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Central Suboptimal H_∞ Filter Design for Linear Time-Varying Systems with State and Measurement Delays

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***Abstract*— This paper presents the central finite-dimensional H_∞ filters for linear systems with state and measurement delays, that are suboptimal for a given threshold γ with respect to a modified Bolza-Meyer quadratic criterion including the attenuation control term with the opposite sign. The paper first presents the central suboptimal H_∞ filter for linear systems with state and measurement delays, which consists, in the general case, of an infinite set of differential equations. Then, the finite-dimensional central suboptimal H_∞ filter is designed in case of linear systems with commensurable state and measurement delays, which contains a finite number of equations for any fixed filtering horizon; however, this number still grows unboundedly as time goes to infinity. To overcome that difficulty, the alternative central suboptimal H_∞ filter is designed for linear systems with state and measurement delays, which is based on the alternative optimal H_2 filter from [39]. Numerical simulations are conducted to verify performance of the designed central suboptimal filters for linear systems with state and measurement delays against the central suboptimal H_∞ filter available for linear systems without delays.**

I. Introduction

Over the past two decades, the considerable attention has been paid to the H_∞ estimation problems for linear and nonlinear systems with and without time delays. The seminal papers in H_∞ control [1] and estimation ([2]–[4]) established a background for consistent treatment of filtering/controller problems in the H_∞-framework. The H_∞ filter design implies that the resulting closed-loop filtering system is robustly stable and achieves a prescribed level of attenuation from the disturbance input to the output estimation error in L_2/l_2-norm. A large number of results on this subject has been reported for systems in the general situation, linear or nonlinear (see ([5]–[13]). For the specific area of linear time-delay systems, the H_∞-filtering problem has also been extensively studied (see [14]–[34]). The sufficient conditions for existence of an H_∞ filter, where the filter gain matrices satisfy Riccati equations, were obtained for linear systems with state delay in [35] and with measurement delay in [36]. However, the criteria of existence and suboptimality of solution for the central H_∞ filtering problems based on the reduction of the original H_∞ problem to the induced H_2 one, similar to those obtained in [1], [4] for linear systems without delay, remain yet unknown for linear systems with state and measurement delays.

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The paper first presents the central suboptimal H_∞ filter for linear systems with state and measurement delays, based on the optimal H_2 filter from [37], which consists, in the general case, of an infinite set of differential equations. In contrast to the results previously obtained for linear systems with state [35] or measurement delay [36], the paper reduces the original H_∞ filtering problem to the corresponding H_2 (mean-square) filtering problem, using the technique proposed in [1]. To the best authors' knowledge, this is the first paper which applies the reduction technique of [1] to linear systems with both, state and measurement, delays. Indeed, application of the reduction technique makes sense, since the optimal filtering equations solving the H_2 (mean-square) filtering problems have been obtained for linear systems with state and measurement delays [38], [37]. Then, the finite-dimensional central suboptimal H_∞ filter is designed in case of linear systems with commensurable state and measurement delays, which contains a finite number of equations for any fixed filtering horizon; however, this number still grows unboundedly as time goes to infinity. To overcome that difficulty, the alternative central suboptimal H_∞ filter is designed for linear systems with state and measurement delays, which is based on the alternative optimal H_2 filter from [39]. The alternative filter contains only two differential equations for determining the estimate and filter gain matrix, regardless of the filtering horizon.

II. H_∞ Filtering Problem Statement for LTV Systems with State and Measurement Delays

Consider the following continuous-time LTV system with state and measurement delays:

$$\mathscr{S}_1: \ \dot{x}(t) = A(t)x(t-h) + B(t)\omega(t), \qquad (1)$$
$$y(t) = C(t)x(t-\tau) + D(t)\omega(t), \qquad (2)$$
$$z(t) = L(t)x(t), \qquad (3)$$
$$x(\theta) = \varphi(\theta), \quad \forall\theta \in [t_0-h, t_0] \qquad (4)$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $z(t) \in \mathbb{R}^q$ is the signal to be estimated, $y(t) \in \mathbb{R}^m$ is the measured output, $\omega(t) \in \mathscr{L}_2^p[0,\infty)$ is the disturbance input. $A(\cdot)$, $B(\cdot)$, $C(\cdot)$, $D(\cdot)$, and $L(\cdot)$ are known continuous functions. $\varphi(\theta)$ is an unknown vector-valued continuous function defined on the initial interval $[t_0-h, t_0]$. The state delay h and measurement delay τ are known.

