Proceedings of the
47th IEEE Conference on Decision and Control
Cancun, Mexico, Dec. 9-11, 2008

Second Cumulant Statistical Control with Indefinite Control Weight

Chang-Hee Won, *IEEE Member*, Ronald W. Diersing, *IEEE Member*, and Hyo-Sung Ahn, *IEEE Member*

***Abstract*— In linear-quadratic-Gaussian control, the positive definiteness of the control weighting matrix in the cost function has been assumed, however, it has been shown that solutions do exist for indefinite control weight matrices for the linear-quadratic-Gaussian case. Here we extend the results to other statistical control methods such as second cumulant statistical control and risk-sensitive control. In this paper, we find the optimal controller where the diffusion term in the state equation depends on the control.**

I. INTRODUCTION

Linear-quadratic-Gaussian (LQG) control [4], second cumulant statistical (SCS) control [10], and risk-sensitive (RS) control [1], [12] has been actively researched in the literature. The control weighting matrices in these control methods, however, have been always assumed to be strictly positive definite. In 1998, LQG control with indefinite control weights was solved by Chen *et al.* when the diffusion term in the state equation is dependent on the control [2]. Moreover, Lim and Zhou extended the problem to include not one, but a number of integral quadratic constraints in [8]. In this research, we extend the LQG results to SCS and RS control.

Statistical control is defined as the minimization of any finite or infinite linear combinations of the cost cumulants, thus LQG, SCS and RS control become special cases of statistical control. In classical LQG control, the first cumulant or the mean of the cost function is minimized. In SCS control, also known as minimal cost variance (MCV) control, the second cumulant, or the variance, of the cost function is minimized. And the minimization of all the “fixed weighted” linear combination of the cost cumulants corresponds to RS control [11].

SCS control is a special case of statistical control, where the second cumulant is optimized. In 1971 Sain and Liberty published an open loop result in minimizing the performance variance while keeping the performance mean close to a prespecified value [10]. The full state feedback SCS control problem is solved in [13], [11]. Nonlinear statistical control was investigated in [14]. More recently, the application of the statistical control concept to game theory has appeared in the literature [3].

This material is supported in part by the National Science Foundation under grant number ECS-0428546

Chang-Hee Won is with the Department of Electrical and Computer Engineering, Temple University, Philadelphia, PA 19122, USA cwon@temple.edu

Ronald Diersing is with Department of Engineering, University of Southern Indiana, Evansville, IN 47712, rwdiersing@usi.edu

Hyo-Sung Ahn is with Department of Mechatronics, Gwangju Institute of Science and Technology (GIST), 1 Oryong-dong, Buk-gu, Gwangju 500-712, Korea, ahnhyosung@gmail.com

In Section II, mathematical preliminaries needed to formulate the indefinite control weight control problem are given. In Section III, SCS control is defined. Hamilton-Jacobi-Bellman (HJB) equations and associated verification theorems are given in Section IV. Then in Section V for a linear system and quadratic cost function, the solution of SCS control with an indefinite control weight is found using the HJB equation. In Section VI, the RS problem with an indefinite control weight is solved using dynamic programming approach. Finally, conclusions are presented in the last section.

II. PROBLEM FORMULATION

In this section we repeat the problem formulation as in [11] for the sake of completeness. Consider the Ito-sense stochastic differential equation,

$$dx(t) = f(t, x(t), u(t))dt + \sigma(t, x(t), u(t))dw(t), \qquad (1)$$

where $t \in T = [t_0, t_F], x(t_0) = x_0, x(t) \in \Re^n$ is the state, x_0 is a random variable which is independent of w, and $w(t)$ is a Brownian motion of dimension d defined on a probability space $(\Omega, \mathcal{F}, P)$, and $u(t) \in U \subset \Re^m$ is the control action. Let p and q be natural numbers. Suppose that $D \subset \Re^p$ and that $h : D \to \Re^q$. Then h is said to belong to $C^j(D)$ if it is continuous. Let $Q_0 = (t_0 \times t_F) \times \Re^n$ and $\bar{Q}_0 = T \times \Re^n$ denote the closure of Q_0. Assume that $f : \bar{Q}_0 \times U \to \Re^n$ is $C^1(\bar{Q}_0 \times U)$; $\sigma : \bar{Q}_0 \times U \to \Re^{n\times d}$ is $C^1(\bar{Q}_0 \times U)$; and $E\{dw\} = 0, E\{dwdw'\} = Idt$. Furthermore we assume that

$$|f(t,0,0)| \le c, \qquad \left|\frac{\partial \sigma(t,x,u)}{\partial x}\right| + \left|\frac{\partial \sigma(t,x,u)}{\partial u}\right| \le \bar{c},$$

$$|\sigma(t,0,0)| \le c \qquad \left|\frac{\partial f(t,x,u)}{\partial x}\right| + \left|\frac{\partial f(t,x,u)}{\partial u}\right| \le \bar{c}$$

for $(t, x, u) \in \bar{Q}_0 \times U, (t, x) \in \bar{Q}_0$, and constants c and $\bar{c}$. The matrix norm notation $|A|$ denotes $\sup_{|x|=1} |Ax|$. In order to control the performance of (1), a memoryless feedback *control law* is introduced in the manner

$$u(t) = k(t, x(t)), \;\; t \in T, \qquad (2)$$

where k is a nonrandom function with random arguments.

