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On the robust stabilization of discrete-time SISO plants with saturating actuators

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Abstract— **This paper proposes the use of a time-varying sliding surface for stabilizing linear, possibly unstable, discrete-time plants subject to saturating actuators, in the presence of bounded matched uncertainties. The present work generalizes our previous contributions in the discrete-time framework. A constructive procedure is given, and a result about semiglobal practical stabilization is given. Simulation results show both the effectiveness of the control technique and the low computational burden required.**

Keywords Saturating Actuators, Discrete-time Sliding Mode Control, Robust Control, Semiglobal stabilization.

I. Introduction

The presence of actuator saturation in control systems, though frequently ignored, is due to inherent (and unavoidable) physical limitations of devices. The relevance of this issue from the practical viewpoint is more and more attracting the attention of control system researchers, as failure in accounting for actuator saturation may lead to severe deterioration of closed loop system performance, even to instability.

In the vast literature addressing the stabilization problem for discrete-time linear systems subject to actuator saturation, two lines of research have been mostly pursued. The first line focusses on the estimation of the asymptotic stability region, which often has a very conservative expression. To reduce this conservatism, estimates are given as solution of suitable LMI optimization problems [5], [14], [6], [8], [1]. The other line of research focuses on the estimation, less conservative as possible, of the null controllable region, i.e. the set of state which can be driven towards the origin of the state space using saturating actuators. In this latter framework, the problem has been completely studied for plants known as Asymptotically Null Controllable with Bounded Controls (ANCBC), for which the null controllable region is the whole state space [11], [12], [16]. Moreover, some results are available for general discrete-time systems about feedback laws achieving semi-global stabilization on the null controllable region. Broadly speaking, such techniques consist either in dividing the null controllable region in polygons and finding suitable controls driving the vertices to the origin [2], or in designing a sequence of feedback laws such that the union of the corresponding invariant sets is an invariant set contained in the domain of attraction [7]. Both techniques, however, require a considerable computational burden also for plants with relatively low order.

Furthermore, the problem of disturbance rejection for linear systems subject to actuator saturation has been investigated only marginally in the discrete time framework. Note that for continuous time plant an interesting research line considers disturbances that are bounded in magnitude. In such context, [15] proved that semiglobal practical stabilization for a linear system subject to actuator saturation and input additive disturbances can be achieved as long as the open loop system is not exponentially unstable. For the same class of systems, Lin [10] constructed nonlinear feedback laws that achieve global practical stabilization. Recently, it has been proved in [4] that a 2-dimensional linear systems subject to actuator saturation and bounded input additive disturbances can be globally practically stabilized by linear state feedback, while a sliding mode approach has been very recently presented [3].

Very few results are available, as far as authors are aware, about the use of quasi sliding modes for controlling plants with saturating actuators. In the continuous time framework, it is worth mentioning the paper by [9], where a family of low-gain based variable structure controllers are built using a standard sliding mode design approach, and the recent paper [3], which proposes a time-varying sliding surface. The present work generalizes our previous contribution [3] in the discrete-time framework, thus requiring a completely different proof of the main result with respect to the continuous time case. It will be shown that ultimate boundedness of a single input discrete-time linear plant can be achieved by means of a time-varying state feedback controller, derived imposing the achievement of a quasi-sliding motion onto a suitable time-varying sliding surface. It will be proved here that a constructive procedure exists for designing the surface as to guarantee the ultimately boundedness of plant trajectories in the presence of bounded matched uncertainties.

II. Problem statement

Consider the following time invariant, uncertain discrete time single input plant described by:

$$\mathbf{x}(k+1) = \mathbf{A}\mathbf{x}(k) + \mathbf{B}(u(k) + d(k)) \qquad (1)$$

where: $\mathbf{x} = [x_1(k) \cdots x_n(k)]^T \in \mathbb{R}^n$ is the state vector (assumed available for measurement), $u(k) \in \mathbb{R}$ is the control input, and $\mathbf{A} \in \mathbb{R}^{n \times n}$ is the state matrix. The uncertain term $d(k) \in \mathbb{R}$ represents external disturbances affecting the system.

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Assumption 1: The uncertain element $d(k)$ is such that it is bounded by a known constant $\bar{\rho}$, i.e. $|d(k)| \leq \bar{\rho}$

Assumption 2: The pair (**A**,**B**) is controllable and, without loss of generality is given in the controllable canonical form, with:

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & \dots \\ & & \dots & \\ a_1 & a_2 & \dots & a_n \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} \mathbf{0}_{(n-1)\times 1} \\ 1 \end{bmatrix}$$

The plant is supposed to be preceded by a saturating device $u(k) = f(v(k))$, such that it holds:

$$u(k) = f(v(k)) = \begin{cases} M & if \;\; v(k) \geq M \\ v(k) & if \;\; -M < v(k) < M \\ -M & if \;\; v(k) \leq -M \end{cases} \tag{2}$$

with threshold $M > 0$ known.

