI (7) admits a solution $S > 0$ and $\Lambda_c \in \boldsymbol{P_\Theta}$.

(iii) The LMI (8) admits a solution $P > 0$ and $\Lambda_o \in \boldsymbol{P_\Theta}$.

With generalized Gramians available, the uncertain system (1) can be readily balanced and then proceeded for model reduction.

Definition 8: An uncertain system of the form (1) is said to be *balanced* if it has generalized observability and controllability Gramians which are identical diagonal matrices.

Procedure 9 (Balanced Truncation):

1) Solve the LMIs (7) and (8) to obtain generalized Gramians $S > 0, P > 0$.
2) Balance S, P by constructing a state transformation matrix T [19] such that

$$TST^* \quad = \quad (T^{-1})^*PT^{-1} = \Sigma = \mathrm{diag}(\Sigma_1, \Sigma_2), \quad (9)$$

where $\Sigma_1 = diag(\gamma_1, ..., \gamma_d)$, $\Sigma_2 = diag(\gamma_{d+1}, ..., \gamma_n)$, $\gamma_1 \geq ... \geq \gamma_d > \gamma_{d+1} \geq ... \geq \gamma_n > 0$.
3) Write the transformed nominal system of (1) as

$$\bar{M} = \left[\begin{array}{c|cc} \bar{A} & \bar{E} & \bar{B} \\ \hline \bar{K} & \mathbf{0}_h & G \\ \bar{C} & D & \mathbf{0}_{l\times m} \end{array} \right], \quad (10)$$

where

$$\bar{A} = TAT^{-1};\ \bar{E} = TE;\ \bar{B} = TB;\ \bar{C} = CT^{-1};\ \bar{K} = KT^{-1}.$$

The sub-matrices of this balanced system $\bar{M}$ corresponding to the matrix Σ_2 in (9) are truncated to obtain the reduced d-th order uncertain system defined by

$$M_r = \left[\begin{array}{c|cc} \bar{A}_r & \bar{E}_r & \bar{B}_r \\ \hline \bar{K}_r & \mathbf{0}_h & G \\ \bar{C}_r & D & \mathbf{0}_{l\times m} \end{array} \right]. \quad (11)$$

4) Write the reduced dimension uncertain system as $\mathcal{G}_{r\Delta} = \mathcal{F}_u(M_r, \Delta), \Delta \in \boldsymbol{\Delta^c}$.

Theorem 10: Consider the uncertain system (1) and suppose that the reduced dimension uncertain system $\mathcal{G}_{r\Delta}$ is obtained as described in Procedure 9. Then $\mathcal{G}_{r\Delta}$ is also balanced and robustly stable. Furthermore,

$$\sup_{\Delta \in \boldsymbol{\Delta^c}} \|\mathcal{G}_\Delta(s) - \mathcal{G}_{r\Delta}(s)\|_\infty \leq 2(\gamma'_1 + \cdots + \gamma'_q), \quad (12)$$

where γ'_i denote the distinct generalized Hankel singular values of $\gamma_{d+1}, \ldots, \gamma_n$, that is, $\gamma'_1 > \gamma'_2 > \ldots > \gamma'_q$ and $\{\gamma_{d+1}, \ldots, \gamma_n\} = \{\gamma'_1, \ldots, \gamma'_q\}$.

III. Contractive Coprime Factor Model Reduction for Uncertain Systems

As introduced in Section II, the main restriction of the balanced truncation technique is the requirement of robust stability on the uncertain systems under consideration. For those uncertain systems which may be robustly unstable, a so-called balanced LQG truncation approach is presented in [20], taking into account of the *closed-loop* control considerations; see [21] for details and discussions on LTI systems. Another popular approach for unstable systems is coprime factorization approach. Coprime factorization of uncertain systems is explored in [16] for discrete-time systems and in [17] for LPD systems, and a model reduction algorithm based on coprime factorization is given in [16]. However, no indication is given in [16], [17] on the contractiveness of the underlying coprime factors. It is well-known that, for continuous-time LTI systems, the balanced LQG approach and coprime factor model reduction approach are actually equivalent; see [22], [23]. This motivates us to follow the ideas in [20] to pursue a contractive coprime factor model reduction method for uncertain systems of the form (1).