It is known that this density satisfies the following backward Fokker-Planck (or Kolmogorov) equation [4]

$$-\frac{\partial p(t, x; s, y; k)}{\partial t} = \mathcal{O}(k)[p(t, x; s, y; k)], \;\; s > t, \qquad (3)$$

where $p(t, x, s, y; k)$ be the probability density corresponding and $\mathcal{O}(k)$ is the backward evolution operator given by

$$\mathcal{O}(k) = \frac{\partial}{\partial t} + \left\langle f, \frac{\partial}{\partial x} \right\rangle + \frac{1}{2}\; tr \;\left(\sigma\sigma' \frac{\partial^2}{\partial x^2}\right) \qquad (4)$$

where

$$\left\langle f, \frac{\partial}{\partial x} \right\rangle = \sum_{i=1}^{n} f_i(t,x,k) \frac{\partial}{\partial x_i} \triangleq \mathcal{O}^{(1)}(k)$$

$$\frac{1}{2}\, tr\left(\sigma\sigma' \frac{\partial^2}{\partial x^2}\right) = \frac{1}{2} \sum_{i,j=1}^{n} (\sigma(t,x,k)\sigma'(t,x,k))_{ij} \frac{\partial^2}{\partial x_i \partial x_j} \triangleq \mathcal{O}^{(2)}(k). \quad (5)$$

In (5), tr denotes the trace operator. The derivative operators are defined so that the i-th element in the n-tuple $(\partial/\partial x)$ is $(\partial/\partial x_i)$, and the ij-th element in the $n \times n$ matrix operator $(\partial^2/\partial x^2)$ is $(\partial^2/\partial x_i \partial x_j)$.

For all $(t,x) \in \bar{Q}_0$, a real valued function $\Phi(t,x)$ on $T \times \Re^n$ satisfies a polynomial growth condition, if there exist constants k_1 and k_2 such that

$$|\Phi(t,x)| \le k_1(1+|x|^{k_2}). \quad (6)$$

Let $C^{1,2}(\bar{Q}_0)$ denote the space of $\Phi(t,x)$ such that Φ and the partial derivatives $\Phi_t, \Phi_{x_i}, \Phi_{x_i x_j}$ for $i,j = 1,\cdots,n$ are continuous on $\bar{Q}_0$. Also let $C_p^{1,2}(\bar{Q}_0)$ denote the space of $\Phi(t,x) \in C^{1,2}(\bar{Q}_0)$ such that $\Phi, \Phi_t, \Phi_{x_i}, \Phi_{x_i x_j}$ for $i,j = 1,\cdots,n$ satisfy a polynomial growth condition. Assumptions $\Phi(t,x) \in C_p^{1,2}(\bar{Q}_0)$, k admissible, and $E\{|x(s)|^m | x(t) = x\}$ bounded for $m = 1,2,\cdots$ and $t \le s \le t_F$ ensure existence of the terms in the right memeber of the Dynkin formula (see [5, pages 128,135,161]),

$$\Phi(t,x) = E_{tx}\left\{ \int_t^{t_F} -\mathcal{O}(k)\Phi(s,x(s))\, ds + \Phi(t_F, x(t_F)) \right\} \quad (7)$$

where E_{tx} denotes a conditional expectation with respect to $x(t) = x$. In order to assess the performance of (1), consider the cost function

$$J(t,x(t),k) = \int_t^{t_F} [L(s,x(s),k(s,x(s)))]\, ds + \psi(x(t_F)). \quad (8)$$

Assume that L and ψ satisfy the polynomial growth conditions

$$|L(t,x,k)| \le c_1\,(1+|x|+|k|)^{c_2}, \ \ \forall (t,x,k) \in \bar{Q}_0 \times U,$$
$$|\psi(x)| \le c_1\,(1+|x|)^{c_2}, \ \ \forall x \in \Re^n, \quad (9)$$

for constants c_1 and c_2. Fleming and Rishel show that a process $x(t)$ from (1), having an admissible controller k, together with the assumption (9), ensure that $E\{J(t,x(t),k)|x(t) = x\}$ is finite [4, page 157]. Furthermore, we assume that σ, L, f, and M are given, and we wish to find k. Now we make a few assumptions:

$$\sigma(t,x,k) = F(t)x(t) + G(t)k(t,x(t)), \quad (10)$$

$$L(t,x,k(t,x)) = h(t,x) + k'(t,x)R(t)k(t,x), \quad (11)$$

$$\psi(x(t_F)) = x'(t_F)Q_F x(t_F), \quad (12)$$

and

$$f(t,x,k(t,x)) = g(t,x) + B(t)k(t,x), \quad (13)$$

where k is an admissible feedback control law; $h : \bar{Q}_0 \to \Re^+$ is $C(\bar{Q}_0)$ and satisfies the polynomial growth conditions assumed for L; and $g : \bar{Q}_0 \to \Re^n$ is $C^1(\bar{Q}_0)$ and satisfies the linear growth condition and the local Lipschitz condition assumed for f. $F(t), G(t)$, and $B(t)$ are continuous real matrices of appropriate dimensions for all $t \in T$. Note that we do not make the usual assumption, $R(t) > 0$.