For the system (1)–(4), the following standard conditions ([4]) are assumed:

- the pair (A,B) is stabilizable; $(\mathscr{C}_1)$
- the pair (C,A) is detectable; $(\mathscr{C}_2)$
- $D(t)B^T(t)=0$ and $D(t)D^T(t)=I_m$. $(\mathscr{C}_3)$

Here, I_m is the identity matrix of dimension $m\times m$. As usual, the first two conditions ensure that the estimation error, provided by the designed H_∞ filter, converge to zero ([40]). The last noise orthonormality condition is technical and corresponds to the condition of independence of the standard Wiener processes (Gaussian white noises) in the stochastic filtering problems ([41]).

Now, consider a full-order $\mathscr{H}_\infty$ filter in the following form $(\mathscr{S}_2)$:

$$\mathscr{S}_2: \dot{x}_f(t)=A(t)x_f(t-h)+K_f(t)[y(t)-C(t)x_f(t-\tau)], \quad (5)$$
$$z_f(t)=L(t)x_f(t), \quad (6)$$

where $x_f(t)$ is the filter state. The gain matrix $K_f(t)$ is to be determined.

Upon transforming the model (1)-(3) to include the states of the filter, the following filtering error system is obtained $(\mathscr{S}_3)$:

$$\mathscr{S}_3: \ \dot{e}(t) = A(t)e(t-h)+B(t)\omega(t)-K_f(t)\tilde{y}(t), \quad (7)$$
$$\tilde{y}(t) = C(t)e(t-\tau)+D(t)\omega(t), \quad (8)$$
$$\tilde{z}(t) = L(t)e(t), \quad (9)$$

where $e(t)=x(t)-x_f(t)$, $\tilde{y}(t)=y(t)-C(t)x_f(t-\tau)$, and $\tilde{z}(t)=z(t)-z_f(t)$.

Therefore, the problem to be addressed is as follows: develop a robust $\mathscr{H}_\infty$ filter of the form (5)-(6) for the LTV system with state delay $(\mathscr{S}_1)$, such that the following two requirements are satisfied:

1) The resulting filtering error dynamics $(\mathscr{S}_3)$ is robustly asymptotically stable in the absence of disturbances, $\omega(t)\equiv 0$;
2) The filtering error dynamics $(\mathscr{S}_3)$ ensures a noise attenuation level γ in an $\mathscr{H}_\infty$ sense. More specifically, for all nonzero $\omega(t)\in\mathscr{L}_2^p[0,\infty)$, the inequality
$$\|\tilde{z}(t)\|_2^2<\gamma^2\left\{\|\omega(t)\|_2^2+\|\varphi(\theta)\|_{2,R,[-h,0]}^2\right\} \quad (10)$$
holds, where $\|f(t)\|_2^2=\int_{t_0}^{\infty}f^T(t)f(t)dt$, $\|\varphi(\theta)\|_{2,R,[t_0-h,t_0]}^2=\int_{t_0-h}^{t_0}\varphi^T(\theta)R\varphi(\theta)d\theta$, R is a positive definite symmetric matrix, and γ is a given real positive scalar.

III. Design of Central H_∞ Filter for LTV Systems with State and Measurement Delays

The proposed design of the central H_∞ filter (see Theorem 4 in [1]) for LTV systems with state and measurement delays is based on the general result (see Theorem 3 in [1]) reducing the H_∞ controller problem to the corresponding H_2 (i.e., optimal linear-quadratic) controller problem. In this paper, only the filtering part of this result, valid for the entire controller problem, is used. Then, the optimal mean-square filter of the Kalman-Bucy type for LTV systems with state and measurement delays [37] is employed to obtain the desired result, which is given by the following theorem.