A. Coprime Factorization of Uncertain Systems

Suppose that the uncertain system (1) is robustly stabilizable and robustly detectable, as stated in Def. 2. Consider the following LQG control and filter Riccati inequalities for the uncertain system (1),

$$\begin{aligned} W(\mathcal{A}_\Delta - \mathcal{B}_\Delta\mathcal{R}_\Delta^{-1}\mathcal{D}_\Delta^*\mathcal{C}_\Delta) + (\mathcal{A}_\Delta - \mathcal{B}_\Delta\mathcal{R}_\Delta^{-1}\mathcal{D}_\Delta^*\mathcal{C}_\Delta)^*W \\ - W\mathcal{B}_\Delta\mathcal{R}_\Delta^{-1}\mathcal{B}_\Delta^*W + \mathcal{C}_\Delta^*\tilde{\mathcal{R}}_\Delta^{-1}\mathcal{C}_\Delta < 0,\ \forall\Delta\in\boldsymbol{\Delta^c}, \end{aligned} \quad (13)$$

$$\begin{aligned} (\mathcal{A}_\Delta - \mathcal{B}_\Delta\mathcal{R}_\Delta^{-1}\mathcal{D}_\Delta^*\mathcal{C}_\Delta)V + V(\mathcal{A}_\Delta - \mathcal{B}_\Delta\mathcal{R}_\Delta^{-1}\mathcal{D}_\Delta^*\mathcal{C}_\Delta)^* \\ - V\mathcal{C}_\Delta^*\tilde{\mathcal{R}}_\Delta^{-1}\mathcal{C}_\Delta V + \mathcal{B}_\Delta\mathcal{R}_\Delta^{-1}\mathcal{B}_\Delta^* < 0,\ \forall\Delta\in\boldsymbol{\Delta^c}, \end{aligned} \quad (14)$$

where $\mathcal{R}_\Delta = I + \mathcal{D}_\Delta^*\mathcal{D}_\Delta$, $\tilde{\mathcal{R}}_\Delta = I + \mathcal{D}_\Delta\mathcal{D}_\Delta^*$.

Definition 11: Given a pair of uncertain systems $\mathcal{M}_\Delta = \mathcal{F}_u(H_M,\Delta), \mathcal{N}_\Delta = \mathcal{F}_u(H_N,\Delta)$, $\Delta\in\boldsymbol{\Delta^c}$, where H_M and H_N are LTI casual systems, $(\mathcal{M}_\Delta,\mathcal{N}_\Delta)$ is said to be a *right coprime factorization* (RCF) of $\mathcal{G}_\Delta$ (1) if the following conditions hold.

1) $\mathcal{M}_\Delta$ and $\mathcal{N}_\Delta$ are robustly stable.
2) For any fixed $\Delta\in\boldsymbol{\Delta^c}$, $\mathcal{M}_\Delta$ is invertible and casual.
3) For any fixed $\Delta\in\boldsymbol{\Delta^c}$, $(\mathcal{M}_\Delta,\mathcal{N}_\Delta)$ is right coprime, and $\mathcal{G}_\Delta = \mathcal{N}_\Delta\mathcal{M}_\Delta^{-1}$.

Furthermore, if $\mathcal{M}_\Delta^*\mathcal{M}_\Delta + \mathcal{N}_\Delta^*\mathcal{N}_\Delta \le I$ for all $\Delta\in\boldsymbol{\Delta^c}$, we say $(\mathcal{M}_\Delta,\mathcal{N}_\Delta)$ is a contractive RCF of $\mathcal{G}_\Delta$ (1).

For an LTI system, it is shown that LQG control and filter algebraic Riccati equations or inequalities are closely related to coprime factorization problems [24], [15] and some special $\mathcal{H}_2$ control problems [25]. In [17], coprime factorizations for LPD systems were discussed, and the problems were also reduced to some $\mathcal{H}_2$ problems for uncertain systems. Following the idea of [17], it is shown in [20], in the context of balanced LQG model reduction, that the solution of the LQG control and filter Riccati inequalities (13) and (14) can be obtained by solving a set of LMIs. We state this result in the following theorem.