Now, we can formulate the following three control problems:

1) LQG : $\min_k \left(E\{J(t,x,k)\}\right)$
2) RS: $\min_k \left(-\frac{1}{\theta} \log E\{\exp(-\theta J(t,x,k))\}\right)$
3) SCS: $\min_k \left(E_{tx}\{J^2(t,x,k)\} - E_{tx}^2\{J(t,x,k)\}\right)$

where E_{tx} represents $E\{\cdot | x(t) = x\}$.

III. Second Cumulant Statistical Control Problem

Second Cumulant Statistical (SCS) Control , which is also called Minimal Cost Variance (MCV) control, is a type of statistical control where we minimize the variance of the cost function while keeping the mean of the cost function at a specified level. In SCS control we define a class of admissible controllers, then the cost variance is minimized within that class of controllers. The full-state-feedback SCS control is solved in [11]. Here we extend the results to include the state and control dependent diffusion term $\sigma(t,x,k)$ and indefinite control weight cost.

The class of admissible control laws, and comparison of control laws within the class, is defined in terms of the first and second moments of (8). Define

$$V_1(t,x;k) = E_{tx}\{J(t,x(t),k)\} \quad (14)$$

$$V_2(t,x;k) = E_{tx}\{J^2(t,x(t),k)\}. \quad (15)$$

Define a function $M : \bar{Q}_0 \to \Re^+$, which is $C^{1,2}(\bar{Q}_0)$, as an *admissible mean cost function* if there exists an admissible control law k such that

$$V_1(t,x;k) = M(t,x) \quad (16)$$

for $t \in T$ and $x \in \Re^n$. Every admissible M defines a class K_M of *control laws k corresponding to M* in the manner that $k \in K_M$ if and only if k is an admissible control law which satisfies (16).

It is now possible to define a SCS control law $k^*_{V|M}$. Let M be an admissible mean cost function, and let K_M be its induced class of admissible control laws. A SCS control law $k^*_{V|M}$ satisfies

$$V_2(t,x;k^*_{V|M}) = V_2^*(t,x) \le V_2(t,x;k), \quad (17)$$

for $t \in T$, $x \in \Re^n$, whenever $k \in K_M$. The corresponding minimal cost variance is given by

$$V^*(t,x) = V_2^*(t,x) - M^2(t,x) \quad (18)$$

for $t \in T$, $x \in \Re^n$.

IV. Hamilton-Jacobi-Bellman Equation for $V^*(t,x)$

We derive a Hamilton-Jacobi-Bellman (HJB) equation for the second cumulant statistical or SCS control problem. This section derives that HJB equation under the assumption that a sufficiently smooth solution exists. A full-state-feedback SCS control law is derived in the sequel utilizing this HJB equation.

One of the main results of this section is summarized in the following theorem, which makes use of the notation $\|a\|_A^2 = a'Aa$.

Theorem 1: Let $M(t,x) \in C_p^{1,2}(\bar{Q}_0)$ be an admissible mean cost function, and let M induce a non-empty class K_M of admissible control laws. Assume the existence of an optimal control law $k = k^*_{V|M}$ and an optimum value function $V^* \in C_p^{1,2}(\bar{Q}_0)$. Then the SCS function V^* satisfies the HJB equation

$$\min_{k\in K_M} \mathcal{O}(k)[V^*(t,x)] + \left\| \frac{\partial M(t,x)}{\partial x} \right\|^2_{\sigma(t,x,k)\sigma'(t,x,k)} = 0, \tag{19}$$

for $(t,x) \in \bar{Q}_0$, together with the terminal condition,

$$V^*(t_F, x) = 0. \tag{20}$$

Proof: See [11] with $\sigma(t,x)$ changed to $\sigma(t,x,k)$. □

Theorem 2: (Verification Theorem). Let M be an admissible mean cost function satisfying $M^2(t,x) \in C_p^{1,2}(Q) \cap C(\bar{Q})$, and let K_M be the associated non-empty class of admissible control laws. Suppose that a nonnegative function $V^*(t,x) \in C_p^{1,2}(Q) \cap C(\bar{Q})$ is a solution to the partial differential equation

$$\min_{k\in K_M} \mathcal{O}(k)[V^*(t,x)] + \left\| \frac{\partial M(t,x)}{\partial x} \right\|^2_{\sigma(t,x,k)\sigma'(t,x,k)} = 0, \tag{21}$$

$\forall (t,x) \in Q$ together with the boundary condition $V^*(t_F,x) = 0$. Then $V^*(t,x) \le V(t,x;k)$ for every $k \in K_M$ and any $(t,x) \in Q$. If in addition such a k satisfies the equation

$$\mathcal{O}(k)[V^*(t,x)] = \min_{\tilde{k}\in K_M} \left\{ \mathcal{O}(\tilde{k})[V^*(t,x)] \right\}$$

for all $(t,x) \in Q$, then $V^*(t,x) = V(t,x;k)$ and $k = k^*_{V|M}$ is an optimal control law.

