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A step-wise procedure for reduced order approximation in the ν-gap metric

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Abstract— **Recent results on reduced-order approximation in the ν-gap metric, characterised in terms of a non-convex feasibility problem, are investigated further. A detailed analysis of the properties of the constituent rank constrained Linear Matrix Inequalities, when the nominal system has an LQG-balanced state-space realisation, reveals that it is possible to construct a feasible point directly for a particular choice of reduced order and ν-gap error. This gives rise to a step-wise procedure, based on constructing an optimal approximant at each step. While as of yet, the freedom in the parameterisation of optimal approximants has not been exploited, the new step-wise technique developed in this paper appears to perform well for numerical examples, yielding approximants between the upper and lower bounds for approximation in the ν-gap metric.**

NOTATION

The symbols $\mathbb{R}$ and $\mathbb{C}$ denote the real and complex numbers, respectively. $\mathbb{F}^{m\times q}$ denotes an m-row by q-column matrix with entries in $\mathbb{F}$, which denotes either $\mathbb{R}$ or $\mathbb{C}$. $\bar{\sigma}(X)$ and $\underline{\sigma}(X)$ respectively denote the maximum and minimum singular values of $X \in \mathbb{F}^{m\times q}$, while $\lambda_i(X)$ represents the i-th eigenvalue (in decreasing order of size) of X. The spectral radius, i.e. maximum eigenvalue, is denoted by $\mathrm{rad}(\cdot)$. A superscript T denotes matrix transpose, whereas $*$ denotes complex conjugate transpose. The superscript symbol $\dagger$ indicates the Moore-Penrose inverse of a matrix, and $\perp$ denotes the orthogonal complement of a matrix.

$\mathcal{R}^{m\times q}$ denotes the transfer functions $P : \mathbb{C} \to \mathbb{C}^{m\times q}$ (a.e.), with a (state-space) realisation of the form $P(s) = C(sI - A)^{-1}B + D$, for appropriate matrices $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times q}$, $C \in \mathbb{R}^{m\times n}$ and $D \in \mathbb{R}^{m\times q}$ (i.e. proper real rational functions). The order of such a realisation, denoted (A, B, C, D) is said to be n and this is minimal if $\begin{bmatrix}\lambda I - A & B\end{bmatrix}$ and $\begin{bmatrix}\lambda I - A^T & C^T\end{bmatrix}$ have full row rank for all $\lambda = \sigma + j\omega$ with $\sigma > 0$. The McMillan degree is defined as the dimension of the A matrix of a minimal state-space realisation. Note that if a state-space realisation is minimal then the order and McMillan degree are the same. Given $P \in \mathcal{R}^{m\times q}$, the conjugate transfer function $P^{\sim} \in \mathcal{R}^{q\times m}$ is defined by $P^{\sim}(s) := P(-s)^T$ (a.e.), so that $P^{\sim}(j\omega) = (P(j\omega))^*$ (a.e.). For notational convenience the input-output dimensions are frequently suppressed.

$\mathcal{RL}_\infty(j\mathbb{R})$ is the space of transfer functions $P \in \mathcal{R}$ that satisfy $\|P\|_\infty := \sup_{\omega\in\mathbb{R}} \bar{\sigma}(P(j\omega)) < \infty$. $\mathcal{RH}_\infty(\mathbb{C}_+)$ is the space of transfer functions $P \in \mathcal{RL}_\infty(j\mathbb{R})$ that are analytic (i.e. have no poles) in the open right-half plane $\mathbb{C}_+$. For $P \in \mathcal{RH}_\infty(\mathbb{C}_+)$, $\|P\|_\infty = \sup_{s\in\mathbb{C}_+} \bar{\sigma}(P(s))$.

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I. INTRODUCTION

The ν-gap metric provides a measure of the difference between open-loop systems from the perspective of closed-loop behaviour [1], [2], [3]. Within the context of reduced order modeling for feedback compensator design, it is therefore sensible to measure modeling approximation errors with the metric. In particular, when seeking to build a model with smaller McMillan degree than a nominal (full-order) model, it makes sense to keep the ν-gap error small.

To date, the only known procedure for reduced-order approximation in the ν-gap metric appeared in [2]. This method relies on a clever application of Hankel norm approximation [4] of a normalised graph symbol of the n-th order nominal system to construct a graph symbol for an optimal ν-gap metric approximant of order $n - r$, where r is the multiplicity of the smallest Hankel singular value of a normalised graph symbol. Utilising the inherent freedom in the Hankel norm construction, the reduced-order graph symbol is endowed with certain properties, e.g. it is normalised and has smaller Hankel singular values. This optimal ν-gap approximation gives rise to a *step-wise* procedure for model order reduction in the ν-gap metric. The properties of the optimal approximation constructed at each step are chosen to yield a reduced-order model that satisfies *a priori* error bounds by applying the metric properties of the ν-gap metric.

While this step-wise method of approximation constructs optimal approximants at each step, the upper and lower error bounds may be loose. In some cases, the reduced-order model obtained may have a ν-gap error near the upper bound, while for other cases it could be towards the lower bound. Most of the freedom in the Hankel norm approximation has already been exploited and it is not clear how any remaining freedom can be exploited in order to sharpen any results.