Theorem 12: [20] If there exist matrices $\bar{P} > 0$, $\bar{\Lambda}_o\in\boldsymbol{P_\Theta}$ and $X\in\mathbf{R}^{m\times n}$ solving the following LMI:

$$\begin{bmatrix} (1,1) & \bar{P}K^* + X^*G^* & X^* & \bar{P}C^* + E\bar{\Lambda}_oD^* \\ \star & -\bar{\Lambda}_o & \mathbf{0}_{h\times m} & \mathbf{0}_{h\times l} \\ \star & \star & -I_m & \mathbf{0}_{m\times l} \\ \star & \star & \star & -I_l + D\bar{\Lambda}_oD^* \end{bmatrix} < 0, \quad (15)$$

where $(1,1) = A\bar{P} + \bar{P}A^* + BX + X^*B^* + E\bar{\Lambda}_oE^*$, then $\bar{P}^{-1}$ verifies (13).

Now we are in the position to state the main results of this paper, regarding to the RCF and contractive RCF of the uncertain system $\mathcal{G}_\Delta$.

Theorem 13: Given an uncertain system $\mathcal{G}_\Delta$ (1) which is robustly stabilizable and robustly detectable, the following statements hold.

(i) There exist matrices $\bar{P} > 0$, $\bar{\Lambda}_o\in\boldsymbol{P_\Theta}$ and $X\in\mathbf{R}^{m\times n}$ solving the LMI (15).

(ii) Let
$$F = X\bar{P}^{-1}, \quad (16)$$
and consider the following system
$$\mathcal{G}_{F\Delta} = \begin{bmatrix} \mathcal{N}_\Delta \\ \mathcal{M}_\Delta \end{bmatrix} = \mathcal{F}_u(M_F,\Delta), \quad (17)$$
$$M_F = \left[\begin{array}{c|cc} A+BF & E & B \\ \hline K+GF & \mathbf{0}_h & G \\ C & D & \mathbf{0}_{l\times m} \\ F & \mathbf{0}_{m\times h} & I_m \end{array}\right]. \quad (18)$$

Then $(\mathcal{M}_\Delta,\mathcal{N}_\Delta)$ is an RCF of the uncertain system $\mathcal{G}_\Delta$.

Proof: (i) By assumption that the uncertain system $\mathcal{G}_\Delta$ in (1) is robustly stabilizable, from Definition 2 and Lemma 3, there exist matrices F, $W > 0$ and $\Theta\in\boldsymbol{P_\Theta}$ such that

$$\begin{aligned} (A+BF)^*W + W(A+BF) \\ + (K+GF)^*\Theta(K+GF) + WE\Theta^{-1}E^*W < 0, \end{aligned}$$

which, by Lemma 3 again, implies the uncertain system $\mathcal{G}_{F\Delta}$ (17), (18) is robust stable. Then we can apply Theorem 7 to $\mathcal{G}_{F\Delta}$ to show that there exist $P > 0$, $\Lambda_o\in\boldsymbol{P_\Theta}$ such that

$$\begin{bmatrix} \mathcal{T}_{11} & PE & C^* & F^* \\ \star & -\Lambda_o & D^* & 0 \\ \star & \star & -I & 0 \\ \star & \star & \star & -I \end{bmatrix} < 0, \quad (19)$$

where

$$\mathcal{T}_{11} = (A+BF)^*P + P(A+BF) + (K+GF)^*\Lambda_o(K+GF).$$

It is easy to derive (15) by letting $\bar{P} = P^{-1}$, $X = F\bar{P}$ and $\bar{\Lambda}_o = \Lambda_o^{-1}$ in (19).

(ii) Suppose that $\bar{P} > 0$, $\bar{\Lambda}_o\in\boldsymbol{P_\Theta}$ and $X\in\mathbf{R}^{m\times n}$ is a feasible solution of the LMI (15). First we show that $\mathcal{G}_{F\Delta}$ is robustly stable. Indeed, by letting $P = \bar{P}^{-1}$, $F = XP$ and $\Lambda_o = \bar{\Lambda}_o^{-1}$, (15) is equivalent to (19). This implies that $\mathcal{G}_{F\Delta}$ is robustly stable by Theorem 7.

To prove that $(\mathcal{M}_\Delta,\mathcal{N}_\Delta)$ is an RCF of $\mathcal{G}_\Delta$, first we show that $(\mathcal{M}_\Delta,\mathcal{N}_\Delta)$ is right coprime.