Proof: See [11] with $\sigma(t,x)$ changed to $\sigma(t,x,k)$. □

Equation (21) in Theorem 2 differs from SCS control of [11] by the σ term depending on k.

V. Solutions of Second Cumulant Statistical Control

We derive the full-state feedback solution of the second cumulant statistical control problem. We assume a linear system and a quadratic cost function in this section. We search for an admissible linear controller that minimizes the cost variance. We consider the class of admissible controls that satisfy the following equation

$$L(t,x,k(t,x)) + \mathcal{O}(k)[M(t,x)] = 0. \tag{22}$$

Here we repeat the results of Liu and Leake [9], [11]. Let $x \in \Re^n$ be a real n-vector, $z(x)$ and $y(x)$ be real r-vector functions, and $\alpha(x)$ be a real function defined on $\Re^n$.

Lemma 3: (Liu and Leake Lemma). Let X be a positive definite symmetric-real matrix. Then $z(x)$ satisfies the condition

$$\langle z(x), Xz(x)\rangle + 2\langle z(x), y(x)\rangle + \alpha(x) = 0 \tag{23}$$

if and only if $\langle y(x), X^{-1}y(x)\rangle \ge \alpha(x)$. In this case, the set of all solutions to (23) is represented by

$$z(x) = \beta H^{-1} a(x) - X^{-1} y(x) \tag{24}$$

where

$$\beta(x) = \left(\langle y(x), X^{-1}y(x)\rangle - \alpha(x)\right)^{\frac{1}{2}}, \tag{25}$$

H is a non-singular matrix such that $X = H'H$, and $a(x)$ is an arbitrary unit vector.

Proof: See [11]. □

Consider an open set, $Q \subset Q_0$.

Theorem 4: Assume $M(t,x) \in C_p^{1,2}(Q) \cap C(\bar{Q})$ and that the above assumptions in Section II are satisfied. Then we have a solution $k(t,x)$, which may or may not be admissible, if and only if (suppressing the arguments)

$$\begin{aligned} &\left(\frac{1}{2}B'\frac{\partial M}{\partial x} + \frac{1}{2}G'\frac{\partial M^2}{\partial x^2}Fx\right)' \left(R + \frac{1}{2}G'\frac{\partial M^2}{\partial x^2}G\right)^{-1} \\ &\left(\frac{1}{2}B'\frac{\partial M}{\partial x} + \frac{1}{2}G'\frac{\partial M^2}{\partial x^2}Fx\right) \\ &\ge \frac{\partial M}{\partial t} + \frac{1}{2}x'F'\frac{\partial^2 M}{\partial x^2}Fx + h + g'\frac{\partial M}{\partial x}. \end{aligned} \tag{26}$$

Then a control law k is in K_M if and only if (1) it is admissible and (2) it is of the form,

$$\begin{aligned} k(t,x) &= \beta(x)H^{-1}a(x) - \frac{1}{2}\left(R + \frac{1}{2}G'\frac{\partial^2 M}{\partial x^2}G\right)^{-1} \\ &\quad \left(B'\frac{\partial M}{\partial x} + G'\frac{\partial^2 M}{\partial x^2}Fx\right), \end{aligned} \tag{27}$$

where $a(x)$ is an arbitrary unit vector, $H'H = R + \frac{1}{2}G'\frac{\partial^2 M(t,x)}{\partial x^2}G$, and

$$\beta(x) = \sqrt{\begin{aligned} &\frac{1}{4}\left(B'\frac{\partial M}{\partial x} + G'\frac{\partial^2 M}{\partial x^2}Fx\right)' \\ &\left(R + G'\frac{\partial^2 M}{\partial x^2}G\right)^{-1}\left(B'\frac{\partial M}{\partial x} + G'\frac{\partial^2 M}{\partial x^2}Fx\right) \\ &-\frac{\partial M}{\partial t} - h - g'\frac{\partial M}{\partial x} - \frac{1}{2}x'F'\frac{\partial^2 M}{\partial x^2}Fx. \end{aligned}} \tag{28}$$

Moreover $\beta(x) = 0$ corresponds to the optimal mean (LQG) cost law.