Consider a vector $\mathbf{C} \in \mathbb{R}^{1\times n}$ of the form

$$\mathbf{C} = \begin{bmatrix} c_1 & c_2 & \dots & c_{n-1} & \epsilon \end{bmatrix} \tag{3}$$

with $c_i, \epsilon \in \mathbb{R}$, $i = 1, \dots, n-1$. In view of the controllability hypothesis, coefficients appearing in the $\mathbf{C}$ vector can be designed such that, when a sliding motion is achieved on the following sliding surface:

$$\hat{\mathbf{s}}(\mathbf{x}(k)) = \mathbf{C}\mathbf{x}(k) = 0 \tag{4}$$

the corresponding reduced order system has assigned stable eigenvalues, and, as a consequence, system (1) is stable, too.

Definition 1: Denote solutions of a general system $\mathbf{x}(k+1) = f(\mathbf{x}(k), k)$ as $\phi(k, k_0, \mathbf{x}(0))$ with initial condition $\mathbf{x}(0)$. Following [13], such solutions are defined *uniformly ultimately bounded* (with bound B) if there exists a $B > 0$ and if corresponding to any $\alpha > 0$ and for every $k_0 \in \mathbb{N}$, there exists a $T = T(\alpha) > 0$ (independent of k_0) such that $||\mathbf{x}(0)|| < \alpha$ implies that $|\phi(k, k_0, \mathbf{x}(0))| < B$ for all $k \geq k_0 + T(\alpha)$.

In the absence of the saturating device (2), i.e. if the control input $u(k)$ could be directly manipulated, the following control law, obtained by imposing the inequality $|\hat{s}(\mathbf{x}(k+1))| < |\hat{s}(\mathbf{x}(k))|$ outside the sector of width $\bar{\rho}$,

$$u_c(k) = \begin{cases} -(\mathbf{CB})^{-1}\mathbf{CA}\mathbf{x}(k) - \theta(|\hat{s}(\mathbf{x}(k))| - \bar{\rho}) \\ \qquad\qquad\qquad\qquad if|\hat{s}(\mathbf{x}(k))| \geq \bar{\rho} \\ -(\mathbf{CB})^{-1}\mathbf{CA}\mathbf{x}(k) \qquad if|\hat{s}(\mathbf{x}(k))| < \bar{\rho} \end{cases} \tag{5}$$

with $|\theta| \leq 1$ would ensure the achievement of a quasi sliding motion on (4), hence plant practical stabilization (i.e. stabilization outside the sector $|\hat{s}(\mathbf{x}(k))| < \bar{\rho}$). Since only the input $v(k)$ is available for direct manipulation, the control problem addressed in this paper consists in finding a feedback controller $v(k)$ guaranteeing the robust practical stabilization of the system (1) in the presence of a saturating nonlinearity in the actuating device.

III. A TIME VARYING SLIDING SURFACE

Define

$$\mathbf{D} = \begin{bmatrix} d_1 & d_2 & \dots & d_{n-1} & 0 \end{bmatrix} \tag{6}$$

and $\bar{\mathbf{C}}(k) \stackrel{\text{def}}{=} (\mathbf{C} + \mathbf{D}\bar{\lambda}^k) = [(c_1 + d_1\bar{\lambda}^k) \;\dots\; (c_{n-1} + d_{n-1}\bar{\lambda}^k) \quad \epsilon]$ (note that $d_n = 0$). Consider the following time-varying sliding surface:

$$\begin{aligned} s(\mathbf{x}(k), \mathbf{x}(0), k) &= \bar{\mathbf{C}}(k)\left[\mathbf{x}(k) - \mathbf{x}(0)\bar{\lambda}^k\right] = \\ &= \left[\; (c_1 + d_1\bar{\lambda}^k)\dots(c_{n-1} + d_{n-1}\bar{\lambda}^k) \;\; \epsilon \;\right] \cdot \\ &\quad \cdot \left[\mathbf{x}(k) - \mathbf{x}(0)\bar{\lambda}^k\right] = 0, \quad |\bar{\lambda}| < 1; \end{aligned} \tag{7}$$

It is straightforward that, for any choice of $d_i \lesseqgtr 0$, $i = 1, \dots n-1$, constraining the system to the surface $s(\mathbf{x}(k), \mathbf{x}(0), k) = 0$ implies plant asymptotical stabilization. Moreover, since $s(\mathbf{x}(0), \mathbf{x}(0), 0) = 0$, the surface (7) is such that no reaching phase exists.

