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Efficiency Loss and Uniform-Price Mechanism

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Abstract— **We investigate the properties of the class of resource allocation mechanisms with uniform pricing scheme. An important consequence of the assumption that agents maximize their profits is that both the resource price and the quantities assigned to agents can be viewed as the implicit functions of the messages communicated to the mechanism. As a result agents' individual decisions become interdependent, which makes each agent capable of anticipating the effects of individual actions on the price of the resource. Focusing our attention on the Nash equilibrium solution concept we discuss the efficiency of equilibrium allocations of the game defined by the allocation mechanism both from the perspective of the global system goals and the individual objectives of agents. Our first contribution is the ranking of three variations of the uniform-price mechanism. We demonstrate the significant role of strategic variables used by agents and analyze mechanism designer's best response to agents' expected price-anticipating behavior. Since the resource allocation model being subject to our considerations can very well serve as a description of the uniform-price auction for divisible resources, the results of this paper can be viewed as an inquiry into properties of this auction format. As a second contribution we show how signals exchanged between agents and the mechanism can be successfully used to reach an equilibrium point in an iterative bidding process.**

I. Introduction

This work builds upon Kelly's [1], [2] and Johari's [3], [4] analysis of the uniform-price resource allocation mechanism designed for dynamic bandwidth allocation in communication networks. In general, however, the results presented below hold for the settings where a divisible resource must be distributed between a small number of consumers maximizing their profits. For such an environment we investigate the consequences of agents' strategic bidding behavior, which is argued to result from agents' perception of the market-clearing resource allocation rules computing a single price to balance the aggregate demand and supply in the system. The price and allocations are determined by the messages reported to the mechanism by agents. The messages reveal agents' demand at the given price of resource and are required to belong to the set of positive real numbers. Since the resource allocation model being subject to our considerations can very well serve as a description of the uniform-price auction for divisible resources, the results of this paper can be viewed as an extension of the results presented by Ausubel and Cramton [5], Engelbrecht-Wiggans and Kahn [6].

As we are about to show, a definition of the resource allocation rules with uniform pricing has a truly remarkable implication. Namely, in light of the assumption that agents maximize their profits, it makes both the price and the quantities the implicit functions of the messages communicated to the mechanism. As a consequence, agents' individual decisions become interdependent, which makes each agent capable of anticipating the effects of his individual actions on the price of the resource. If the number of agents is small enough, so that each individual can have a considerable influence on the total cost of allocation, then the decision problem faced by each agent becomes a noncooperative game of incomplete information. In the paragraphs which follow we investigate the properties of solutions to the game. Focusing our attention on the pure strategy Nash equilibrium solution concept we discuss the efficiency of equilibrium allocations both from the perspective of the global system goals and the individual objectives of agents. In particular, we present a ranking of different variations of the uniform-price mechanism with elastic supply and emphasize the role of strategic variables used by agents. We also show when the signals exchanged between agents and the mechanism can be successfully used to reach an equilibrium point in an iterative bidding process.

In Section II we define the resource allocation setting being subject to our investigations. Next we discuss the Nash equilibrium conditions given by Johari [3], [4] for the class of uniform-price scalar strategy mechanisms. These conditions are then used to establish the result of Proposition 2. It characterizes the equilibrium allocations in terms of the relationship between marginal utility gains and price elasticity. The relationship is next shown in Proposition 3 to imply the ranking of allocations attainable in equilibria of the games induced by the uniform price mechanism with payments and allocation levels as strategic variables. We show that from the viewpoint of a price-taking resource manager it is reasonable to require from the price-anticipating agents to submit in their bids the total payments they are willing to make for allocations, rather than the demanded allocation levels. On the other hand, from the viewpoint of each agent, the best way to improve profits is to bid strategically in the game where signals reported to the resource manager represent the amount of resource demanded at a given price. Finally, in Section VI we give a bidding procedure which is proved in Proposition 4 to converge to Nash equilibrium of the game defined by resource allocation rules of Section II.