Proof: Rewriting (22),

$$\begin{aligned} 0 &= \frac{\partial M(t,x)}{\partial t} + L(t,x,k(t,x)) \\ &\quad + \frac{1}{2}\,tr\left(\sigma(t,x,k)\sigma'(t,x,k)\frac{\partial^2 M(t,x)}{\partial x^2}\right) \\ &\quad + f'(t,x,k(t,x))\frac{\partial M(t,x)}{\partial x}, \end{aligned} \tag{29}$$

we obtain

$$\begin{aligned} 0 &= \frac{\partial M}{\partial t} + h + k'\left(R + \frac{1}{2}G'\frac{\partial^2 M}{\partial x^2}G\right)k + g'\left(\frac{\partial M}{\partial x}\right) \\ &\quad + k'\left(B'\frac{\partial M}{\partial x} + G'\frac{\partial^2 M}{\partial x^2}Fx\right) + \frac{1}{2}x'F'\frac{\partial^2 M}{\partial x^2}x. \end{aligned} \quad (30)$$

One can then solve the above equation for k, using the method of Liu and Leake; see Lemma 3. We may identify from (23) the following:

$$z \Leftrightarrow k \quad (31)$$

$$X \Leftrightarrow \left(R + \frac{1}{2}G'\frac{\partial^2 M}{\partial x^2}G\right) \quad (32)$$

$$y \Leftrightarrow \frac{1}{2}B'\frac{\partial M}{\partial x} + \frac{1}{2}G'\frac{\partial^2 M}{\partial x^2}Fx \quad (33)$$

$$\alpha \Leftrightarrow \frac{\partial M}{\partial t} + h + g'\frac{\partial M}{\partial x} + \frac{1}{2}x'F'\frac{\partial^2 M}{\partial x^2}Fx \quad (34)$$

Accordingly, we must satisfy

$$y'X^{-1}y \geq \alpha, \quad (35)$$

which gives the desired equation (26).

Incorporating (30) into (26) gives

$$\begin{aligned} &k'\left(R + \frac{1}{2}G'\frac{\partial^2 M}{\partial x^2}G\right)k + k'\left(B'\frac{\partial M}{\partial x} + G'\frac{\partial^2 M}{\partial x^2}Fx\right) \\ &\geq -\frac{1}{4}\left(B'\frac{\partial M}{\partial x} + G'\frac{\partial^2 M}{\partial x^2}Fx\right)'\left(R + \frac{1}{2}G'\frac{\partial^2 M}{\partial x^2}G\right)^{-1} \\ &\left(B'\frac{\partial M}{\partial x} + G'\frac{\partial^2 M}{\partial x^2}Fx\right) \end{aligned}$$

The left hand side of the above inequality is $-\alpha$ (see (30)), and the right hand side is $-\langle y, X^{-1}y\rangle$. Thus if k is the minimal mean cost law then $\beta = 0$ and $M(t,x) = V_1(t,x;k) = V_1^*(t,x)$. If k is not the minimal mean cost law, and (26) is satisfied, then $\beta > 0$. We use (24) to obtain the control law k as

$$\begin{aligned} k = \beta(x)H^{-1}a(x) - \frac{1}{2}\left(R + \frac{1}{2}G'\frac{\partial^2 M}{\partial x^2}G\right)^{-1} \\ \left(B'\frac{\partial M}{\partial x} + G'\frac{\partial^2 M}{\partial x^2}Fx\right). \end{aligned}$$

□

For a wide variety of problems, therefore, we have shown that sub-optimal mean control introduces the possibility of reducing variance, i.e., we have some freedom in selecting $a(x)$.

To find the solution of the SCS control problem, we rewrite the HJB equation (21) of Theorem 2 as

$$\begin{aligned} \frac{-\partial V^*(t,x)}{\partial t} &= \min_{k \in K_M}\left\{ f'(t,x,k(t,x))\frac{\partial V^*(t,x)}{\partial x} \right. \\ &\quad + \frac{1}{2}\,tr\left(\sigma(t,x,k)\sigma'(t,x,k)\frac{\partial^2 V^*(t,x)}{\partial x^2}\right) \\ &\quad \left. + \left\|\frac{\partial M(t,x)}{\partial x}\right\|^2_{\sigma(t,x,k)\sigma'(t,x,k)} \right\} \end{aligned} \quad (36)$$

with boundary condition $V^*(t_F, x) = 0$.

Theorem 5: Assume that the conditions of Theorem 2 and Theorem 4 are satisfied. Then a nonlinear optimal SCS control law is of the form

$$\begin{aligned} k^*_{V|M} &= \frac{E_1 - \frac{1}{2}D_1^{-1}C_1}{\left\|E_1 - \frac{1}{2}D_1^{-1}C_1\right\|_{\frac{X}{\beta}}} - \frac{1}{2}\left(R + \frac{1}{2}G'\frac{\partial^2 M}{\partial x^2}G\right)^{-1} \\ &\quad \left(B'\frac{\partial M}{\partial x} + G'\frac{\partial^2 M}{\partial x^2}Fx\right), \end{aligned} \quad (37)$$

where

$$\begin{aligned} C_1 &= B'\frac{\partial V}{\partial x} + G'Fx\,tr\frac{\partial^2 V}{\partial x^2} + G'Fx\,tr\left(\frac{\partial M}{\partial x}\frac{\partial M}{\partial x}'2\right), \\ D_1 &= \frac{1}{2}G'G\,tr\frac{\partial^2 V}{\partial x^2} + G'G\,tr\left(\frac{\partial M}{\partial x}\frac{\partial M}{\partial x}'\right), \\ E_1 &= \frac{1}{2}\left(R + \frac{1}{2}G'\frac{\partial^2 M}{\partial x^2}G\right)^{-1}\left(B'\frac{\partial M}{\partial x} + G'\frac{\partial^2 M}{\partial x^2}Fx\right), \\ X &= R + \frac{1}{2}G'\frac{\partial^2 M}{\partial x^2}G. \end{aligned} \quad (38)$$