Theorem 1. I. The central H_∞ filter for the unmeasured state (1) over the observations (2), ensuring the H_∞ noise attenuation condition (10) for the output estimate $z_f(t)$, is given by the equations for the state estimate $x_f(t)$ and the output estimate $z_f(t)$

$$\dot{x}_f(t)=A(t)x_f(t-h)+P_0(t)C^T(t)[y(t)-C(t)x_f(t-\tau)], \quad (11)$$
$$z_f(t)=L(t)x_f(t), \quad (12)$$

with the initial condition $x_f(\theta)=0$ for $\forall\theta\in[t_0-h,t_0]$, and the system of the equations for the matrices $P_k(t)$, $k=\ldots,-1,0,1,\ldots$,

$$dP_k(t)/dt=A(t)P_{k-1}(t-h)+P_{k+1}(t)A^T(t-\tau-kh)+ \quad (13)$$
$$(1/2)[B(t)B^T(t-\tau-kh)+B(t-\tau-kh)B^T(t)]-$$
$$(1/2)[P_0(t)C^T(t)(D(t)D^T(t-\tau-kh))^{-1}C(t-\tau-kh)\times$$
$$P_0^T(t-\tau-kh)-\gamma^{-2}P_0(t)L^T(t)L(t-\tau-kh)P_0(t-\tau-kh)-$$
$$\gamma^{-2}P_0(t-\tau-kh)L^T(t-\tau-kh)L(t)P_0(t)+$$
$$P_0(t-\tau-kh)C^T(t-\tau-kh)(D(t-\tau-kh)D^T(t))^{-1}C(t)P_0^T(t)].$$

with the initial conditions $P_0(t_0)=R^{-1}$ and $P_k(\theta)=0$, $k\neq 0$,

$$\theta\in[max\{t_0-h,t_0+\tau+(k-1)h\},max\{t_0+\tau+kh,t_0\}].$$

II. If the state delay h in (1) and the measurement delay τ in (2) are commensurable, that is, $\tau=qh$, $q=1,2,\ldots$ is a natural number, then the equation (11) for the state estimate $x_f(t)$ and the system of equations (13) for the matrices $P_k(t)$, $k=-q,-q+1,\ldots,0,1,\ldots$, take the following simplified form

$$\dot{x}_f(t)=A(t)x_f(t-h)+P_0(t)C^T(t)[y(t)-C(t)x_f(t-qh)], \quad (14)$$

$$dP_k(t)/dt=A(t)P_{k-1}(t-h)+P_{k+1}(t)A^T(t-(q+k)h)+ \quad (15)$$
$$(1/2)[B(t)B^T(t-(q+k)h)+B(t-(q+k)h)B^T(t)]-$$
$$(1/2)[P_0(t)C^T(t)(D(t)D^T(t-(q+k)h))^{-1}\times$$
$$C(t-(q+k)h)P_0^T(t-(q+k)h)-$$
$$\gamma^{-2}P_0(t)L^T(t)L(t-(q+k)h)P_0(t-(q+k)h)-$$
$$\gamma^{-2}P_0(t-(q+k)h)L^T(t-(q+k)h)L(t)P_0(t)+$$
$$P_0(t-(q+k)h)C^T(t-(q+k)h)(D(t-(q+k)h)D^T(t))^{-1}\times$$
$$C(t)P_0^T(t)],\quad k=-q+1,\ldots,0,1,\ldots,$$
$$dP_{-q}(t)/dt=A(t)P_{-q+1}^T(t)+P_{-q+1}(t)A^T(t)+B(t)B^T(t)+$$
$$\gamma^{-2}P_0(t)L^T(t)L(t)P(t)-P_0(t)C^T(t)C(t)P_0^T(t),\quad k=-q,$$

with the same initial conditions as in (11),(13). If the current filtering horizon t belongs to the semi-open interval $(t_0+(k+q)h,t_0+(k+q+1)h]$, where h is the state delay in (1), then the number of equations in (15) is equal to $k+q$.

Proof. I. First of all, note that the filtering error system (7)-(9) is already in the form used in Theorem 3 from [1]. Hence, according to Theorem 3 from [1], the H-infinity filtering part of this H_∞ controller problem would be equivalent to the H_2 (i.e., optimal mean-square) filtering problem, where the worst disturbance $w_{worst}(t) = \gamma^{-2}B^T(t)Q(t)e(t)$ is realized, and $Q(t)$ is the solution of the equation for the corresponding H_2 (optimal linear-quadratic) control gain. Therefore, the system, for which the equivalent H_2 (optimal mean-square) filtering problem is stated, takes the form

$$\mathscr{S}_4 : \dot{e}(t) = A(t)e(t-h) + \gamma^{-2}B(t)B^T(t)Q(t)e(t) - K_f(t)\tilde{y}(t), \quad (16)$$