Define a new uncertain system as follows,

$$\mathcal{G}_{L\Delta} = [-\mathcal{Y}_\Delta,\ \mathcal{X}_\Delta] = \mathcal{F}_u(M_L, \Delta), \tag{20}$$

$$M_L = \left[\begin{array}{c|ccc} A+LC & E+LD & L & -B \\ \hline K & \mathbf{0} & \mathbf{0} & -G \\ F & \mathbf{0} & \mathbf{0} & I \end{array}\right], \tag{21}$$

where, similar to F, L is obtained from the dual result of (16) by the robust detectability assumption of $\mathcal{G}_\Delta$. Similarly, $\mathcal{G}_{L\Delta}$ is robustly stable. It is not difficult to show that

$$[-\mathcal{Y}_\Delta,\ \mathcal{X}_\Delta] \begin{bmatrix} \mathcal{N}_\Delta \\ \mathcal{M}_\Delta \end{bmatrix} = I. \tag{22}$$

Therefore, $(\mathcal{M}_\Delta, \mathcal{N}_\Delta)$ is right coprime.

Next, we show that $\mathcal{M}_\Delta$ is invertible for all $\Delta \in \boldsymbol{\Delta^c}$. Similarly, it is easy to verify that the following system

$$\bar{\mathcal{M}}_\Delta = \mathcal{F}_u(H, \Delta), \qquad H = \left[\begin{array}{c|cc} A & E & B \\ \hline K & \mathbf{0} & G \\ -F & \mathbf{0} & I \end{array}\right]$$

is the right inverse of $\mathcal{M}_\Delta$. Note that $\bar{\mathcal{M}}_\Delta$ is well-defined since $\mathcal{G}_\Delta$ (1) is well-defined.

Considering the uncertain system $\mathcal{G}_\Delta$ (1), define

$$v(t) = u(t) - Fx(t). \tag{23}$$

It is easy to derive

$$\begin{bmatrix} \mathcal{N}_\Delta \\ \mathcal{M}_\Delta \end{bmatrix} v = \begin{bmatrix} y \\ u \end{bmatrix}. \tag{24}$$

From (24), it is straightforward that $\mathcal{G}_\Delta = \mathcal{N}_\Delta \mathcal{M}_\Delta^{-1}$, which completes the proof. ■

Theorem 14: Given a robustly stabilizable and detectable uncertain system $\mathcal{G}_\Delta$ (1), suppose there exist matrices $\bar{P} > 0$, $\bar{\Lambda}_o \in \boldsymbol{P_\Theta}$ and $X \in \mathbf{R}^{m\times n}$ solving the LMI (15). Let

$$R = I + G^* \bar{\Lambda}_o^{-1} G, \tag{25}$$

$$F^c = -R^{-1}(B^* \bar{P}^{-1} + G^* \bar{\Lambda}_o^{-1} K), \tag{26}$$

and consider the following system,

$$\mathcal{G}_{F\Delta}^c = \begin{bmatrix} \mathcal{N}_\Delta^c \\ \mathcal{M}_\Delta^c \end{bmatrix} = \mathcal{F}_u(M_F^c, \Delta), \tag{27}$$

$$M_F^c = \left[\begin{array}{c|cc} A+BF^c & E & BR^{-\frac{1}{2}} \\ \hline K+GF^c & \mathbf{0}_h & GR^{-\frac{1}{2}} \\ C & D & \mathbf{0}_{l\times m} \\ F^c & \mathbf{0}_{m\times h} & R^{-\frac{1}{2}} \end{array}\right]. \tag{28}$$

Then $(\mathcal{M}_\Delta^c, \mathcal{N}_\Delta^c)$ is a contractive RCF of the uncertain system $\mathcal{G}_\Delta$.

Proof: Suppose the LMI (15) has a feasible solution $\bar{P}$, X and $\bar{\Lambda}_o$. By Schur complements, (15) is equivalent to

$$\begin{bmatrix} \mathcal{M}_{11} & \bar{P}C^* + E\bar{\Lambda}_o D^* \\ \star & -I_l + D\bar{\Lambda}_o D^* \end{bmatrix} < 0, \tag{29}$$

where

$$\mathcal{M}_{11} = A\bar{P} + \bar{P}A^* + E\bar{\Lambda}_o E^* + (X - F^c\bar{P})^* R (X - F^c \bar{P}) - \bar{P}(F^c)^* R F^c \bar{P} + \bar{P}K^* \bar{\Lambda}_o^{-1} K \bar{P}.$$

Here R and F^c are defined in (25) and (26).