II. Model

Consider a case of $\mathcal{L}$ rational (preference maximizing) agents competing for a single divisible resource. Let $x_i \in \mathbb{R}_+$

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denote the amount of the resource allocated to the agent i and suppose that:

Assumption II-A: For each $i \in \mathcal{L}$, over the domain $x_i \geq 0$ the utility function $U_i(x_i)$ is concave, strictly increasing, and continuous, and over the domain $x_i > 0$, $U_i(x_i)$ is continuously differentiable. Furthermore, the right directional derivative at 0, denoted $U_i'(0^+)$, is finite.

The total cost of assigning the amount $y = \sum_{i\in\mathcal{L}} x_i$ of the resource to the agents is given by $C(y)$, which is assumed to be incurred by a single, strategically neutral resource manager, i.e. a manager acting as *price-taker* maximizing his profits:

$$\Pi(\mu, y) = \mu y - C(y), \tag{1}$$

where μ is a fixed unit price of the resource established within the market-clearing process according to the rules of a resource allocation mechanism selected by the manager. Since manager's profits are maximized when marginal production cost is equal to the price, it is natural to assume that:

Assumption II-B: There exists a continuous, convex and strictly increasing function $p(y)$ such that $p(0) \geq 0$ and such that:

$$C(y) = \int_0^y p(s)ds. \tag{2}$$

Thus total cost function $C(y)$ is strictly convex and strictly increasing over $y \geq 0$.

For further development it is also necessary to define the price elasticity:

$$\varepsilon(y) \equiv \frac{y}{p(y)}\frac{\partial p(y)}{\partial y} \tag{3}$$

measuring the marginal change in $\log(p(y))$ resulting from the marginal change in $\log(y)$. Furthermore, we define:

$$\beta(y) \equiv \frac{\varepsilon(y)}{(1+\varepsilon(y))}. \tag{4}$$

If $p(y)$ is not differentiable at y, then the corresponding right and left directional derivatives of p define $\varepsilon^+(y) \equiv y/p(y)\partial^+p(y)/\partial y$ and $\varepsilon^-(y) \equiv y/p(y)\partial^-p(y)/\partial y$ respectively. Definitions of β^+ and β^- follow immediately. Notice that, by the assumption of continuity and convexity of p, directional derivatives exist [7], [8].

There are three key assumptions that we follow. First, the aggregate utility $\sum_{i\in\mathcal{L}} U_i(x_i)$ of the consumers less the aggregate cost $C(y)$ of allocation gives a value interpreted as a total welfare measure related to the allocation y. Second, the mechanism sets a single market-clearing price for a resource unit that is assumed to be equal to the marginal production cost and that ensures that demand equals supply, i.e. that total amount of resource available to agents at the price is allocated to them. Third, agents anticipate the effects of their actions on the market-clearing price.

III. Payment Bidding

Consider a resource allocation game in which, having $\mathbf{w}_{-i} = (w_1, ..., w_{i-1},\ w_{i+1}, ..., w_L)$ fixed, each player i selects a signal $w_i \geq 0$ to maximize his payoff function:

$$Q_i(x_i(w_i, \mathbf{w}_{-i}), w_i) = U_i(x_i(w_i, \mathbf{w}_{-i})) - w_i. \tag{5}$$

The signal w_i can be interpreted as the total payment that the agent i is willing to make for the amount x_i of the resource. Agent i does not know $\mathbf{w}_{-i}$, but knows that the price $p(y(\mathbf{w}))$ and the amount of the resource he receives $x_i(\mathbf{w})$ depend on it.

The following assumption is crucial from the viewpoint of the developed framework. It states that the allocations assigned to agents constitute a solution to the market-clearing equation, i.e. the equation according to which supply of the resource equals aggregate demand at the market-clearing price.