$$\tilde{y}(t) = C(t)e(t-\tau) + \gamma^{-2}D(t)B^T(t)Q(t)e(t), \quad (17)$$

$$\tilde{z}(t) = L(t)e(t). \quad (18)$$

As follows from Theorem 3 from [1] and Theorem 1 in [37], the H_2 (optimal mean-square) estimate equations for the error states (16) and (18) are given by

$$\mathscr{S}_5 : \dot{e_f}(t) = A(t)e_f(t-h) - K_f(t)\tilde{y}(t) + P(t)C^T(t)[\tilde{y}(t) - C(t)e_f(t-\tau)], \quad (19)$$

$$\tilde{z}_f(t) = L(t)e_f(t), \quad (20)$$

where $e_f(t)$ and $\tilde{z}_f(t)$ are the H_2 (optimal mean-square) estimates for $e(t)$ and $\tilde{z}(t)$, respectively. In the equation (19), $P(t)$ is the solution of the equation for the corresponding H_2 (optimal mean-square) filter gain, where, according to Theorem 3 from [1], the observation matrix $C(t)$ should be changed to $C(t) - \gamma^{-1}L(t)$ ($L(t)$ is the output matrix in (3)).

It should be noted that, in contrast to Theorem 3 from [1], no correction matrix $Z_\infty(t) = [I_n - \gamma^{-2}P(t)Q(t)]^{-1}$ appears in the last innovations term in the right-hand side of the equation (19), since there is no need to make the correction related to estimation of the worst disturbance $w_{worst}(t)$ in the error equation (16). Indeed, as stated in ([4]), the desired estimator must be unbiased, that is, $\tilde{z}_f(t) = 0$. Since the output error $\tilde{z}(t)$, satisfying (18), also stands in the criterion (10) and should be minimized as much as possible, the worst disturbance $w_{worst}(t)$ in the error equation (16) should be plainly rejected and, therefore, does not need to be estimated. Thus, the corresponding H_2 (optimal mean-square) filter gain would not include any correction matrix $Z_\infty(t)$. The same situation can be observed in Theorems 1–4 in [4]. However, if not the output error $\tilde{z}(t)$ but the output $z(t)$ itself would stand in the criterion (10), the correction matrix $Z_\infty(t) = [I_n - \gamma^{-2}P(t)Q(t)]^{-1}$ should be included.

Taking into account the unbiasedness of the estimator (19)-(20), it can be readily concluded that the equality $K_f(t) = P(t)C^T(t)$ must hold for the gain matrix $K_f(t)$ in (5). Thus, the filtering equations (5)-(6) take the final form (11)-(12), with the initial condition $x_f(\theta) = 0$ for $\forall\theta \in [t_0 - h, t_0]$, which corresponds to the central H_∞ filter (see Theorem 4 in [1]). It is still necessary to indicate the equations for the corresponding H_2 (optimal mean-square) filter gain matrix $P(t) = P_0(t)$. In accordance with Theorem 1 from [37], the filter gain matrix $P(t) = P_0(t)$ is given by one of the equations (13), where $k = 0$, with the initial condition $P(t_0) = R^{-1}$, which corresponds to the central H_∞ filter (see Theorems 3 and 4 in [4]). Note that the observation matrix $C(t)$ is changed to $C(t) - \gamma^{-1}L(t)$ according to Theorem 3 from [1]. Then, in view of Theorem 1 from [37], the equations (13) for complementary matrices $P_k(t)$, $k \neq 0$, should be added to obtain a closed system of the filtering equations.

II. In the case of commensurable delays in the state and observation equations (1),(2), the filtering equations (14),(15) directly follow from the results of Subsection 3.1 in [37] and the preceding discussion. It should be noted that, for every fixed t, the number of equations in (15), that should be taken into account to obtain a closed system of the filtering equations, is not equal to infinity, since the matrices $A(t)$, $B(t)$, $C(t)$, $D(t)$, and $L(t)$ are not defined for $t < t_0$. Therefore, if the current filtering horizon t belongs to the semi-open interval $(t_0 + (k+q)h, t_0 + (k+q+1)h]$, where h is the delay value in the equations (1),(2) the number of equations in (15) is equal to $k+q$. ■

Remark 1. The convergence properties of the obtained estimate (14) are given by the standard convergence theorem (see, for example, [40]): if in the system (1),(2) the pair $(A(t)\Psi(t-h,t), B(t))$ is uniformly completely controllable and the pair $(C(t), A(t)\Psi(t-h,t))$ is uniformly completely observable, where $\Psi(t,\tau)$ is the state transition matrix for the equation (1) (see [42] for definition of matrix Ψ), and the inequality $C^T(t)D^T(t)D(t-qh)C(t-qh) - \gamma^{-2}L^T(t)L(t-qh) > 0$ holds, then the error of the obtained filter (14),(15) is uniformly asymptotically stable. As usual, the uniform complete controllability condition is required for assuring non-negativeness of the matrix $P_0(t)$ (13) and may be omitted, if the matrix $P_0(t)$ is non-negative definite in view of its intrinsic properties.