Recently, in [5], a new characterisation of the ν-gap metric was presented in terms of Linear Matrix Inequalities (LMIs) and a rank constraint based on operator theoretic results developed in [6]. In this paper, a slight modification of this new characterisation of reduced-order approximation in the ν-gap metric is presented in Section II in terms of a non-strict norm bound on a Linear Fractional Transformation (LFT). The existence of the factor forming this LFT is covered in Section III. A particular choice of this factor is exploited in a way that allows use of standard LMI analysis techniques in Section IV. The construction of feasible points to the rank

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constrained LMIs obtained leads to a new technique for stepwise model order reduction in the ν-gap metric, as developed in Section V. A numerical example is presented in Section VI to illustrate the algorithm.

II. Reduced order approximation in the ν-gap metric

Given a $P_0 \in \mathcal{R}$ and a minimal state-space realisation (A, B, C, D) of order n, a natural model reduction problem is to seek an order $k < n$ realisation such that the corresponding transfer matrix $P_1 \in \mathcal{R}$ satisfies $\delta_\nu(P_0, P_1) < \beta$ for some (small) constant β. To this end, the ν-gap metric formula [2]

$$\delta_\nu(P_0, P_1) = \inf_{Q, Q^{-1} \in \mathcal{L}_\infty, \, \mathrm{wno}\det(Q)=0} \|G_0 - G_1 Q\|_\infty, \quad (1)$$

suggests the problem of reduced-order approximation can be approached via Hankel norm approximation of a minimal realisation for the graph symbol G_0 of P_0, as discussed in [7], [2]. Indeed, it can be shown that [2, Thm 8.6]

$$\sigma_{k+1} \leq \inf_{P_1 \in \mathfrak{P}(k)} \delta_\nu(P_0, P_1) \leq u_k, \quad (2)$$

with $s_k := \sum_{i=k+1}^{n-r+1} \arcsin \sigma_i$ and

$$u_k := \begin{cases} \sin(s_k), & \text{if } s_k < \frac{\pi}{2}; \\ 1, & \text{otherwise,} \end{cases}$$

where $\mathfrak{P}(k)$ denotes the transfer matrices $P_1 \in \mathcal{R}$ which have a minimal realisation of order less than or equal to $k < n$, and the n Hankel singular values σ_i of the normalised right graph symbol G_0 are such that

$$1 > \sigma_1 \geq \ldots \geq \sigma_{n-r} > \sigma_{n-r+1} = \ldots = \sigma_n.$$

Note that the upper and lower bounds coincide when $k = n - r$, where r is the multiplicity of the smallest Hankel singular value σ_n. In [5], a characterisation of the ν-gap metric in terms of strict LMIs and a rank constraint (LMI-rank) was presented. Towards using this new characterisation to recover a result of the type given by (2), whereby for $k = n - r$, the lower bound on the ν-gap metric is achievable, it makes sense to seek an LMI-rank characterisation in terms of non-strict inequalities. The following lemma, underpinning the subsequent development, is a slight modification to [5, Theorem 3], where the strict norm-bound is now non-strict (see [8, Theorem 3.3] for a proof, which is essentially the same as the corresponding result in [5]).

Lemma 1: Given $P_0, P_1 \in \mathcal{R}$ and a

$$\beta < b_{\mathrm{opt}}(P_0) := \sup_{C \in \mathcal{R} \text{ s.t. } [P_0, C] \in \mathcal{RH}_\infty(\mathbb{C}_+)} b(P_0, C),$$

let $R, R^{-1} \in \mathcal{RH}_\infty(\mathbb{C}_+)$ be such that

$$\begin{bmatrix} \beta I & \tilde{G}_0^{\sim} \end{bmatrix} J_G \begin{bmatrix} \beta I \\ \tilde{G}_0 \end{bmatrix} = R^{\sim} J_R R, \quad (3)$$

where J_G is a signature matrix of the form $\left[\begin{smallmatrix} I & 0 \\ 0 & -I \end{smallmatrix}\right]$ and J_R is of the form $\left[\begin{smallmatrix} -I & 0 \\ 0 & I \end{smallmatrix}\right]$. Then the following are equivalent:

1) $\delta_\nu(P_0, P_1) \leq \beta$;

2) The LFT

$$F(R, P_1) := (R_{11} P_1 + R_{12})(R_{21} P_1 + R_{22})^{-1} \quad (4)$$

is such that $F(R, P_1) \in \mathcal{RH}_\infty(\mathbb{C}_+)$ and $\|F(R, P_1)\|_\infty \leq 1$.

III. Existence and selection of the J-spectral factor

In this section the conditions for the existence of an $R, R^{-1} \in \mathcal{RH}_\infty(\mathbb{C}_+)$ satisfying Lemma 1 are established, utilising a minimal state-space realisation (A, B, C, D) of $P_0 \in \mathcal{R}^{m \times q}$. The existence of such an R for the case $D = 0$ was previously established in [5], and although the $D \neq 0$ case required here essentially follows a similar line of proof, a particular choice of R is required to ensure that the state-space representation of the LFT $F(R, P_1)$ can be expressed as an affine function of the state-space matrices of P_1. To this end, let the symmetric matrices $X > 0$ and $Y > 0$ be stabilising solutions of the generalised control and filtering Riccati equations