What motivates the introduction of the vanishing term $\mathbf{D}\bar{\lambda}^k\mathbf{x}$ with respect to standard surfaces is the need of modulating the control input in order to cope with the saturation limitation. Roughly speaking, we are aiming at constraining the system on a sliding surface which, besides being asymptotically stabilizing, has a tunable part such that the control input is able to constrain the plant state on the sliding hyperplane without violating the saturation bounds. The following section is therefore devoted to show that the coefficients of the $\mathbf{D}$ vector can always be found as to satisfy the saturation limits, still preserving the persistence of the sliding motion.

For the surface (7), the control input ensuring the achievement of a finite-time sliding motion is, similarly to (5):

$$\begin{aligned} \epsilon v(k) = &-\sum_{i=1}^{n-1}(c_i + d_i\bar{\lambda}^{k+1})x_{i+1}(k) - \epsilon\sum_{i=1}^{n} a_i x_i(k) + \\ &+ \varphi(\mathbf{x}(0), k) - \begin{cases} \theta(|s(k)| - \bar{\rho}) & if|s(k)| \geq \bar{\rho} \\ 0 & if|s(k)| < \bar{\rho} \end{cases} \end{aligned} \tag{8}$$

where, with some abuse of notation, the variable $s(\mathbf{x}(k), \mathbf{x}(0), k)$ has been denoted by $s(k)$, and with:

$$\varphi(\mathbf{x}(0), k) = \sum_{i=1}^{n-1}(c_i + d_i\bar{\lambda}^{k+1})x_i(0)\bar{\lambda}^{k+1} + \epsilon x_n(0)\bar{\lambda}^{k+1}.$$

Lemma 3.1: It is given the uncertain system (1) driven by the feedback controller (8) under Assumptions 1, 2. For any initial condition $\mathbf{x}(0)$, there exists a constant $\Delta_F^{(max)} \in \mathbb{R}^+$, depending on the chosen $\mathbf{x}(0)$, such that:

$$||\mathbf{x}(k)|| \leq \Delta_F^{(max)}, \quad \forall k \tag{9}$$

Proof. As already discussed, for the stabilizing surface (7) no reaching phase exists. Hence, the plant is in quasi sliding motion from $k = 0$, and the dynamics of the state variables are governed by sliding mode. It follows that state trajectories are always bounded.

IV. THE CONTROL LAW

The constraint induced by saturation (2) requires:

$$|v(k)| \leq M \quad \forall k \geq 0 \tag{10}$$

Taking the worst case and considering (3), (6), (8) it follows that the condition (10) can be rewritten as:

$$\left|\bar{\lambda}\sum_{i=1}^{n}\left(c_i + d_i\bar{\lambda}^{k+1}\right)x_i(0)\bar{\lambda}^{k+1} - \epsilon\sum_{i=1}^{n}a_ix_i - \sum_{i=1}^{n-1}\bar{c}_ix_{i+1}\right| + |s(\mathbf{x}(k),\mathbf{x}(0),k)| \leq M|\epsilon| \tag{11}$$

Considering the expression of $|s(\mathbf{x}(k),\mathbf{x}(0),k)|$

$$|s(\mathbf{x}(k),\mathbf{x}(0),k)| \leq \sum_{i=1}^{n-1}(|c_i|+|d_i|)\Delta_F^{(max)} + \epsilon(\Delta_F^{(max)} + |x_n(0)|) + \sum_{i=1}^{n-1}(|c_i|+|d_i|)|x_i(0)| \tag{12}$$

and taking again the worst case, one has:

$$\sum_{i=1}^{n-1}\left((1+\bar{\lambda})|c_i| + (1+\bar{\lambda}^2)|d_i|\right)\frac{|x_i(0)|}{\Delta_F^{(max)}} + |\epsilon|(1+\sum_{i=1}^{n}|a_i|) + \sum_{i=1}^{n-1}\left(2|c_i| + (\bar{\lambda}+1)|d_i|\right) + \epsilon(\bar{\lambda}+1)\frac{|x_n(0)|}{\Delta_F^{(max)}} \leq \frac{M}{\Delta_F^{(max)}}|\epsilon| \tag{13}$$

The following Theorem provides a stabilizing controller designed as to fulfill the constraint $|v(k)| \leq M$ associated to saturation.

Theorem 1: It is given the uncertain system (1) preceded by the saturating device (2), under Assumptions 1, 2. For any given $\mathbf{x}(0)$, proper coefficients d_i, $i = 1,\ldots,n-1$, and a suitable $|\bar{\lambda}| < 1$ can always be found such that the feedback controller (8) guarantees that plant trajectories are uniformly ultimately bounded.