Remark 2. According to the comments in Subsection V.G in [1], the obtained central H_∞ filter (14),(15) presents a natural choice for H_∞ filter design among all admissible H_∞ filters satisfying the inequality (10) for a given threshold γ, since it does not involve any additional actuator loop (i.e., any additional external state variable) in constructing the filter gain matrix. Moreover, the obtained central H_∞ filter (11)–(14) has the suboptimality property, i.e., it minimizes the criterion $J = \|\tilde{z}(t)\|_2^2 - \gamma^2\{\|\omega(t)\|_2^2 + \|\varphi(\theta)\|_{2,R,[-h,0]}^2\}$ for such positive $\gamma > 0$ that the inequality $C^T(t)D^T(t)D(t-qh)C(t-qh) - \gamma^{-2}L^T(t)L(t-qh) > 0$ holds.

Remark 3. Following the discussion in Subsection V.G in [1], note that the complementarity condition always holds for the obtained H_∞ filter (11)–(14), since the positive definiteness of the initial condition matrix R implies the positive definiteness of the filter gain matrix gain $P_0(t)$ as the solution of (15). Therefore, the stability failure is the only reason why the obtained filter can stop working.

IV. Alternative Central H_∞ Filter for LTV Systems with State and Measurement Delays

Consider now another design for the central H_∞ filter for LTV systems with commensurable state and measurement delays in (1),(2), which is based on the alternative H_2 (optimal mean-square) filter obtained in [39]. In doing so, the system of the equations (14),(15) for determining the filter gain matrix $P_0(t)$, whose number grows as the filtering horizon tends to infinity, is replaced by the unique equation for $P_0(t)$, which includes the state transition matrix $\Psi(t,\tau)$ for the time-delay equation (1) (see [42] for the definition). The result is given by the following theorem.

Theorem 2. The alternative "central" H_∞ filter for the unmeasured state (1) over the observations (2), ensuring the H_∞ noise attenuation condition (10) for the output estimate $z_f(t)$, is given by the equations (14) for the state estimate $x_f(t)$, the equation (12) for the output estimate $z_f(t)$, and the equation for the filter gain matrix $P_0(t)$

$$dP(t) = A(t)(\Psi(t-h,t))P_0(t) + P_0(t)(\Psi(t-h,t))^T A^T(t) + \tag{21}$$
$$(1/2)[B(t)B^T(t-qh) + B(t-qh)B^T(t)] -$$
$$(1/2)[P_0(t)C^T(t)(D(t)D^T(t-qh))^{-1}C(t-qh)P_0^T(t-qh) -$$
$$\gamma^{-2}P_0(t)L^T(t)L(t-qh)P_0(t-qh) -$$
$$\gamma^{-2}P_0(t-qh)L^T(t-qh)L(t)P_0(t) +$$
$$P_0(t-qh)C^T(t-qh)(D(t-qh)D^T(t))^{-1}C(t)P_0^T(t)],$$

with the initial condition $P(t_0) = R^{-1}$.

Proof. In view of Theorem 1 in [39], the alternative equation for determining the H_2 (optimal mean-square) filter gain matrix $P_0(t)$ in the estimate equation (15) is given by the equation (21), with the initial condition $P(t_0) = R^{-1}$, which corresponds to the central H_∞ filter (see Theorems 3 and 4 in [4]). The observation matrix $C(t)$ is changed to $C(t) - \gamma^{-1}L(t)$ according to Theorem 3 from [1].■

Note the designed alternative filter contains only two differential equations, the estimate equation (14) and the gain matrix equation (21), regardless of the filtering horizon. This presents a significant advantage in comparison to the preceding filter (14),(12),(15) consisting of a variable number of the gain matrix equations, which is specified by the ratio between the current filtering horizon and the delay value in the state equation and unboundedly grows as the filtering horizon tends to infinity. This advantage seems to be even more significant upon recalling that the state space of the time-delay system (1) is infinite-dimensional [42].