$$\begin{aligned} X(A - BS^{-1}D^T C) + (A - BS^{-1}D^T C)^T X \\ + C^T \tilde{S}^{-1} C - XBS^{-1}B^T X = 0 \end{aligned} \quad (5)$$

and

$$\begin{aligned} (A - BD^T \tilde{S}^{-1} C) Y + Y (A - BD^T \tilde{S}^{-1} C)^T \\ - Y C^T \tilde{S}^{-1} C Y + BS^{-1} B^T = 0, \end{aligned} \quad (6)$$

where $S := I + D^T D$ and $\tilde{S} := I + DD^T$.[1] The positive square roots of the eigenvalues of the product of the solutions to (5) and (6), $\upsilon_i = \lambda_i^{1/2}(YX)$, are called the *LQG-characteristic values* of P_0. The realisation (A, B, C, D) is an *LQG-balanced realisation* [10], [11] if $X = Y = \mathrm{diag}(\upsilon_i) = \Upsilon$, are the solutions to (5) and (6) respectively, with $\upsilon_1 \geq \upsilon_2 \geq \cdots \geq \upsilon_n > 0$. The Hankel singular values of a normalised graph symbol are related to the LQG-characteristic values by $\sigma_i^2 = \frac{\upsilon_i^2}{1 + \upsilon_i^2} < 1$, for $i \in \{1, n\}$ [12].

By [9, Thm. 13.37], a state-space realisation for the transfer matrix $\Omega_0 := \left[\begin{smallmatrix} \beta I_{m+q} \\ \tilde{G}_0 \end{smallmatrix}\right]$ in Lemma 1 is given by

$$\begin{aligned} \Omega_0(s) &= \left(\begin{array}{c|c} A + LC & \begin{bmatrix} -L & B + LD \end{bmatrix} \\ \hline \begin{matrix} 0_{(m+q)\times n} \\ \tilde{S}^{-1/2} C \end{matrix} & \begin{matrix} \beta I_{m+q} \\ \begin{bmatrix} -\tilde{S}^{-1/2} & \tilde{S}^{-1/2} D \end{bmatrix} \end{matrix} \end{array}\right) \\ &=: \left(\begin{array}{c|c} A_\Omega & B_\Omega \\ \hline C_\Omega & D_\Omega \end{array}\right), \end{aligned} \quad (7)$$

where

$$L := -(BD^T + YC^T) S^{-1}. \quad (8)$$

The existence of $R, R^{-1} \in \mathcal{RH}_\infty(\mathbb{C}_+)$ such that (3) holds can be now established for $\beta < b_{\mathrm{opt}}(P_0)$, using the same

[1] Such solutions always exist, because (A, B, C, D) is a minimal realisation [9, Corol. 13.8]

steps as in Section IV of [5], choosing $R := \left[\begin{smallmatrix} 0 & I_m \\ I_q & 0 \end{smallmatrix}\right]\Pi$, where

$$\begin{aligned} \Pi &:= \left(\begin{array}{c|c} A_\Omega & B_\Omega \\ \hline L_\Pi & \Pi_\infty \end{array}\right), \\ L_\Pi &:= J_{qm}^{-1}\Pi_\infty^{-T}(D_\Omega^T J_{lm} C_\Omega + B_\Omega^T Z), \\ Z &:= \frac{\beta^2}{1-\beta^2} X (I - \frac{\beta^2}{1-\beta^2} YX)^{-1} = Z^T, \end{aligned} \quad (9)$$

and Π_∞ is any non-singular matrix such that

$$\begin{aligned} D_\Omega^T J_{lm} D_\Omega &= \begin{bmatrix} \beta^2 I_m - \tilde{S}^{-1} & \tilde{S}^{-1}D \\ D^T\tilde{S}^{-1} & \beta^2 I_q - D^T\tilde{S}^{-1}D \end{bmatrix} \\ &= \Pi_\infty^T J_{qm} \Pi_\infty. \end{aligned} \quad (10)$$

In order to apply standard LMI analysis to the LFT $F(R, P_1)$, a specific choice of Π_∞ must be made. Let

$$\begin{aligned} \Pi_\infty &= \begin{bmatrix} \Pi_{\infty 11} & \Pi_{\infty 12} \\ \Pi_{\infty 21} & \Pi_{\infty 22} \end{bmatrix} \\ &:= \begin{bmatrix} 0 & \beta(I_q + D^T(I_m - \beta^2\tilde{S})^{-1}D)^{\frac{1}{2}} \\ (\tilde{S}^{-1} - \beta^2 I_m)^{\frac{1}{2}} & -(\tilde{S}^{-1} - \beta^2 I_m)^{-\frac{1}{2}}\tilde{S}^{-1}D \end{bmatrix} \end{aligned}$$

which is a non-singular solution of (10) if $\overline{\sigma}(D) < \sqrt{1-\beta^2}/\beta$, where $S = I + D^T D$ and $\tilde{S} = I + DD^T$. Note that $\Pi_{\infty 12}$ and $\Pi_{\infty 21}$ are non-singular. For notational convenience define

$$\begin{aligned} M_1 &:= -\Pi_{\infty 12}^{-T}(\Pi_{\infty 22}^T \Pi_{\infty 21}^{-T} + D^T), \\ M_2 &:= \Pi_{\infty 12}^{-T},\ M_3 := \Pi_{\infty 21}^{-T}, \end{aligned} \quad (11)$$

The following lemma yields a state-space representation for R and the LFT $F(R, P_1)$.