Proof. The proof is constructive. Define m_j, $j = 1,\ldots,n-1$ such that

$$\sum_{j=1}^{n} m_j^{-1} \leq 1. \tag{14}$$

Condition (13) is equivalent to:

$$\sum_{i=1}^{n-1}\left((1+\bar{\lambda})|c_i| + (1+\bar{\lambda}^2)|d_i|\right)\frac{|x_i(0)|}{\Delta_F^{(max)}} + |\epsilon|(1+\sum_{i=1}^{n}|a_i|) + \sum_{i=1}^{n-1}\left(2|c_i| + (\bar{\lambda}+1)|d_i|\right) + \epsilon(\bar{\lambda}+1)\frac{|x_n(0)|}{\Delta_F^{(max)}} \leq M_1|\epsilon|\sum_{j=1}^{n}m_j^{-1} \tag{15}$$

being $M_1 \stackrel{\text{def}}{=} \dfrac{M}{\Delta_F^{(max)}}$. Condition (15) can be split into n chained inequalities, to be fulfilled simultaneously.

- Consider first $i = 1$, and choose d_1 such that:

$$|d_1| < \frac{|\epsilon|\left(\dfrac{M_1}{m_1} - (1+|a_1|)\right) - (\bar{\lambda}+1)|c_1|\dfrac{|x_1(0)|}{\Delta_F^{(max)}} + K_1}{(1+(1+\bar{\lambda}^2)\dfrac{|x_1(0)|}{\Delta_F^{(max)}})} \tag{16}$$

where a same constant K_1, to be determined, has been added and subtracted in (15). Condition (16) requires:

$$m_1 > \frac{M_1}{1+|a_1|}; \quad K_1 > (\bar{\lambda}+1)|c_1|\frac{|x_1(0)|}{\Delta_F^{(max)}} \stackrel{\text{def}}{=} K_1^*$$

and

$$|\epsilon| < \frac{K_1 - (\bar{\lambda}+1)|c_1|\dfrac{|x_1(0)|}{\Delta_F^{(max)}}}{(1+|a_1|) - \dfrac{M_1}{m_1}} \stackrel{\text{def}}{=} Q_1. \tag{17}$$

Taking into account condition (16), inequality (15) is fulfilled if:

$$|c_1|\left(2 - \bar{\lambda}\lambda_1\frac{|x_1(0)|}{\Delta_F^{(max)}}\right) + |\epsilon|\bar{\lambda}\left(\frac{M_1}{m_1} - (1+|a_1|)\right) + 2|c_2| + |\epsilon||a_2| + \lambda_1|c_2|\frac{|x_2(0)|}{\Delta_F^{(max)}} + (1 + (1+\bar{\lambda}^2)\frac{|x_2(0)|}{\Delta_F^{(max)}})|d_2| + \bar{\lambda}|d_2| + \lambda_1K_1 + \sum_{i=3}^{n}|\epsilon||a_i| + \sum_{i=3}^{n-1}\left(2|c_i| + (1+\bar{\lambda})|d_i|\right) + \sum_{i=3}^{n-1}\left(\lambda_1|c_i| + (1+\bar{\lambda}^2)|d_i|\right)\frac{|x_i(0)|}{\Delta_F^{(max)}} + \epsilon\lambda_1\frac{|x_n(0)|}{\Delta_F^{(max)}} \leq M_1|\epsilon|\sum_{j=2}^{n}m_j^{-1} \tag{18}$$

with: $\lambda_1 = (\bar{\lambda}+1)$.

- Consider $i = 2$ and choose d_2 such that:

$$|d_2| < \frac{1}{1+(1+\bar{\lambda}^2)\dfrac{|x_2(0)|}{\Delta_F^{(max)}}} \cdot \left\{|\epsilon|\left(M_1\mu_2 - \nu_2\right) - |c_1|\left(2 - \bar{\lambda}\lambda_1\frac{|x_1(0)|}{\Delta_F^{(max)}}\right) - \lambda_1|c_2|\frac{|x_2(0)|}{\Delta_F^{(max)}} - \lambda_1K_1 + K_2\right\} \tag{19}$$

being:

$$\mu_2 \stackrel{\text{def}}{=} \frac{1}{m_2} - \bar{\lambda}\mu_1; \quad \mu_1 \stackrel{\text{def}}{=} \frac{1}{m_1}; \tag{20}$$

$$\nu_2 \stackrel{\text{def}}{=} |a_2| - \bar{\lambda}\nu_1; \quad \nu_1 \stackrel{\text{def}}{=} 1+|a_1| \tag{21}$$