It is obvious that $\bar{P}$, $X^c = F^c \bar{P}$ and $\bar{\Lambda}_o$ also satisfy (29), therefore satisfies the LMI (15). It follows from Theorem 13 that $(\mathcal{M}_\Delta^c, \mathcal{N}_\Delta^c)$ is an RCF of $\mathcal{G}_\Delta$. Note that here $(\mathcal{M}_\Delta^c, \mathcal{N}_\Delta^c)$ are scaled by $R^{-\frac{1}{2}}$.

To prove that $(\mathcal{M}_\Delta^c, \mathcal{N}_\Delta^c)$ is contractive, that is $\|\mathcal{G}_{F\Delta}^c\| \le 1$, we show an equivalent claim that $\|\mathcal{G}_{F\Delta}^c\| < \beta$ for any $\beta > 1$. By [18, Proposition 9.9], this claim is equivalent to finding a $\Theta \in \boldsymbol{P_\Theta}$ such that

$$\left\| \begin{bmatrix} \Theta & 0 \\ 0 & I \end{bmatrix} M_F^c \begin{bmatrix} I & 0 \\ 0 & \beta^{-1} I \end{bmatrix} \begin{bmatrix} \Theta & 0 \\ 0 & I \end{bmatrix}^{-1} \right\| < 1, \tag{30}$$

where M_F^c is defined in (28). By the KYP lemma, e.g. see [18, Lemma 7.4], (30) is equivalent to finding $\hat{P} > 0$, $\Theta \in \boldsymbol{P_\Theta}$ such that

$$\begin{bmatrix} \Phi_{11} & \hat{P}(K+GF^c)^*\Theta & \hat{P}C^* & \hat{P}(F^c)^* & E\Theta^{-1} & \beta^{-1}BR^{-1/2} \\ \star & -I & 0 & 0 & 0 & \beta^{-1}\Theta G R^{-1/2} \\ \star & \star & -I & 0 & D\Theta^{-1} & 0 \\ \star & \star & \star & -I & 0 & \beta^{-1}R^{-1/2} \\ \star & \star & \star & \star & -I & 0 \\ \star & \star & \star & \star & \star & -I \end{bmatrix} < 0,$$

where $\Phi_{11} = (A+BF^c)\hat{P} + \hat{P}(A+BF^c)^*$. The above inequality is further equivalent to

$$\begin{bmatrix} \mathcal{N}_{11} & \hat{P}C^* + E\Theta^{-2}D^* & \mathcal{N}_{13} \\ \star & -I + D\Theta^{-2}D^* & 0 \\ \star & \star & -\beta^2 R + I + G^*\Theta^2 G \end{bmatrix} < 0, \tag{31}$$

where

$$\mathcal{N}_{11} = (A+BF^c)\hat{P} + \hat{P}(A+BF^c)^* + E\Theta^{-2}E^* + \hat{P}(F^c)^* F^c \hat{P} + \hat{P}(K+GF^c)^*\Theta^2(K+GF^c)\hat{P},$$

$$\mathcal{N}_{13} = B + \hat{P}(F^c)^* + \hat{P}(K+GF^c)^*\Theta^2 G.$$

It is easy to verify that $\hat{P} = \bar{P}$, $\Theta = \bar{\Lambda}_o^{-\frac{1}{2}}$ satisfy (31). Indeed, substituting $\hat{P} = \bar{P}$, $\Theta = \bar{\Lambda}_o^{-\frac{1}{2}}$ into (31), we have $\mathcal{N}_{13} = 0$ and $\mathcal{N}_{11} = A\bar{P} + \bar{P}A^* + E\bar{\Lambda}_o E^* - \bar{P}(F^c)^* R F^c \bar{P} + \bar{P}K^* \bar{\Lambda}_o^{-1} K \bar{P}$. Noting from (29), (31) holds. This completes the proof. ■

B. Contractive Coprime Factor Model Reduction

Having constructed the contractive RCF (27) of the underlying uncertain system, we are now ready to apply the balanced truncation technique [3] to the derived RCF (27). In this regards, we concentrate on finding the generalized Gramians of the RCF (27).

Theorem 15: Given that all the conditions in Theorem 14 are satisfied, the following statements hold.