Lemma 2: Given a minimal state-space realisation (A, B, C, D) for $P_0 \in \mathcal{R}^{m\times q}$ and a real $\beta < \min\{b_{\rm opt}(P_0), 1/\sqrt{1+\overline{\sigma}(D)^2}\}$, let the matrix L be defined by (8), Z be defined by (9) and (M_1, M_2, M_3) be defined as in (11). Then the transfer matrix

$$R(s) := \left(\begin{array}{c|cc} A_R & B_{R1} & B_{R2} \\ \hline C_{R1} & D_{R11} & D_{R12} \\ C_{R2} & 0 & D_{R22} \end{array}\right),$$

with

$$\begin{aligned} A_R &:= A + LC, \\ \begin{bmatrix} B_{R1} & B_{R2} \end{bmatrix} &:= \begin{bmatrix} -L & B + LD \end{bmatrix}, \\ \begin{bmatrix} C_{R1} \\ C_{R2} \end{bmatrix} &:= \begin{bmatrix} -M_3 & 0 \\ M_1 & M_2 \end{bmatrix} \begin{bmatrix} \tilde{S}^{-1}C - L^T Z \\ B^T Z \end{bmatrix}, \\ \begin{bmatrix} D_{R11} & D_{R12} \\ D_{R21} & D_{R22} \end{bmatrix} &:= \begin{bmatrix} M_3^{-T} & -M_3^{-T}(M_1^T M_2^{-T} + D) \\ 0 & M_2^{-T} \end{bmatrix}, \end{aligned}$$

is such that $R, R^{-1} \in \mathcal{RH}_\infty(\mathbb{C}_+)$ and (3) is satisfied. Moreover, for any transfer matrix $P_1 \in \mathcal{R}^{m\times q}$, with a realisation $(\hat{A}, \hat{B}, \hat{C}, \hat{D})$, the linear fractional transformation $F(R, P_1)$ in (4) has a realisation (A_F, B_F, C_F, D_F) that exhibits an affine dependence on the realisation for P_1. In particular, when $\hat{A} \in \mathbb{R}^{k\times k}$,

$$\left(\begin{array}{c|c} A_F & B_F \\ \hline C_F & D_F \end{array}\right) = \left(\begin{array}{c|c} \bar{A} + \underline{B}\hat{\Phi}\underline{C} & \bar{B} + \underline{B}\hat{\Phi}\underline{D}_{22} \\ \hline \bar{C} + \underline{D}_{11}\hat{\Phi}\underline{C} & \bar{D} + \underline{D}_{11}\hat{\Phi}\underline{D}_{22} \end{array}\right),$$

where $\hat{\Phi} = \left[\begin{smallmatrix} \hat{A} & \hat{B} \\ \hat{C} & \hat{D} \end{smallmatrix}\right]$,

$$\begin{aligned} &\bar{A} := \begin{bmatrix} \bar{A}_{11} & 0 \\ 0 & 0_k \end{bmatrix},\ \bar{B} := \begin{bmatrix} -B_{R2}D_{R22}^{-1} \\ 0_{k\times q} \end{bmatrix},\ \underline{B} := \begin{bmatrix} 0 & B_{R1} \\ I_k & 0 \end{bmatrix}, \\ &\bar{C} := \begin{bmatrix} -C_{R1} + D_{R12}D_{R22}^{-1}C_{R2} & 0_{m\times k} \end{bmatrix},\ \bar{D} := D_{R12}D_{R22}^{-1}, \\ &\underline{C} := \begin{bmatrix} 0 & I_k \\ -D_{R22}^{-1}C_{R2} & 0 \end{bmatrix}, \\ &\underline{D}_{11} := \begin{bmatrix} 0_{m\times k} & -D_{R11} \end{bmatrix},\ \underline{D}_{22} := \begin{bmatrix} 0_{k\times q} \\ -D_{R22}^{-1} \end{bmatrix}, \end{aligned}$$

and $\bar{A}_{11} := A_R - B_{R2}D_{R22}^{-1}C_{R2}$.

Proof: The proof follows by performing standard state-space manipulations to yield the state-space realisation for $F(R, P_1)$. See [8, Section 3.2] for further details. ■

IV. A RANK-CONSTRAINED LMI CHARACTERISATION

In this section, Lemma 1 and Lemma 2 are exploited, via standard LMI analysis [13], to obtain a rank constrained LMI sufficient condition which characterises the existence of a transfer matrix $\hat{P} \in \mathfrak{P}(k)$ that lies within a specified ν-gap distance of a nominal transfer matrix $P \in \mathfrak{P}(n)$. The development of this sufficient condition follows closely that in [5]. When $k < n$, $\mathfrak{P}(k) \subset \mathfrak{P}(n)$, and this characterisation is of a *reduced-order* model $\hat{P}$ that approximates P to within a given bound on the ν-gap error.