And the optimal cost function V^* satisfies the partial differential equation

$$\begin{aligned} -\frac{\partial V^*}{\partial t} &= \left[\frac{-X(-D_1^{-1} + 2E_1)}{\frac{1}{\sqrt{\beta}}\| - D_1^{-1} + 2E_1\|_X} - \frac{1}{2}\left(R + \frac{1}{2}G'\frac{\partial^2 M}{\partial x^2}G\right)^{-1}\right. \\ &\quad \left.\left(B'\frac{\partial M}{\partial x} + G'\frac{\partial^2 M}{\partial x^2}Fx\right)\right]' \\ &\quad \left\{C_1 + D_1\left[\frac{-X(-D_1^{-1} + 2E_1)}{\frac{1}{\sqrt{\beta}}\| - D_1^{-1} + 2E_1\|_X}\right.\right. \\ &\quad \left.\left. -\frac{1}{2}\left(R + \frac{1}{2}G'\frac{\partial^2 M}{\partial x^2}G\right)^{-1}\left(B'\frac{\partial M}{\partial x} + G'\frac{\partial^2 M}{\partial x^2}Fx\right)\right]\right\} \end{aligned} \quad (39)$$

Proof: Omitted for brevity. □

We will need the following series expansion theorem.

Theorem 6: If $\|\cdot\|$ denotes any matrix norm for which $\|I\| = 1$ and if $\|M\| < 1$, then $(I + M)^{-1}$ exists,

$$(I + M)^{-1} = I - M + M^2 - \dots$$

and

$$\|(I + M)^{-1}\|] \leq \frac{1}{1 - \|M\|}.$$

Proof: See [6, p. 384].

Our next step is to estimate $\beta(x)$ in the case in which the dynamical system is linear and the cost function accumulates at a quadratic rate. We assume that the average cost function is in a quadratic form.

$$M(t,x) = x'\mathcal{M}(t)x + m(t); \quad (40)$$

so that we can obtain the explicit evaluations

$$\frac{\partial^2 M(t,x)}{\partial x^2} = 2\mathcal{M}(t), \quad \frac{\partial M(t,x)}{\partial x} = 2\mathcal{M}(t)x, \text{ and} \quad (41)$$

$$\frac{\partial M(t,x)}{\partial t} = x'\dot{\mathcal{M}}(t)x + \dot{m}(t), \quad (42)$$

with which one obtains the following lemma.

Lemma 7: With the same assumptions as Theorem 4 and equation (40). Then for the special case

$$\begin{aligned} h(t,x) &= x'(t)Q(t)x(t), \quad Q(t) \geq 0, \\ g(t,x) &= A(t)x(t), \end{aligned} \tag{43}$$

of (11) and (13), we find $\beta(x)$ in (27) to be

$$\beta(x) = \|x\|_{\mathcal{R}(t)}, \tag{44}$$

where (suppressing the argument t)

$$\begin{aligned} \mathcal{R} &\triangleq \mathcal{M}B(R+G'\mathcal{M}G)^{-1}B'\mathcal{M} \\ &+ \mathcal{M}B\left(R+G'\mathcal{M}G\right)^{-1}G'\mathcal{M}F \\ &+ F'\mathcal{M}G\left(R+G'\mathcal{M}G\right)^{-1}B'\mathcal{M} \\ &+ F'\mathcal{M}G(R+G'\mathcal{M}G)^{-1}G'\mathcal{M}F \\ &- \dot{\mathcal{M}} - Q - (A'\mathcal{M}+\mathcal{M}A) - F'\mathcal{M}F. \end{aligned} \tag{45}$$

A particular case of this situation occurs when

$$V^*(t,x) = x'\mathcal{V}(t)x + v(t)$$

in which case the optimal linear SCS control law (37) can be rewritten as

$$\begin{aligned} k^*_{V|M} &= -\left[(1-\gamma)(R+G'\mathcal{M}G)^{-1}(B'\mathcal{M}+G'\mathcal{M}F)\right. \\ &\quad \left. -2\gamma G'F\, tr\mathcal{V} + \gamma(G'G\, tr\mathcal{V})^{-1}B'\mathcal{V}\right]x. \end{aligned} \tag{46}$$

Proof: Omitted for brevity. □

Let us now restrict the class of controllers, K_M, to be vector space morphisms. To denote this, we replace the notation K_M by K_{ML}. It follows from the work of Liberty and Hartwig [7] that M and V are then quadratic, which is consistent with the assumptions and results in the foregoing lemma. It is straightforward to see that equation (46) defines a homogeneous mapping, by $k^*_{V|M}(t,\alpha x) = \alpha k^*_{V|M}(t,x)$. Indeed, the result follows by the definition

$$f(x) = \|x\|_{\mathcal{R}} / \|x\|_{\Xi_2} \tag{47}$$

on the domain in which the denominator does not vanish, together with the observation that $f(\alpha x) = \alpha f(x)$ on this domain. The question of whether or not $k^*_{V|M}(t,x)$ is a morphism under the addition of vectors are shown in [11].