(i) $\bar{P}^{-1}$ is a generalized observability Gramian for the uncertain system $\mathcal{G}_{F\Delta}^c$ (27).

(ii) The LMI

$$\begin{bmatrix} (1,1) & S(K+GF^c)^* & B \\ \star & -\bar{\Lambda}_c & G \\ \star & \star & -R \end{bmatrix} < 0 \tag{32}$$

has a feasible solution $S > 0$, $\bar{\Lambda}_c \in \boldsymbol{P_\Theta}$, where $(1,1) = S(A+BF^c)^* + (A+BF^c)S + E\bar{\Lambda}_c E^*$. Furthermore, S is a generalized controllability Gramian for the uncertain system $\mathcal{G}_{F\Delta}^c$ in (27).

Proof: (i) Suppose that $\bar{P} > 0$, $\bar{\Lambda}_o \in \boldsymbol{P_\Theta}$ and $X \in \mathbf{R}^{m\times n}$ is a feasible solution of the LMI (15). Following the proof of Theorem 13, the LMI (15) can be rewritten as (19) by letting $P = \bar{P}^{-1}$, $F = XP$ and $\Lambda_o = \bar{\Lambda}_o^{-1}$. Then $P = \bar{P}^{-1}$ is a generalized observability Gramian for $\mathcal{G}_{F\Delta}$ in (17) by invoking Proposition 6.

From the proof of Theorem 14, $\bar{P}$, $X^c = F^c\bar{P}$ and $\bar{\Lambda}_o$ also satisfy (15), and $\mathcal{G}^c_{F\Delta}$ is the corresponding RCF. Therefore, based on the above derivation, $\bar{P}^{-1}$ is a generalized observability Gramian for $\mathcal{G}^c_{F\Delta}$ in (27).

(ii) Since $\mathcal{G}^c_{F\Delta}$ is robustly stable, invoking Theorem 7 and Proposition 5, it is straightforward that the LMI (32) is feasible, and S is a generalized controllability Gramian for $\mathcal{G}^c_{F\Delta}$. ■

The above theorem provides a numerical way to compute generalized Gramians $P = \bar{P}^{-1}$ and S for the contractive RCF $(\mathcal{M}^c_\Delta, \mathcal{N}^c_\Delta)$ of the uncertain system $\mathcal{G}_\Delta$. Then balanced truncation approach in Procedure 9 is readily applied to $\mathcal{G}^c_{F\Delta}$ to obtain a reduced dimensional RCF $(\mathcal{M}^c_{r\Delta}, \mathcal{N}^c_{r\Delta})$, and consequently a corresponding reduced uncertain system $\mathcal{G}_{r\Delta} = \mathcal{N}^c_{r\Delta}(\mathcal{M}^c_{r\Delta})^{-1}$; see [15] for the similar approach to LTI systems. It can be shown, as in [23] for LTI systems and in [16] for the discrete-time uncertain systems, that this approach leads to the same reduced uncertain system $\mathcal{G}_{r\Delta}$ if we apply balanced truncation (via balancing P,S) directly to the original uncertain system $\mathcal{G}_\Delta$; see Theorem 17 below. We first summarize the proposed coprime factor model reduction algorithm as follows.

Procedure 16 (Coprime Factor Model Reduction):

1) Solve the LMI (15) to obtain $\bar{P}$ and $\bar{\Lambda}_o$. Define R as in (25), F^c as in (26) and $P = \bar{P}^{-1}$;
2) Solve the LMI (32) to obtain S and $\bar{\Lambda}_c$;
3) Apply Steps 2-4 in Procedure 9 to the uncertain system $\mathcal{G}_\Delta$ (1) to obtain the reduced dimension uncertain system as $\mathcal{G}_{r\Delta} = \mathcal{F}_u(M_r, \Delta), \Delta \in \boldsymbol{\Delta^c}$.