Theorem 3: Given a transfer matrix $P \in \mathcal{R}^{m\times q}$, with a minimal realisation (A, B, C, D) of order n (i.e. $A \in \mathbb{R}^{n\times n}$), and given a number $0 < \beta < \min\{b_{\rm opt}(P), 1/\sqrt{1+\overline{\sigma}(D)^2}\}$, let the matrix L be as defined in (8) and Z be as defined in (9). Then there exists a $\hat{P} \in \mathfrak{P}(k) \subset \mathcal{R}^{m\times q}$, where $k < n$, such that $\delta_\nu(P, \hat{P}) \le \beta$ if there exists matrices $\Xi = \Xi^T > 0$ and $\Psi = \Psi^T > 0$ such that

$$\Xi A_\Xi + A_\Xi^T \Xi + S_\Xi \le 0 \quad (12)$$

$$A_\Psi \Psi + \Psi A_\Psi^T + T_\Psi \le 0 \quad (13)$$

$$\begin{bmatrix} \Xi & I \\ I & \Psi \end{bmatrix} \ge 0 \quad (14)$$

$$\text{and} \quad \text{rank}(\Xi - \Psi^{-1}) \le k, \quad (15)$$

where

$$\begin{aligned} A_\Xi &:= A + LC,\ A_\Psi := A_\Xi + \begin{bmatrix} L & B \end{bmatrix} \bar{S} \begin{bmatrix} L^T Z - \tilde{S}^{-1}C \\ B^T Z \end{bmatrix}, \\ S_\Xi &:= \begin{bmatrix} L^T Z - \tilde{S}^{-1}C \\ B^T Z \end{bmatrix}^T \bar{S} \begin{bmatrix} L^T Z - \tilde{S}^{-1}C \\ B^T Z \end{bmatrix}, \\ T_\Psi &:= -\begin{bmatrix} L & B \end{bmatrix} \bar{S} \begin{bmatrix} L^T \\ B^T \end{bmatrix}, \\ \bar{S} &:= \frac{1}{1-\beta^2} \begin{bmatrix} \tilde{S} & D \\ D^T & I_q - \frac{1}{\beta^2} S^{-1} \end{bmatrix}, \end{aligned}$$

$S := I + D^T D$ and $\tilde{S} := I + DD^T$.

Proof: The proof follows using standard LMI analysis [14] of the state-space realisation of the LFT $F(R, \hat{P})$, Schur complement arguments, the non-strict version of the

Elimination Lemma given in [15] and the sufficient non-strict bounded real LMI condition in [8, Lemma B.4]. See [8, Section 3.3] for details. ∎

Given a feasible point in the set of matrices defined by the rank constrained LMI condition, a state-space realisation for $\hat{P}$ can be constructed explicitly, from a family of realisations parameterised in an affine fashion.

Theorem 4: For any matrices $\Xi > 0$ and $\Psi > 0$ that satisfy the conditions in Theorem 3, and defining

$$\Xi_2\Xi_2^T := \Xi - \Psi^{-1} \geq 0,$$

$$\underline{X} := \begin{bmatrix} \Xi & \Xi_2 \\ \Xi_2^T & I \end{bmatrix}, \quad H := \begin{bmatrix} \bar{A}^T\underline{X} + \underline{X}\bar{A} & \underline{X}\bar{B} & \bar{C}^T \\ \bar{B}^T\underline{X} & -I & \bar{D} \\ \bar{C} & \bar{D} & -I \end{bmatrix},$$

$$U_{\underline{X}} := \begin{bmatrix} \underline{B}^T\underline{X} & 0 & \underline{D}_{11}^T \end{bmatrix}, \quad V := \begin{bmatrix} \underline{C} & \underline{D}_{22} & 0 \end{bmatrix},$$

where the matrices $\bar{A}$, $\bar{B}$, and so on, are defined in Section III, a realisation $(\hat{A}, \hat{B}, \hat{C}, \hat{D})$ of order k for $\hat{P}$ such that $\delta_\nu(P, \hat{P}) \leq \beta$ is given by

$$\hat{\Phi} = \begin{bmatrix} \hat{A} & \hat{B} \\ \hat{C} & \hat{D} \end{bmatrix} := \hat{\Phi}_0 + \hat{\Phi}_1\Gamma\hat{\Phi}_2,$$

where

$$\hat{\Phi}_0 := (V_W U_{\underline{X}}^T)^{-1}(H_{23}H_{33}^\dagger H_{13}^T - H_{12}^T)(VU_W^T)^{-1}$$

$$\hat{\Phi}_1 := (V_W U_{\underline{X}}^T)^{-1}(H_{23}H_{33}^\dagger H_{23}^T - H_{22})^{1/2}$$

$$\hat{\Phi}_2 := (H_{12}H_{33}^\dagger H_{13}^T - H_{11})^{1/2}(VU_W^T)^{-1}$$

$$W^\perp := \begin{bmatrix} U_{\underline{X}}^T & V^T \end{bmatrix}^\perp, \; U_W := \begin{bmatrix} U_{\underline{X}}^T & W^{\perp T} \end{bmatrix}^\perp$$

$$V_W := \begin{bmatrix} V^T & W^{\perp T} \end{bmatrix}^\perp$$

$$\begin{bmatrix} H_{11} & H_{12} & H_{13} \\ H_{12}^T & H_{22} & H_{23} \\ H_{13}^T & H_{23}^T & H_{33} \end{bmatrix} := \begin{bmatrix} U_W \\ V_W \\ W^\perp \end{bmatrix} H \begin{bmatrix} U_W \\ V_W \\ W^\perp \end{bmatrix}^T$$

and Γ is an arbitrary matrix such that $\|\Gamma\| \leq 1$.