Condition (19) requires:

$$|\epsilon| < \frac{K_2 - \lambda_1|c_2|\dfrac{|x_2(0)|}{\Delta_F^{(max)}} - |c_1|\gamma_1 - \lambda_1K_1}{(\nu_2 - M_1\mu_2)} \stackrel{\text{def}}{=} Q_2 \tag{22}$$

where $\gamma_i \stackrel{\text{def}}{=} 2 - \bar{\lambda}\lambda_1\dfrac{|x_i(0)|}{\Delta_F^{(max)}}$, provided that K_2 is large enough

$$K_2 > \lambda_1|c_2|\frac{|x_2(0)|}{\Delta_F^{(max)}} + |c_1|\gamma_1 + \lambda_1K_1 \stackrel{\text{def}}{=} K_2^* \tag{23}$$

and provided that $M_1\mu_2 - \nu_2 < 0$. To this purpose, one can impose $\mu_2 < 0$ and $\nu_2 > 0$, corresponding to:

$$\bar\lambda < \frac{|a_2|}{1+|a_1|} \stackrel{\text{def}}{=} \bar\lambda_2; \quad m_2 > \frac{m_1}{\bar\lambda} \tag{24}$$

Taking into account (19) and (18), (15) is fulfilled if:

$$\gamma_2|c_2| - \bar\lambda|c_1|\gamma_1 + |\epsilon|\bar\lambda\left(M_1\mu_2 - \nu_2\right) - \lambda_1\bar\lambda K_1 + \lambda_1 K_2 + \left(\lambda_1|c_3| + (1+\bar\lambda^2)|d_3|\right)\frac{|x_3(0)|}{\Delta_F^{(max)}} + |\epsilon||a_3| + \lambda_1|d_3| + + 2|c_3| + \sum_{i=4}^{n-1}(2|c_i| + \lambda_1|d_i|) + \sum_{i=4}^{n-1}\left(\lambda_1|c_i| + (1+\bar\lambda^2)|d_i|\right)\cdot \frac{|x_i(0)|}{\Delta_F^{(max)}} + |\epsilon|\sum_{i=4}^{n}|a_i| + \epsilon\lambda_1\frac{|x_n(0)|}{\Delta_F^{(max)}} \le M_1|\epsilon|\sum_{j=3}^{n} m_j^{-1}. \tag{25}$$

- Consider $i = 3$ and choose d_3 such that:

$$|d_3| < \frac{1}{1+(1+\bar\lambda^2)\frac{|x_3(0)|}{\Delta_F^{(max)}}}\cdot\left\{|\epsilon|\left(M_1\mu_3 - \nu_3\right) - + \bar\lambda|c_1|\gamma_1 - |c_2|\gamma_2 - \lambda_1|c_3|\frac{|x_3(0)|}{\Delta_F^{(max)}} - \lambda_1 K_2 + K_3\right\} \tag{26}$$

being:

$$\mu_3 \stackrel{\text{def}}{=} \frac{1}{m_3} - \bar\lambda\mu_2; \quad \nu_3 \stackrel{\text{def}}{=} |a_3| - \bar\lambda\nu_2; \tag{27}$$

Differently from the previous case, where $\mu_2 < 0$ and $\nu_2 > 0$, the condition $M\mu_3 - \nu_3 < 0$ has now to be imposed explicitly, i.e.

$$\bar\lambda(\nu_2 - M\mu_2) < |a_3| - \frac{M}{m_3} \tag{28}$$

where, setting $m_3 > \frac{M}{|a_3|}$ and recalling the step 2, both members are positive. Substituting the expressions (20), one gets

$$\bar\lambda(\bar\lambda(\frac{M}{m_1} - 1 - |a_1|) + |a_2| - \frac{M}{m_2}) < |a_3| - \frac{M}{m_3} \tag{29}$$

and, setting $m_1 > \frac{M}{1+|a_1|}$, a strongest condition is

$$\bar\lambda(|a_2| - \frac{M}{m_2}) < |a_3| - \frac{M}{m_3} \tag{30}$$

providing, for $m_2 > \frac{M}{|a_2|}$,

$$\bar\lambda < \frac{|a_3| - \frac{M}{m_3}}{(|a_2| - \frac{M}{m_2})} \stackrel{\text{def}}{=} \bar\lambda_3 > 0 \tag{31}$$

Condition (26) requires:

$$|\epsilon| < \frac{K_3 - \lambda_1|c_3|\frac{|x_3(0)|}{\Delta_F^{(max)}} + \bar\lambda|c_1|\gamma_1 - |c_2|\gamma_2 + \lambda_1 K_2}{(\nu_3 - M_1\mu_3)} \stackrel{\text{def}}{=} Q_3 \tag{32}$$

provided that K_3 is large enough

$$K_3 > \lambda_1|c_3|\frac{|x_3(0)|}{\Delta_F^{(max)}} - \bar\lambda|c_1|\gamma_1 + |c_2|\gamma_2 + \lambda_1 K_2 \stackrel{\text{def}}{=} K_3^*. \tag{33}$$