V. EXAMPLE

This section presents an example of designing the central H_∞ filter for a linear state with delay over linear delayed observations and comparing it to the best H_∞ filter available for a linear system without delays, that is the filter obtained in Theorems 3 and 4 from [4].

Let the unmeasured state $x(t) = [x_1(t), x_2(t)] \in R^2$ with delay (a mechanical oscillator with a delayed force input and delayed observations) be given by

$$\dot{x}_1(t) = x_2(t-5), \tag{22}$$
$$\dot{x}_2(t) = -x_1(t-5) + w_1(t),$$

with an unknown initial condition $x(\theta) = \varphi(\theta)$, $\theta \in [-5,0]$, the scalar observation process satisfy the equation

$$y(t) = x_1(t-5) + w_2(t), \tag{23}$$

and the scalar output be represented as

$$z(t) = x_1(t). \tag{24}$$

Here, $w(t) = [w_1(t), w_2(t)]$ is an L_2^2 disturbance input. It can be readily verified that the noise orthonormality condition (see Section 2) holds for the system (22)–(24).

The filtering problem is to find the H_∞ estimate for the linear state with delay (22) over delayed linear observations (23), which satisfies the noise attenuation condition (10) for a given γ, using the designed H_∞ filter (14),(15) or the alternative H_∞ filter (14),(21). The filtering horizon is set to $T = 8$. Note that since $8 \in [1\times 5, 2\times 5]$, where 5 is the delay value in the state and observation equations (22),(23), only the first two of the equations (15), for $k = -1, 0$, along with the equations (14), should be employed.

The filtering equations (14) and the first two of the equations (15) take the following particular form for the system (22),(23)

$$\dot{x}_{f_1}(t) = x_{f_2}(t-5) + P_{0_{11}}(t)[y(t) - x_{f_1}(t-5)], \tag{25}$$
$$\dot{x}_{f_2}(t) = -x_{f_1}(t-5) + P_{0_{12}}(t)[y(t) - x_{f_1}(t-5)],$$

with the initial condition $x_f(\theta) = 0$, $\theta \in [-5,0]$;

$$\dot{P}_{0_{11}}(t) = P_{-1_{12}}(t-5) + P_{1_{12}}(t) - (1-\gamma^{-2})P_{0_{11}}(t)P_{0_{11}}(t-5), \tag{26}$$
$$\dot{P}_{0_{12}}(t) = P_{-1_{22}}(t-5) - P_{1_{11}}(t) - \frac{1}{2}(1-\gamma^{-2})\times$$
$$[P_{0_{11}}(t)P_{0_{12}}(t-5) + P_{0_{12}}(t)P_{0_{11}}(t-5)],$$
$$\dot{P}_{0_{21}}(t) = -P_{-1_{11}}(t-5) + P_{1_{22}}(t) - \frac{1}{2}(1-\gamma^{-2})\times$$
$$[P_{0_{11}}(t)P_{0_{12}}(t-5) + P_{0_{12}}(t)P_{0_{11}}(t-5)],$$
$$\dot{P}_{0_{22}}(t) = 1 - P_{-1_{12}}(t-5) - P_{1_{21}}(t) -$$
$$(1-\gamma^{-2})P_{0_{12}}(t)P_{0_{12}}(t-5),$$

with the initial condition $P_0(0) = R^{-1}$, $P_0(\theta) = 0$, $\theta \in [-5,0)$; and

$$\dot{P}_{-1_{11}}(t) = 2P_{0_{12}}(t) - (1-\gamma^{-2})P_{0_{11}}^2(t), \tag{27}$$
$$\dot{P}_{-1_{12}}(t) = -P_{0_{11}}(t) + P_{0_{22}}(t) - (1-\gamma^{-2})P_{0_{11}}(t)P_{0_{12}}(t),$$
$$\dot{P}_{-1_{12}}(t) = 1 - 2P_{0_{21}}(t) - (1-\gamma^{-2})P_{0_{12}}^2(t),$$

with the initial condition $P_{-1}(0) = 0$; finally, $P_1(\theta) = 0$, $\theta \in [5,8)$.