Thus the solution to the full-state-feedback SCS control problem with linear controller is then given by the following theorem.

Theorem 8: Assume $V^*(t,x) \in C_p^{1,2}(Q_0) \cap C(\bar{Q}_0)$ and the same assumptions as in Theorem 5 and Lemma 7. Then for $k \in K_{ML}$, there exists a linear SCS controller, if and only if there exist solutions M and V to the pair of matrix differential equations

$$\begin{aligned} \dot{\mathcal{V}} &= -2\Psi'B'V - 2\Psi'G'F\, tr\mathcal{V} + \Psi G'G\, tr\mathcal{V}\Psi \\ &\quad +2A'\mathcal{V} + tr\mathcal{V}F'F, \end{aligned} \tag{48}$$

$$0 = F - G\Psi, \tag{49}$$

and

$$\begin{aligned} \dot{\mathcal{M}} &= (\mathcal{M}B + F'\mathcal{M}G)\left(R+G'\mathcal{M}G\right)^{-1}(B'\mathcal{M}+G'\mathcal{M}F) \\ &\quad -Q - (A'\mathcal{M}+\mathcal{M}A) - F'\mathcal{M}F \\ &\quad -\gamma^2\Big\{\big[(R+G'\mathcal{M}G)^{-1}(B'\mathcal{M}+G'\mathcal{M}F) \\ &\quad -(G'G\, tr\mathcal{V})^{-1}B'\mathcal{V} + 2G'F\, tr\mathcal{V}\big]'\,(R+G'\mathcal{M}G) \\ &\quad \big[(R+G'\mathcal{M}G)^{-1}(B'\mathcal{M}+G'\mathcal{M}F) \\ &\quad -(G'G\, tr\mathcal{V})^{-1}B'\mathcal{V} + 2G'F\, tr\mathcal{V}\big]\Big\} \end{aligned} \tag{50}$$

where

$$\begin{aligned} \Psi &= (1-\gamma)(R+G'\mathcal{M}G)^{-1}(B'\mathcal{M}+G'\mathcal{M}F) \\ &\quad -2\gamma G'F\, tr\mathcal{V} + \gamma(G'G\, tr\mathcal{V})^{-1}B'\mathcal{V}, \end{aligned}$$

and with boundary conditions $\mathcal{M}(t_F) = Q_F$ and $\mathcal{V}(t_F) = 0$, for a suitable positive time function $\gamma(t)$. In such a case, the controller is given by equation (46).

Proof: Omitted for brevity. □

VI. Risk-Sensitive Control

Another special case of statistical control is risk-sensitive (RS)control. In RS control, all the cumulants of the cost function are optimized. We consider the minimization of the following cost function,

$$\Psi(t,x) = -\frac{1}{\theta}\log E\{\exp(-\theta J(t,x,k))\}, \tag{51}$$

and we obtain the following corresponding Hamilton-Jacobi-Bellman (HJB) equation.

$$\begin{aligned} 0 &= \mathcal{O}(k)[\Psi(t,x)] \\ &\quad -\frac{1}{2}\, tr(\Psi_x'\sigma(t,x,k)\sigma'(t,x,k)\Psi_x) + L(t,x,k(t,x)) \end{aligned} \tag{52}$$

Because all the cumulants of J are in quadratic form, we assume that a solution of the form,

$$\Psi(t,x) = x'\mathcal{P}x + p.$$

For the minimal RS control problem, we obtain the following solution.

Theorem 9: The optimal linear controller of the RS control problem is given by

$$\begin{aligned} k^* &= -\sum_{n=0}^{\infty}(-R^{-1}G'\mathcal{P}G)^n R^{-1}(B'\mathcal{P}+G'\mathcal{P}F)x \\ &= -K(t)x(t) \end{aligned} \tag{53}$$

Proof: Omitted for brevity. □

Now, we are ready to find the corresponding Riccati type equations. The solution to the full state feedback RS problem with indefinite control weight is given by the following theorem.