Assumption III-A: For all $\mathbf{w} = (w_1, \ldots, w_L) \geq 0$, the aggregate allocation

$$y(\mathbf{w}) = \sum_{i\in\mathcal{L}} x_i(\mathbf{w})$$

is the unique solution to the market-clearing equation:

$$\sum_{i\in\mathcal{L}} w_i = y(\mathbf{w})p(y(\mathbf{w})). \tag{6}$$

Furthermore, for each $i \in \mathcal{L}$:

$$x_i(\mathbf{w}) = \begin{cases} \dfrac{w_i}{p(y(\mathbf{w}))} & w_i > 0; \\ 0 & w_i = 0, \end{cases} \tag{7}$$

and there exists $k \in L$ such that $U_k'(0^+) > p(0)$.

The following result is due to Johari [4]:

Proposition 1 (Johari): If assumptions II-A-III-A hold, then $\mathbf{w}$ is a Nash equilibrium of the game defined by $(Q_i(x_i(\mathbf{w}), w_i))_{i\in\mathcal{L}}$ if and only if $\sum_{i\in\mathcal{L}} w_i > 0$, and with $y \equiv y(\mathbf{w})$ and $x_i \equiv x_i(\mathbf{w})$ the following two conditions hold:

$$U_i'(x_i)\left(1 - \beta^+(y)\frac{x_i}{y}\right) \leq p(y); \tag{8}$$

$$U_i'(x_i)\left(1 - \beta^-(y)\frac{x_i}{y}\right) \geq p(y). \tag{9}$$

Conversely, if $\mathbf{x} = (x_1, ..., x_L) \geq 0$ and $y > 0$ satisfy (8)-(9) then $\mathbf{w} = p(y)\mathbf{x}$ is a Nash equilibrium with $x_i \equiv x_i(\mathbf{w})$ and $y \equiv y(\mathbf{w}) = \sum_{i\in\mathcal{L}} x_i(\mathbf{w})$.

The result shows that the solutions to the payoff maximization problem:

$$w_i \in \arg\max_{s\in\mathbb{R}_+} Q_i(x_i(s, \mathbf{w}_{-i}), s), \tag{10}$$

calculated individually by each agent $i \in \mathcal{L}$ subject to the constraints (6) and (7), constitute a Nash equilibrium of the game induced by the market-clearing equation given by (6).

Indeed, since $x_i = w_i/p(y(w_1, ..., w_i, ..., w_L))$ whenever $w_i > 0$, it is straightforward to conclude that each agent can approximate his influence on the total supply of the resource. By assumption II-B function p is continuous, convex and strictly increasing, which implies that $g(y) = yp(y)$ is strictly increasing, strictly convex, continuous and thus invertible. As a consequence $y(\mathbf{w}) = g^{-1}(\sum_{i\in\mathcal{L}} w_i)$ is strictly increasing and strictly concave function of $\sum_{i\in\mathcal{L}} w_i$, and therefore directionally differentiable. Marginal changes in supply, caused by agent i's unilateral deviation from the consumption level determined by his willingness to pay w_i, are therefore given by:

$$\frac{\partial^+ y(\mathbf{w})}{\partial w_i} = \left(p(y(\mathbf{w})) + y(\mathbf{w})\frac{\partial^+ p(y(\mathbf{w}))}{\partial w_i}\right)^{-1};$$

$$\frac{\partial^- y(\mathbf{w})}{\partial w_i} = \left(p(y(\mathbf{w})) + y(\mathbf{w})\frac{\partial^- p(y(\mathbf{w}))}{\partial w_i}\right)^{-1},$$

which follows from directional differentiation of (6) with respect to w_i. It is important to observe that no assumption on differentiability of $p(y)$ has been made, so right and left directional derivatives are not necessarily equal; see [4] for the general treatment.

Since the total production level $y(\mathbf{w})$ determines the price of the resource, an agent capable of calculating the above derivatives of $y(\mathbf{w})$ is also able to anticipate the marginal price changes and, as a consequence, to improve his individual profits. At this point it should be noticed that the price of the resource and the total production level are observable variables; their values, as well as the form of the market-clearing equation (6), are a common knowledge. It is therefore reasonable to assume that each agent will expect other agents to perform the similar price-anticipating reasoning. As a result, decision problem faced by each agent indeed has a nature of noncooperative game.