The estimates obtained upon solving the equations (25)–(27) are compared to the conventional H_∞ filter estimates, obtained in Theorems 3 and 4 from [4], which satisfy the following equations, where the gain matrix equation is a Riccati one and the equations for matrices $P_i(t), i \geq 1$, are not employed:

$$\dot{m}_{f_1}(t) = m_{f_2}(t-5) + P_{11}(t)[y(t) - m_{f_1}(t-5)], \tag{28}$$
$$\dot{m}_{f_2}(t) = -m_{f_1}(t-5) + P_{12}(t)[y(t) - m_{f_1}(t-5)],$$

with the initial condition $m_f(\theta) = 0$, $\theta \in [-5,0]$;

$$\dot{P}_{11}(t) = 2P_{12}(t) - (1-\gamma^{-2})P_{11}^2(t), \tag{29}$$

$$\dot{P}_{12}(t) = -P_{11}(t) + P_{22}(t) - (1-\gamma^{-2})P_{11}(t)P_{12}(t),$$
$$\dot{P}_{22}(t) = 1 - 2P_{12}(t) - (1-\gamma^{-2})P_{12}^2(t),$$

with the initial condition $P(0) = R^{-1}$.

Finally, the previously obtained estimates are compared to the alternative H_∞ filter estimates satisfying the equations (14),(21). The equation (14) for the estimate $x_f(t)$ remains the same as (25), with $P_0(t) = P(t)$, and the gain matrix equation (20) takes the following particular form for the system (22),(23)

$$\dot{P}_{11}(t) = 2\Psi_{22}(t-5,t)P_{12}(t) - (1-\gamma^{-2})P_{11}(t)P_{11}(t-5), \tag{30}$$
$$\dot{P}_{12}(t) = -\Psi_{11}(t-5,t)P_{11}(t) + \Psi_{22}(t-5,t)P_{22}(t) - \frac{1}{2}(1-\gamma^{-2})[P_{11}(t)P_{12}(t-5) + P_{11}(t-5)P_{12}(t)],$$
$$\dot{P}_{22}(t) = 1 - 2\Psi_{11}(t-5,t)P_{12}(t) - (1-\gamma^{-2})P_{12}(t)P_{12}(t-5),$$

with the initial condition $P(0) = R^{-1}$, where it is taken into account that the state transition matrix $\Psi(\tau,t)$ for the linear time-delay state (22) is calculated as a diagonal matrix according to the algorithm suggested in Section 4.

Numerical simulation results are obtained solving the systems of filtering equations (25)–(27), (28)–(29), and (25),(30). The obtained estimate values are compared to the real values of the state vector $x(t)$ in (22).

For each of the three filters (25)–(27), (28)–(29), and (25),(30) and the reference system (22) involved in simulation, the following initial values are assigned: $\varphi_1(\theta) = 1$, $\varphi_2(\theta) = 1$, $\theta \in [-5,0]$; $R = I_2 = diag[1\ 1]$. The L_2 disturbance $w(t) = [w_1(t), w_2(t)]$ is realized as $w_1(t) = 1/(1+t)^2$, $w_2(t) = 2/(2+t)^2$. Since $C(t) = L(t) = [1\ 0]$ in (22),(23) and the minimum achievable value of the threshold γ is equal to $\|L\|/\|C\| = 1$, the value $\gamma = 1.1$ is assigned for the simulations.

The following graphs are obtained: graphs of the output H_∞ estimation error $z(t) - z_f(t)$ corresponding to the estimate $x_f(t)$ satisfying the equations (25)–(27) (Fig. 1); graphs of the output H_∞ estimation error $z(t) - z_f(t)$ corresponding to the conventional estimate $m_f(t)$ satisfying the equations (28)–(29) (Fig. 2); graphs of the output H_∞ estimation error $z(t) - z_f(t)$ corresponding to the alternative estimate $x_f(t)$ satisfying the equations (25),(30) (Fig. 3). The graphs of the output estimation errors are shown in the entire simulation interval from $t_0 = 0$ to $T = 8$. Figures 1–3 also demonstrate the dynamics of the noise-output H_∞ norms corresponding to the shown output H_∞ estimation errors in each case.