Theorem 17: Suppose that all the conditions in Theorem 14 are satisfied, and that the reduced dimension uncertain system $\mathcal{G}_{r\Delta} = \mathcal{F}_u(M_r, \Delta), \Delta \in \boldsymbol{\Delta^c}$, where M_r is defined in (11), is obtained as described in Procedure 16. Consider the following system,

$$\mathcal{G}^c_{rF\Delta} = \begin{bmatrix} \mathcal{N}^c_{r\Delta} \\ \mathcal{M}^c_{r\Delta} \end{bmatrix} = \mathcal{F}_u(M^c_{rF}, \Delta), \quad (33)$$

$$M^c_{rF} = \left[\begin{array}{c|cc} \bar{A}_r + \bar{B}_r\bar{F}^c_r & \bar{E}_r & \bar{B}_r R^{-\frac{1}{2}} \\ \hline \bar{K}_r + G\bar{F}^c_r & \mathbf{0}_h & GR^{-\frac{1}{2}} \\ \bar{C}_r & D & \mathbf{0}_{l\times m} \\ \bar{F}^c_r & \mathbf{0}_{m\times h} & R^{-\frac{1}{2}} \end{array}\right], \quad (34)$$

where $\bar{F}^c_r = -R^{-1}(\bar{B}^*_r\Sigma_1 + G^*\bar{\Lambda}_o^{-1}\bar{K}_r)$, Σ_1 is defined in (9). Then the following statements hold.

(i) $(\mathcal{M}^c_{r\Delta}, \mathcal{N}^c_{r\Delta})$ is a contractive RCF of $\mathcal{G}_{r\Delta}$.

(ii)

$$\sup_{\Delta\in\boldsymbol{\Delta^c}} \|\mathcal{G}^c_{F\Delta}(s) - \mathcal{G}^c_{rF\Delta}(s)\|_\infty \le 2(\gamma'_1 + \cdots + \gamma'_q), \quad (35)$$

where γ'_i denote the distinct values of $\gamma_{d+1}, \ldots, \gamma_n$, that is, $\gamma'_1 > \gamma'_2 > \ldots > \gamma'_q$ and $\{\gamma_{d+1}, \ldots, \gamma_n\} = \{\gamma'_1, \ldots, \gamma'_q\}$.

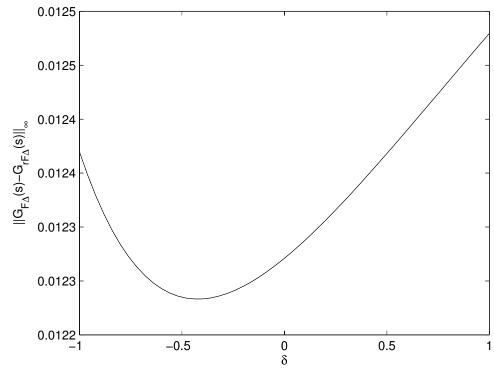

Fig. 1. H_∞-norm of the coprime factor error system as a function of δ.

Proof: (i) As described in Procedure 16, let T be the transformation matrix to balance P,S and $\mathcal{P} = [I \quad \mathbf{0}]$ be the corresponding truncation matrix. Then

$$\begin{aligned} TST^* &= (T^{-1})^*PT^{-1} = \mathrm{diag}(\Sigma_1, \Sigma_2), \\ \bar{A}_r &= \mathcal{P}TAT^{-1}\mathcal{P}^*, \quad \bar{E}_r = \mathcal{P}TE, \quad \bar{B}_r = \mathcal{P}TB, \\ \bar{C}_r &= CT^{-1}\mathcal{P}^*, \quad \bar{K}_r = KT^{-1}\mathcal{P}^*. \end{aligned}$$

Defining $\bar{X}_r = XT^*\mathcal{P}^*$, it is easy to verify, by performing a congruence transformation with $diag(\mathcal{P}T, I, I, I)$ on (15), that the LMI (15) also holds if $A,E,B,C,K,\bar{P},X$ are replaced by $\bar{A}_r, \bar{E}_r, \bar{B}_r, \bar{C}_r, \bar{K}_r, \Sigma_1^{-1}, \bar{X}_r$ respectively. Therefore, by Theorem 14, $(\mathcal{M}^c_{r\Delta}, \mathcal{N}^c_{r\Delta})$ is a contractive RCF of $\mathcal{G}_{r\Delta}$.