Proof: See [8, Theorem 3.13]. ∎

In the next section, it is shown how to constructively obtain, for sufficiently large β, feasible Ξ and Ψ satisfying Theorem 3 by exploiting the structure of the LMIs in the case $k = n - r$, where r is the multiplicity of the smallest Hankel singular value of a normalised graph symbol of $P \in \mathcal{R}$. This gives rise to the step-wise model order reduction procedure developed therein.

V. A STEP-WISE PROCEDURE FOR MODEL ORDER REDUCTION

In this section, a step-wise technique for model order reduction in the ν-gap metric is developed, utilising an explicitly constructed feasible point to Theorem 3 at each step. In particular, if an initial LQG-balancing of the state-space realisation of P is performed, feasible Ξ, Ψ can be explicitly expressed in terms β and the LQG-characteristic values $\upsilon_i : 1 \leq i \leq n$.

Lemma 5: Given $P \in \mathcal{R}$ with a state-space realisation that is LQG-balanced with LQG-characteristic values $\upsilon_1 \geq \ldots \geq \upsilon_{n-r} > \upsilon_{n-r+1} = \ldots = \upsilon_n > 0$, the positive definite matrices

$$\Xi = (1 + \upsilon_n^2) \times \mathrm{diag}\left(\frac{\upsilon_i}{(1 + \upsilon_i^2)(1 - \upsilon_n^2\upsilon_i^2)}\right)$$

$$\Psi = (1 + \upsilon_n^2) \times \mathrm{diag}\left(\frac{\upsilon_i(1 - \upsilon_n^2\upsilon_i^2)}{(1 + \upsilon_i^2)\upsilon_n^2}\right)$$

satisfy the LMI and rank conditions of Theorem 3 for $\beta = \frac{\upsilon_n}{\sqrt{1+\upsilon_n^2}}$ and $k = n - r$.

Proof: The proof follows using the definitions of Ξ and Ψ, some lengthy algebra, and the observation that

$$\Xi A_\Xi + A_\Xi^T\Xi + S_\Xi = 0$$

$$A_\Psi\Psi + \Psi A_\Psi^T + T_\Psi = 0.$$

So Ξ and Ψ satisfy the LMIs but not necessarily the rank constraint in Theorem 3 for any $\beta < b_{\mathrm{opt}}(P)$. However, setting $\beta = \frac{\upsilon_n}{\sqrt{1+\upsilon_n^2}} = \sigma_n$, gives

$$\Xi\Psi = \frac{(1 + \upsilon_n^2)^2}{\upsilon_n^2} \times \mathrm{diag}\left(\frac{\upsilon_i^2}{(1 + \upsilon_i^2)^2}\right),$$

which has r eigenvalues equal to one, and $n - r$ eigenvalues greater than one. Thus the rank constraint (15) is also satisfied for $k = n - r$ for this value of β. ∎

The feasible Ξ, Ψ from Lemma 5 in fact characterise the $\hat{P} \in \mathfrak{P}(k)$ that satisfy the *optimal* lower ν-gap error bound (2) for a given full-order model $P \in \mathfrak{P}(n)$, and $k = n - r$, where r is the multiplicity of the smallest LQG-characteristic value of P.

Theorem 6: Given a $P \in \mathcal{R}$ with McMillan degree n, with LQG characteristic values $\upsilon_1 \geq \ldots \geq \upsilon_{n-r} > \upsilon_{n-r+1} = \ldots = \upsilon_n > 0$, there exists a $\hat{P} \in \mathfrak{P}(k)$, where $k = n - r$, such that

$$\delta_\nu(P, \hat{P}) = \frac{\upsilon_n}{\sqrt{1 + \upsilon_n^2}} = \sigma_n,$$

where σ_n is the smallest Hankel singular value of a right (or left) normalised graph symbol of P. Furthermore, such $\hat{P}$ are given by Theorem 4.

Proof: Lemma 5 gives feasible Ξ and Ψ for Theorem 3, when $\beta = \frac{\upsilon_n}{\sqrt{1+\upsilon_n^2}} = \sigma_n$. Thus, by Theorem 4, a $\hat{P}$ of order $k = n - r$ can be constructed, with an arbitrary choice of $\|\Gamma\| \leq 1$ such that $\delta_\nu(P, \hat{P}) \leq \frac{\upsilon_n}{\sqrt{1+\upsilon_n^2}} = \sigma_n$. But, by (2) which states that the ν-gap between P and $\hat{P}$ in this case cannot be smaller than σ_n, this inequality must be an equality. ∎

Reduced-order transfer matrices satisfying Theorem 6 can be parameterised by Theorem 4 in terms of an arbitrary matrix $\|\Gamma\| \leq 1$– i.e. there is freedom in the choice of such a $\hat{P}$. Although the exploitation of this is not yet fully understood note the affine dependence of the realisation matrix $\hat{\Phi}$ on the "free" parameter Γ.