Taking into account (26) and (34), (15) is fulfilled if:

$$\bar\lambda\gamma_2|c_2| - \bar\lambda^2|c_1|\gamma_1 + |\epsilon|\bar\lambda\left(M_1\mu_3 - \nu_3\right) - \lambda_1\bar\lambda K_2 + \bar\lambda K_3 + \left(\lambda_1|c_4| + (1+\bar\lambda^2)|d_4|\right)\frac{|x_4(0)|}{\Delta_F^{(max)}} + |\epsilon||a_4| + \lambda_1|d_4| + + 2|c_4| + \sum_{i=5}^{n-1}(2|c_i| + \lambda_1|d_i|) + \sum_{i=5}^{n-1}\left(\lambda_1|c_i| + (1+\bar\lambda^2)|d_i|\right)\cdot \frac{|x_i(0)|}{\Delta_F^{(max)}} + \epsilon\lambda_1\frac{|x_n(0)|}{\Delta_F^{(max)}} \le M_1|\epsilon|\sum_{j=4}^{n} m_j^{-1} \tag{34}$$

- The above procedure can be generalized for any $i = r \le n-1$. Choose d_i such that:

$$|d_r| < \frac{1}{1+(1+\bar\lambda^2)\frac{|x_r(0)|}{\Delta_F^{(max)}}}\cdot\left\{|\epsilon|\left[M_1\mu_r - \nu_r\right] + -\lambda_1|c_r|\frac{|x_r(0)|}{\Delta_F^{(max)}} - \sum_{\ell=1}^{r-1}|c_\ell|\left(-\bar\lambda\right)^{r-\ell-1}\gamma_\ell - \lambda_1 K_{r-1} + K_r\right\} \tag{35}$$

and:

$$\mu_r \stackrel{\text{def}}{=} \frac{1}{m_r} - \bar\lambda\mu_{r-1}; \quad r = 2\ldots n. \tag{36}$$

$$\nu_r \stackrel{\text{def}}{=} |a_r| - \bar\lambda\nu_{r-1} \quad r = 2\ldots n-2; \tag{37}$$

$$\nu_{n-1} \stackrel{\text{def}}{=} |a_n| + |a_{n-1}| + \lambda_1\frac{|x_n(0)|}{\Delta_F^{(max)}} - \bar\lambda\nu_{n-2}; \quad \nu_n = \bar\lambda\nu_{n-1} \tag{38}$$

Condition (35) requires:

$$|\epsilon| < \frac{-\sum_{\ell=1}^{r-1}|c_\ell|\left(-\bar\lambda\right)^{r-\ell-1}\gamma_\ell - \lambda_1 K_{r-1} + K_r}{(\nu_r - M_1\mu_r)} + - \frac{\bar\lambda|c_r||\frac{|x_r(0)|}{\Delta_F^{(max)}}|}{(\nu_r - M_1\mu_r)} \stackrel{\text{def}}{=} Q_r \tag{39}$$

and: $M_1\mu_r - \nu_r < 0$, which implies the following conditions:

$$\begin{cases} \bar\lambda < \frac{|a_r|}{\nu_{r-1}} \stackrel{\text{def}}{=} \bar\lambda_r; \quad m_r > \frac{m_{r-1}}{\bar\lambda} \quad r\ even \\ \bar\lambda < \frac{|a_r| - \frac{M}{m_r}}{|a_r| - \frac{M}{m_{r-1}}} \stackrel{\text{def}}{=} \bar\lambda_r; \quad r\ odd \\ m_r > \frac{M}{|a_r|} \quad \forall r \end{cases} \tag{40}$$

$$K_r > \sum_{\ell=1}^{r-1}|c_\ell|\left(-\bar\lambda\right)^{r-\ell-1}\gamma_\ell + \bar\lambda|c_r||\frac{|x_r(0)|}{\Delta_F^{(max)}}| + \lambda_1 K_{r-1} \stackrel{\text{def}}{=} K_r^* \tag{41}$$

- Finally, taking into account (35), (36) for $r = n$ and (41), the last condition to be fulfilled in order to guarantee (15) is the following:

$$\bar{\lambda}\lambda_1 K_{n-2} > \sum_{\ell=1}^{n-1} |c_\ell| \left(-\bar{\lambda}\right)^{n-\ell-1} \gamma_\ell + \lambda_1 K_{n-1} \quad (42)$$

  which provides a further constraint on K_{n-2}. Note that (42) is the second condition imposed on K_{n-2}, and needs to be satisfied together with (35) for $r = n-2$. Condition (42) does not provide any further constraint on ϵ.