Theorem 10: There exist a linear controller, if and only if there exist solution $\mathcal{P}$ to the following Riccati type equations;

$$\begin{aligned} 0 \;=\; & \dot{\mathcal{P}} + \mathcal{P}A + A'\mathcal{P} + F'\mathcal{P}F + Q - 2(\mathcal{P}B + F'\mathcal{P}G) \\ & \left[\sum_{n=0}^{\infty}(-R^{-1}G'\mathcal{P}G)^n R^{-1}(B'\mathcal{P} + G'\mathcal{P}F)\right] \\ & + \left[\sum_{n=0}^{\infty}(-R^{-1}G'\mathcal{P}G)^n R^{-1}(B'\mathcal{P} + G'\mathcal{P}F)\right] \\ & (G'\mathcal{P}G + R) \\ & \left[\sum_{n=0}^{\infty}(-R^{-1}G'\mathcal{P}G)^n R^{-1}(B'\mathcal{P} + G'\mathcal{P}F)\right] = 0 \end{aligned} \tag{54}$$

and

$$0 = \mathcal{P}'F - [\sum_{n=0}^{\infty}(-R^{-1}G'\mathcal{P}G)^n R^{-1}(B'\mathcal{P} + G'\mathcal{P}F)G'\mathcal{P}] \tag{55}$$

Proof: Omitted for brevity. □

VII. CONCLUSIONS

The statistical optimal controller for the second cumulant case, SCS, and infinite cumulant case, RS, are found when the diffusion term in the state equation is dependent on control. The assumption that the control weighting matrix has to be positive definite has not been used. Thus, the control weighting can be negative definite and the optimal controller may exist. The optimal controllers are summerized in the following table.

Optimal Controllers
$k^*_{LQG} = -(R + G'\mathcal{M}G)^{-1}(B'\mathcal{M} + G'\mathcal{M}F)x$
$k^*_{RS} = -\sum_{n=0}^{\infty}(-R^{-1}G'\mathcal{P}G)^n R^{-1}(B'\mathcal{P} + G'\mathcal{P}F)x$
$k^*_{V\|M} = -[(1-\gamma)(R + G'\mathcal{M}G)^{-1}(B'\mathcal{M} + G'\mathcal{M}F) - 2\gamma G'F\, tr\mathcal{V} + \gamma(G'G\, tr\mathcal{V})^{-1}B'\mathcal{V}]x$

REFERENCES

[1] A. Bensoussan and J. H. van Schuppen, "Optimal Control of Partially Observable Stochastic Systems with an Exponential-of-Integral Performance Index," *SIAM Journal on Control and Optimization*, Volume 23, pp. 599–613, 1985.

[2] S. Chen, X. Li, X.Y. Zhou, "Stochastic Linear Quadratic Regulators with Indefinite Control Weight Costs," *SIAM Journal of Control and Optimization*, Vol. 36, No. 5, pp. 1685-1702, September 1998.

[3] R. Diersing, M. Sain, and C.-H. Won, "Discrete-Time Bi-Cumulant Minimax and Nash Games," *Peceedings of 46th IEEE Conference on Decision and Control*, New Orleans, December 2007.

[4] W. H. Fleming and R. W. Rishel, *Deterministic and Stochastic Optimal Control*. New York: Springer-Verlag, 1975.

[5] W. H. Fleming and H. M. Soner, *Controlled Markov Processes and Viscosity Solutions*. New York: Springer-Verlag, 1992.

[6] P. Lancaster and M. Tismenetsky, *The Theory of Matrices*, Secon Edition, Academic Press, Inc., San Diego, 1985.

[7] S. R. Liberty and R. C. Hartwig, "On the Essential Quadratic Nature of LQG Control-Performance Measure Cumulants," *Information and Control*, Volume 32, Number 3, pp. 276–305, 1976.

[8] A. E. B. Lim and Xun Yu Zhou, "Stochastic Optimal LQR Control with Integral Quadratic Constaints and Indefinite Control Weights," *IEEE Transactions on Automatic Control*, Vol. 44, No. 7, pp. 1359–1369, 1999.

[9] R. W. Liu and J. Leake, "Inverse Lyapunov Problems," Technical Report No. EE-6510, Department of Electrical Engineering, University of Notre Dame, August 1965.

[10] M. K. Sain and S. R. Liberty, "Performance Measure Densities for a Class of LQG Control Systems," *IEEE Transactions on Automatic Control*, AC-16, Number 5, pp. 431–439, October 1971.

[11] M. K. Sain, C.-H. Won, B. F. Spencer, Jr. and S. R. Liberty, "Cumulants and Risk-Sensitive Control: A Cost Mean and Variance Theory with Application to Seismic Protection of Structures," *Advances in Dynamic Games and Applications*, J. A. Filar, K. Mizukami, V. Gaitsgory (Eds.), Birkhauser Boston, 1999.

[12] P. Whittle, "A Risk-Sensitive Maximum Principle: The Case of Imperfect State Observation," *IEEE Transactions on Automatic Control*, Vol. 36, No. 7, pp. 793–801, July 1991.

[13] C.-H. Won, *Cost Cumulants in Risk Sensitive and Minimal Cost Variance Control*, Ph.D. Dissertation, Notre Dame, 1995.

[14] C.-H. Won, "Nonlinear n-th Cost Cumulant Control and Hamilton-Jacobi-Bellman Equations for Markov Diffuion Process, " *Peceedings of 44th IEEE Conference on Decision and Control*, Seville, Spain, pp. 4524-4529, 2005.

[15] W. M. Wonham, "Stochastic Problems in Optimal Control," *1963 IEEE Int.Conv.Rec.*, part 2, pp. 114–124, 1963.