A sequential step-wise scheme is now introduced, based on constructing a feasible Ξ and Ψ at each step, to obtain a reduced order model with a computable ν-gap error bound for a desired $k < n - r$. The reduced order model constructed at each step is optimal in that the lower bound

in (2) is achieved. To this end, consider a $P \in \mathfrak{P}(n)$ and applying Lemma 5 to repeatedly construct a sequence of feasible points to satisfy Theorem 3, generating a sequence of approximants $\hat{P}_0, \cdots, \hat{P}_j$ such that

$$\text{order}\{\hat{P}_j\} = n - \sum_{i=0}^{j-1} r_i, \quad \hat{P}_0 := P,$$

where $\text{order}\{\cdot\}$ denotes the order of a transfer matrix and r_j is the multiplicity of the smallest LQG-characteristic value for the model $\hat{P}_j$. For simplicity, if $r_i = 1$ for all i, then the number of iterations required is $n - k$, and

$$\text{order}\{\hat{P}_j\} = n - j.$$

As the ν-gap metric obeys the triangle inequality (and a tighter trigonometric inequality involving arcsines [1, Theorem 3.1]), it can be readily seen that the repeated application of Lemma 5 yields the following bound

$$\delta_\nu(P, \hat{P}) \leq \sin\left(\sum_{i=0}^{n-k} \arcsin(\delta_\nu(\hat{P}_i, \hat{P}_{i+1}))\right) \leq \sum_{i=0}^{n-k} \delta_\nu(\hat{P}_i, \hat{P}_{i+1}). \quad (16)$$

The following algorithm summarises the step-wise technique.

Algorithm 7: Given a minimal LQG-balanced realisation (A, B, C, D) for $P \in \mathcal{R}^{m\times q}$, of order n, a reduced-order $\hat{P} \in \mathcal{R}^{m\times q}$, of order k, that satisfies (16) can be constructed in the following manner:

1) Set $i = 0$, $n_0 = n$, $\hat{P}_0 = P$;
2) Set $\beta_i = \frac{\upsilon_{n_i}}{\sqrt{1+\upsilon_{n_i}^2}}$, where υ_{n_i} has multiplicity r_i, and $n_i = \text{order}(\hat{P}_i)$;
3) Construct Ξ, Ψ using Lemma 5 for $k_i := n_i - r_i$;
4) Construct $\hat{\Phi}_{i+1} = \left[\begin{smallmatrix} \hat{A}_{i+1} & \hat{B}_{i+1} \\ \hat{C}_{i+1} & \hat{D}_{i+1} \end{smallmatrix}\right]$ by Theorem 4 with $\Gamma = 0$;
5) Perform LQG-balancing on $\hat{P}_{i+1} := (\hat{A}_{i+1}, \hat{B}_{i+1}, \hat{C}_{i+1}, \hat{D}_{i+1})$;
6) Store $\delta_\nu(\hat{P}_i, \hat{P}_{i+1}) \equiv \beta_i$;
7) If $n_{i+1} := \text{order}(\hat{P}_{i+1}) \leq k$, set $\hat{P} = \hat{P}_{i+1}$, stop;
8) Set $i = i + 1$; loop from step 2;

Note that, while producing a reduced-order model that satisfies the ν-gap metric error bound in (16), this bound is not an *a priori* bound. In step 4, the "central" solution $\hat{\Phi}$ from Theorem 4 is used, with $\Gamma = 0$. The key to obtaining an a priori upper bound on the ν-gap error would be to somehow exploit the freedom associated with Γ in Theorem 4 so that at each step there is a reduction in the size of the LQG-characteristic values of the synthesised realisation of $\hat{P}_{i+1}$. This is a topic of ongoing research.

VI. Examples

In this section, the algorithm developed in Section V will be put to use for reduced-order modeling in the ν-gap metric. In particular, the step-wise technique described by Algorithm 7 will be compared to a step-wise iterative method based on Hankel norm approximation of a normalised graph symbol from [2, Section 8.2] (HankG) within the context of an example system. In particular, we consider a reduced-order approximation of a MIMO model of a distillation column with I/O delays, taken from MATLAB (see [16] for more details). The distillation column is used to separate a mix of methanol and water (the feed) into bottom products (mostly water) and a methanol-saturated distillate. The regulated output variables are:

- Percentage X_D of methanol in the distillate;
- Percentage X_B of methanol in the bottom products.

The goal is to maximise X_D by designing a controller that adjusts the reflux flow rate R and the steam flow rate S in the re-boiler. To obtain a linearised model around the steady-state operating conditions, the transient responses to pulses in steam and reflux flow are fitted by first-order plus delay models. The resulting transfer function model is

$$\begin{bmatrix} X_D(s) \\ X_B(s) \end{bmatrix} = \begin{bmatrix} \frac{12.8e^{-1s}}{16.7s+1} & \frac{-18.9e^{-3s}}{21.0s+1} \\ \frac{6.6e^{-7s}}{10.9s+1} & \frac{-19.4e^{-3s}}{14.4s+1} \end{bmatrix} \begin{bmatrix} R(s) \\ S(s) \end{bmatrix} =: P_{td} \begin{bmatrix} R(s) \\ S(s) \end{bmatrix},$$

where the exponential transfer function e^{-sT} is the Laplace transform of a T second delay. This irrational transfer function P_{td} can be approximated by a rational transfer function P using a Padé [17] approximation. Choosing :

- a 3-rd order Padé approximation for the 1 second and 3 second delays;
- a 5-th order Padé approximation for the 7 second delay;

yields an approximate transfer matrix P with a minimal state-space realisation of order 15. It is desired to approximate P, using the ν-gap metric as a measure of approximation error, for instance in order to design a low-order feedback controller with sufficient performance with the full-order model.