Summing up, all components of the control input v can be computed choosing

- the parameter $|\bar{\lambda}| < 1$ such that:

$$\bar{\lambda} < \min\{1, \min_i \bar{\lambda}_i, \quad i = 2, \ldots, n\} \quad (43)$$

- parameters K_i fulfilling both (41) and (42)

$$K_i > \max_i K_i^*, \quad , \quad i = 1, \ldots, n-1 \quad (44)$$

- parameters $|d_i|$, $i = 1, \ldots, n-1$, fulfilling conditions (16), (35), (42), and
- the parameter ϵ such that:

$$|\epsilon| < \min\{Q_1, Q_2, \ldots, Q_{n-1}\} \stackrel{\text{def}}{=} Q. \quad (45)$$

$\diamondsuit$

According to the proof of Theorem 1, the following operative procedure can be given for the determination of the coefficients d_i, $i = 1, \ldots, n-1$, of the vector $\mathbf{D}$ in (7).

1) Set $\bar{\lambda} < \min\{1, \bar{\lambda}_2, \bar{\lambda}_3, \ldots, \bar{\lambda}_n\}$.
2) Fix $m_1 = n$, and compute all the further m_i according to (40).
3) Compute ν_i and μ_i, $i = 1, \ldots, n$ according to (36), (37), (38).
4) Determine numerically an (even rough) estimate of the bounding constant $\Delta_F^{(max)}$, based on the initial condition and the assigned eigenvalues.
5) Compute all K_i's, $i = 1, \ldots, n-1$, according to (41) and considering also (42).
6) Compute all Q_i's, $i = 1, \ldots, n-1$, according to (17),(22), (39).
7) Choose ϵ according to (45), and finally, select d_i, $i = 1, \ldots, n-1$, according to (16), (19), etc.

V. Simulation Results

In order to validate previous theoretical results, the proposed control approach, based on sliding surface (7) and control law (8), has been applied by simulation to the plant used by [1] with: $n = 3$, $a_1 = 1; a_2 = -3; a_3 = 3$. The control input u feeding the plant is the output of a saturation device, with threshold $M = 1$. A disturbance term of the form $d(k) = A\sin(\omega k)$ has been supposed to perturb the system, with $|A| \leq 0.1$ and $\omega = 0.2$. The following set of parameters were found: $\bar{\lambda} = 0.34$, $m_1 = 3$; $m_2 = 9.82$; $m_3 = 29.9$. The vector $\mathbf{C}$ has been selected as $\mathbf{C} = \left[\begin{array}{ccc} 0.005 & -0.15 & 1 \end{array}\right]$ while the vector $\mathbf{D}$ has been designed according to Theorem 1 as $\mathbf{D} = \left[\begin{array}{ccc} -7 & 4 & 0 \end{array}\right]$. The reported simulations have been performed with initial conditions $\mathbf{x}(0) = [1\ 1\ 1]^T$. Results have been reported in Figures 1-4. Figures 1,2 shows the state variables $x_1(k)$ and $x_2(k)$, while Figures 3,4 display the control input $v(k)$ (which coincides exactly with $u(k)$ since the saturation threshold is never violated) and the sliding surface $s(\mathbf{x}(k), \mathbf{x}(0), k)$ respectively. It can be easily verified that the controller proposed here is able to keep both the input $v(k)$ (and consequently $u(k)$) far below the saturation threshold by the proposed algorithm, as theoretically proved, and a substantial improvement of the control activity is achieved. In order to perform a comparison, the same plant has been driven by a standard sliding mode controller built using the standard surface (4). Results have been reported in Figures 5-8, showing the state variables $x_1(k)$ and $x_2(k)$, the control input $u(k)$, and the sliding surface (4)respectively. The improvement of the control activity is evident and the "unavailable" control input shows values remarkably larger that in the previous case.

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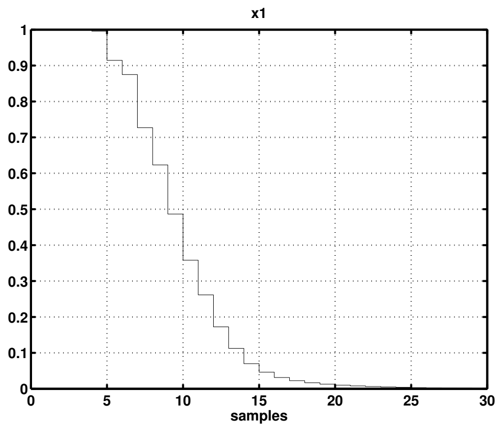