(ii) From Theorem 15, $\mathcal{G}^c_{F\Delta}$ can be balanced by the transformation matrix T with balanced Gramian $\mathrm{diag}(\Sigma_1, \Sigma_2)$. Also note that

$$M^c_{rF} = \left[\begin{array}{c|cc} \mathcal{P}T(A+BF^c)T^{-1}\mathcal{P}^* & \mathcal{P}TE & \mathcal{P}TBR^{-\frac{1}{2}} \\ \hline (K+GF^c)T^{-1}\mathcal{P}^* & \mathbf{0}_h & GR^{-\frac{1}{2}} \\ CT^{-1}\mathcal{P}^* & D & \mathbf{0}_{l\times m} \\ F^cT^{-1}\mathcal{P}^* & \mathbf{0}_{m\times h} & R^{-\frac{1}{2}} \end{array}\right].$$

Now invoke Theorem 10 to obtain error bound (35). This completes the proof. ■

IV. Example

Consider the following uncertain system of the form (1) with $\Delta = \delta \in [-1, 1]$, and

$$A = \begin{bmatrix} -100 & 0 & 0 \\ 1 & -200 & 0 \\ 0 & 1 & 300 \end{bmatrix}, \quad E = \begin{bmatrix} 0.1 \\ 0.1 \\ 0.1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix},$$

$$K = [1 \quad 1 \quad 1], \quad C = [1 \quad 1 \quad 1], \quad G = 1, \quad D = 0.1. \quad (36)$$

It is obvious that this uncertain system is robustly unstable. Therefore, the balanced truncation method [3] is not applicable here. Now we apply the coprime factor model reduction approach in Procedure 16 to this unstable uncertain system.

Solving the LMI (15), we obtain $\bar{\Lambda}_o = 8.218$ and

$$P = \bar{P}^{-1} = \begin{bmatrix} 0.008 & 0.005 & -0.045 \\ 0.005 & 0.007 & 1.456 \\ -0.045 & 1.456 & 745.306 \end{bmatrix}.$$

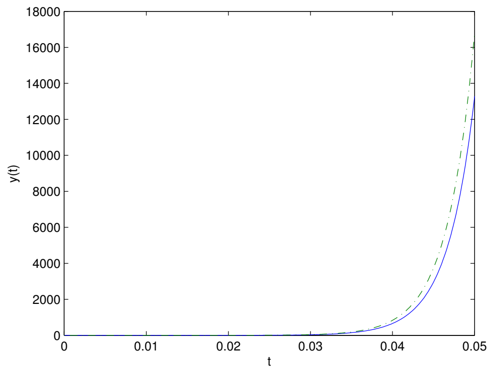

Fig. 2. Step responses of the original uncertain system $\mathcal{G}_\Delta$ and the reduced system $\mathcal{G}_{r\Delta}$ at $\delta = 1.0$.

Solving the LMI (32), we obtain $\bar{\Lambda}_c = 7.820$ and

$$S = \begin{bmatrix} 3.994 & 2.757 & 0.187 \\ 2.757 & 2.134 & 0.155 \\ 0.187 & 0.155 & 1.325 \end{bmatrix} \times 10^{-3}.$$

Then the balanced Gramian is

$$\Sigma = \mathrm{diag}(0.9941, 0.0081, 0.0003).$$

Truncating the last 2 states, the reduced dimension uncertain system model is defined by

$$\bar{A}_r = 300.006, \quad \bar{E}_r = -2.743, \quad \bar{B}_r = -27.434,$$
$$\bar{K}_r = -0.046, \quad \bar{C}_r = -0.046,$$

and the error bound on the coprime factors is given by

$$\sup_{\delta\in[-1,1]} \|\mathcal{G}^c_{F\Delta}(s) - \mathcal{G}^c_{rF\Delta}(s)\|_\infty \leq 2(0.0081 + 0.0003) = 0.0168. \quad (37)$$

Figure 1 shows the actual H_∞-norm of the coprime factor error system as a function of δ, which is less than the upper bound given in (37). The comparison of the step responses between the original uncertain system $\mathcal{G}_\Delta$ (solid line) and the reduced system $\mathcal{G}_{r\Delta}$ (dashed line) at $\delta = 1.0$ is given in Figure 2.

V. Conclusions

This paper presents a contractive coprime factor model reduction approach for a class of continuous-time uncertain systems of LFT form with norm bounded structured uncertainty. A systematic approach is proposed for coprime factorization and contractive coprime factorization of the underlying uncertain systems. Compared to the balanced truncation approach, the proposed coprime factor approach overcomes the robust stability restriction on the underlying systems. Our method is based on the use of LMIs to construct the desired reduced dimension uncertain system model. Error bound for the coprime factors is also derived.

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