The bounds for the best possible approximation in the ν-gap metric are calculated via (2) for this system and given in Table I. It can be seen that a reduction to a 9-th order system would appear to provide a good balance between the order of the approximation and the ν-gap error.

	$\inf_{\hat{P}\in\mathfrak{P}(k)} \delta_\nu(P, \hat{P})$	
Order (k)	Lower bound	Upper bound
12	0.002954	0.005614
11	0.005375	0.010988
10	0.044564	0.055538
9	0.050754	0.106143
8	0.142019	0.246283
7	0.201343	0.436380

TABLE I

Upper and lower ν-gap bounds for distillation column.

Table II gives the results of applying both step-wise model order reduction procedures, HankG and Algorithm 7, to the distillation column, which has $b_{\text{opt}}(P) = 0.2851$. These are also shown graphically in Figure 1. Note that Algorithm 7 yields approximants nearer the lower error bound even

Order (k)	Algorithm 7		HankG
	$\delta_\nu(P,\hat{P})$	$\sin\sum_i \arcsin(\delta_\nu(\hat{P}_i,\hat{P}_{i+1}))$	$\delta_\nu(P,\hat{P})$
12	0.0036	0.0047	0.0056
11	0.0061	0.0088	0.0101
10	0.0483	0.0531	0.0537
9	0.0557	0.0784	0.1028
8	0.1805	0.2130	0.2258
7	0.2766	0.3457	0.4142

TABLE II

Comparison of step-wise methods for approximating distillation column.

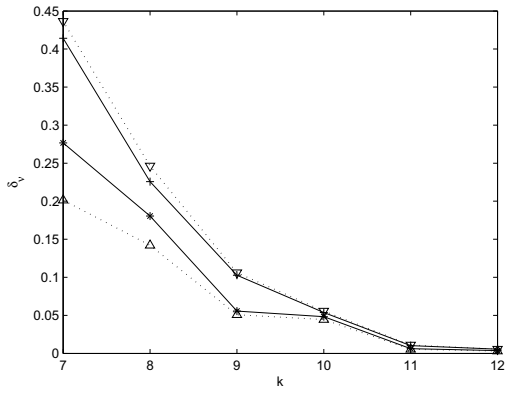

Fig. 1. *ν-gap approximation bounds for distillation column : ($\triangle$) $\inf_{\hat{P}\in\mathfrak{P}(k)}\delta_\nu(P,\hat{P})$ lower bound; ($\triangledown$) $\inf_{\hat{P}\in\mathfrak{P}(k)}\delta_\nu(P,\hat{P})$ upper bound; (+) $\delta_\nu(P,\hat{P})$ using HankG; (*) $\delta_\nu(P,\hat{P})$ using Alg. 7.*

though we are just simplistically using $\Gamma = 0$ at each step, while HankG yields approximants towards the upper error bound.

A plot of the function $\kappa(P,\hat{P})(j\omega)$, which represents the point-wise chordal distance [2] between the full-order nominal model and the approximate model, is shown in Figure 2. Due to the fact that

$$\delta_\nu(P,\hat{P}) = \sup_{\omega\in\mathbb{R}} \kappa(P,\hat{P})(j\omega),$$

which along with Figure 2, shows that a smaller ν-gap error is indeed possible if the value of $\kappa(j\omega)$ is sacrificed slightly around cross-over. Although HankG has smaller κ around cross-over, i.e. better approximation, it yields a higher ν-gap error than Algorithm 7 overall.

VII. Conclusions

While as of yet, the freedom in the parameterisation of optimal approximants has not been exploited, the new step-wise technique for approximation in the ν-gap metric presented in this paper appears to perform well for numerical examples, yielding approximants between the upper and lower approximation bounds. In particular, for SISO systems, the algorithm from [2] and the new step-wise method yield identical results– see [8]. For MIMO systems, numerical experience generally shows improvement in the actual ν-gap error obtained over [2], so far without exploiting any of the freedom in the optimal approximant at each step. Further work lies in choosing a particular optimal approximant

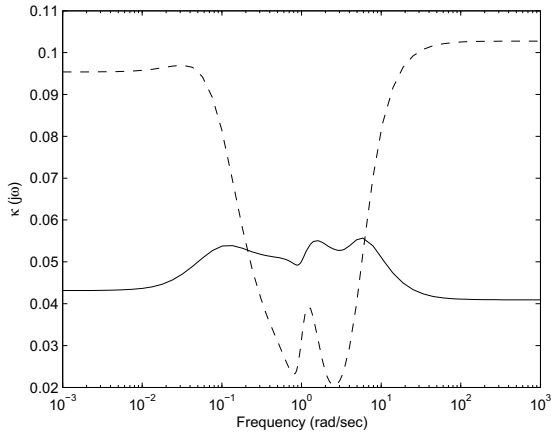

Fig. 2. *$\kappa(P,\hat{P})(j\omega)$ for 9-th order approximation of distillation column : (dashed) HankG; (solid) Alg. 7.*

to satisfy additional desired properties or achieve certain objectives.

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