The following values of the noise-output H_∞ norm $\|T_{zw}\|^2 = \|z(t) - z_f(t)\|_2^2 / (\|\omega(t)\|_2^2 + \|\varphi(\theta)\|_{2,R,[-h,0]}^2)$ are obtained for the simulated disturbances $w_1(t)$ and $w_2(t)$ at the final time $T = 8$: $\|T_{zw}\| = 0.513$ for the H_∞ estimation error $z(t) - z_f(t)$ corresponding to the estimate $x_f(t)$ satisfying the equations (25)–(27), $\|T_{zw}\| = 1.7318$ for H_∞ estimation error $z(t) - z_f(t)$ corresponding to the conventional estimate $m_f(t)$ satisfying the equations (28)–(29), and $\|T_{zw}\| = 0.5461$ for H_∞ estimation error $z(t) - z_f(t)$ corresponding to the alternative estimate $x_f(t)$ satisfying the equations (25),(30).

It can be concluded that the central suboptimal multi-equational H_∞ filter (25)–(27) and the central suboptimal alternative H_∞ filter (25),(30) provide reliably convergent behavior of the output estimation error, yielding very small values of the corresponding H_∞ norms, even in comparison to the assigned threshold value $\gamma = 1.01$. In contrast, the conventional central H_∞ filter (28)–(29) provides divergent behavior of the output estimation error, yielding a larger value of the corresponding H_∞ norm, which exceeds the assigned threshold. Thus, the simulation results show definite advantages of the designed central suboptimal H_∞ filters for linear systems with state and measurement delays, in comparison to the previously known conventional H_∞ filter.

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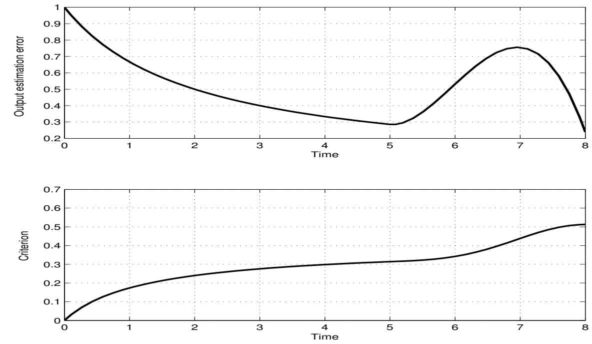

Fig. 1. **Above.** Graph of the output H_∞ estimation error $z(t)-z_f(t)$ corresponding to the estimate $x_f(t)$ satisfying the equations (25)–(27), in the simulation interval [0,8].**Below.** Graph of the noise-output H_∞ norm corresponding to the shown output H_∞ estimation error, in the simulation interval [0,8].

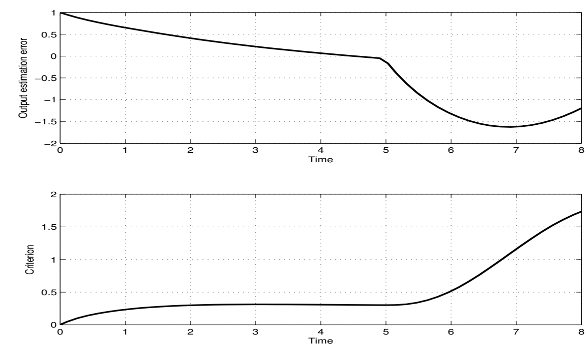

Fig. 2. **Above.** Graph of the output H_∞ estimation error $z(t)-z_f(t)$ corresponding to the estimate $x_f(t)$ satisfying the equations (28)–(29), in the simulation interval [0,8].**Below.** Graph of the noise-output H_∞ norm corresponding to the shown output H_∞ estimation error, in the simulation interval [0,8].

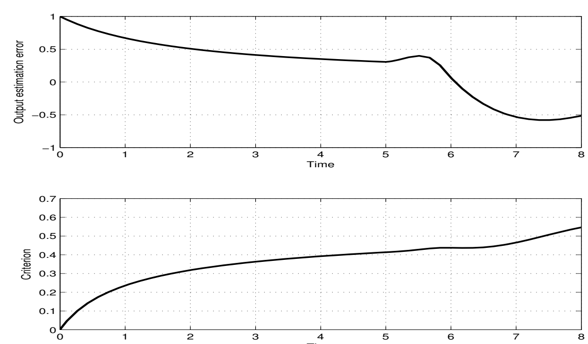

Fig. 3. **Above.** Graph of the output H_∞ estimation error $z(t)-z_f(t)$ corresponding to the estimate $x_f(t)$ satisfying the equations (25),(30), in the simulation interval [0,8].**Below.** Graph of the noise-output H_∞ norm corresponding to the shown output H_∞ estimation error, in the simulation interval [0,8].