Fig.1 - State variable $x_1(k)$

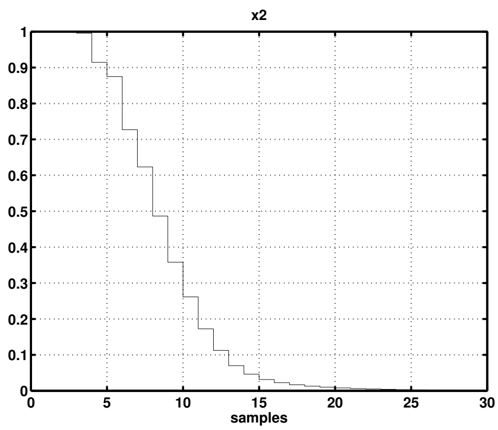

Fig.2 - State variable $x_2(k)$

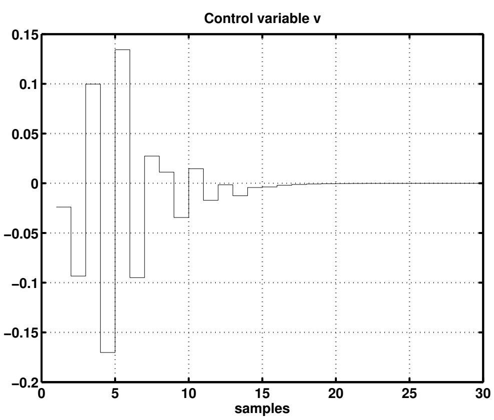

Fig.3 - Control input $v(k)$

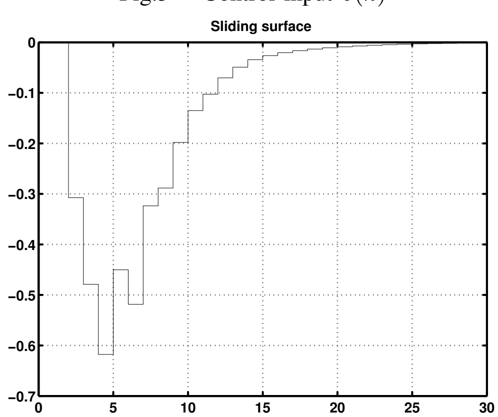

Fig.4 - Sliding surface $s(\mathbf{x}(k), \mathbf{x}(0), k)$

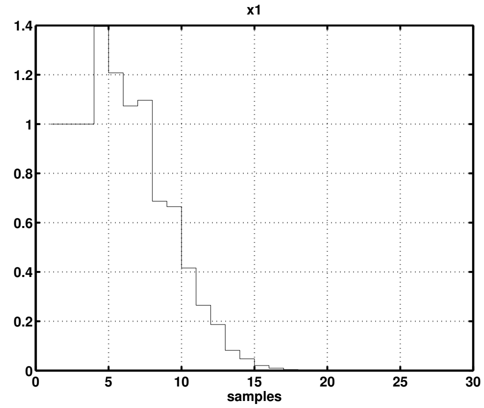

Fig.5 - State variable $x_1(k)$: plant driven by the controller built with (4)

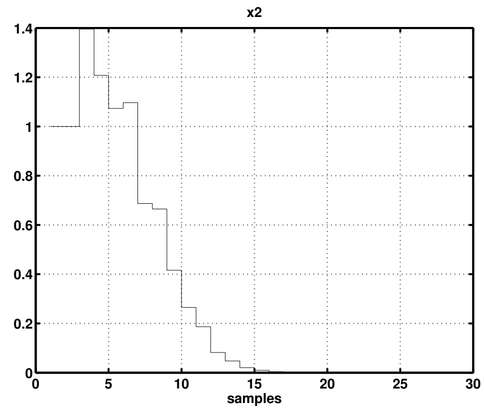

Fig.6 - State variable $x_2(k)$: plant driven by the controller built with (4)

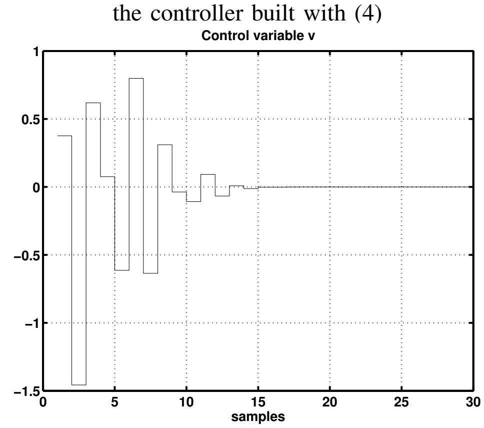

Fig.7 - Control input $v(k)$: plant driven by the controller built with (4)

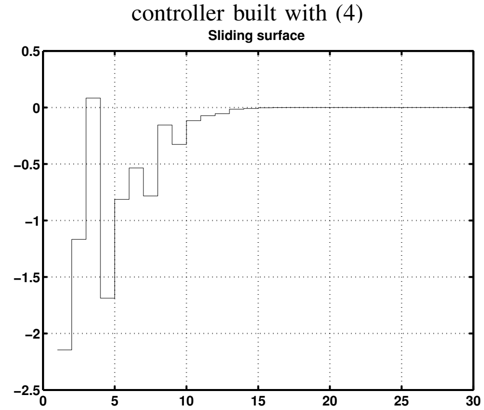

Fig.8 - Sliding surface (4)