From the first order optimality conditions for payoff maximization problem (10) it immediately follows that given a fixed vector $\mathbf{w}_{-i} = (w_1, ..., w_{i-1}, w_{i+1}, ..., w_L)$ agent i's best response to other agents' decisions must satisfy the next two conditions:

$$\frac{\partial^+ Q_i(x_i(w_i, \mathbf{w}_{-i}), w_i)}{\partial w_i} = U_i'(x_i(\mathbf{w}))\frac{\partial^+ x_i(\mathbf{w})}{\partial w_i} - 1 \leq 0;$$

$$\frac{\partial^- Q_i(x_i(w_i, \mathbf{w}_{-i}), w_i)}{\partial w_i} = U_i'(x_i(\mathbf{w}))\frac{\partial^- x_i(\mathbf{w})}{\partial w_i} - 1 \geq 0.$$

Proposition 1 shows that under assumptions II-A-III-A the above best response conditions, admitting optimal solution to be situated at a nondifferentiable point of $p(y)$, constitute a Nash equilibrium $\mathbf{w} \neq 0$ of the resource allocation game; if $\mathbf{w} \neq 0$ satisfies the conditions of Proposition 1 then no agent can improve his payoff by unilaterally deviating from $\mathbf{w}$. Notice that with $\mathbf{w}_{-k} = 0$, by assumption III-A, if $\mathbf{w}_k \to 0^+$ then $(x_k(\mathbf{w})/w_k)^{-1} \to p(0) < U_k'(0^+)$, which implies that at least one agent k can improve his payoff by deviating from $w_k = 0$. Differentiating (7) with respect to w_i and substituting directional derivatives of $y(\mathbf{w})$ yields (8)-(9).

Efficiency Loss

The direct implication of Proposition 1 is that the allocations obtained at Nash equilibrium point defined by (8)-(9) are not Pareto-optimal. To see this, consider the resource allocation problem faced by the resource manager:

$$\max_{\mathbf{x}\in\mathbb{R}_+^L} \left[\sum_{i\in\mathcal{L}} U_i(x_i) - C(y)\right], \quad s.t.\ y = \sum_{i\in\mathcal{L}} x_i. \qquad (11)$$

By assumption II-A, function U_i is concave, strictly increasing and differentiable over $x_i > 0$ for all $i \in \mathcal{L}$. By assumption II-B, function C is convex and differentiable. As a result, the necessary and sufficient conditions for optimality of $\mathbf{x} = (x_1, ..., x_L)$ are given, for every $i \in \mathcal{L}$, by:

$$\begin{cases} x_i\Big[U_i'(x_i) - p(y)\Big] = 0; \\ U_i'(x_i) - p(y) \leq 0. \end{cases} \qquad (12)$$

If agent i is assigned a positive amount $\bar{x}_i$ of the resource, then the corresponding marginal increase in utility $U_i'(\bar{x}_i)$ he receives must be equal to the market-clearing price $p(\bar{y}) > 0$, where $\bar{y} = \sum_{i\in\mathcal{L}} \bar{x}_i$. If, on the other hand, no amount of the resource is allocated to agent i, then the market-clearing price must be greater than or equal to the marginal utility of agent i at zero. Pareto-optimality of the allocations satisfying (12) is implied by the following result:

Lemma 1: Optimal solution to:

$$\max\left\{\sum_{i=1}^{m} f_i(\mathbf{x})\colon \mathbf{x} \in \mathbb{R}^n\right\} \qquad (13)$$

is an an efficient solution to:

$$\max\left\{(f_1(\mathbf{x}), ..., f_m(\mathbf{x}))\colon \mathbf{x} \in \mathbb{R}^n\right\} \qquad (14)$$

where $f_i\colon \mathbb{R}^n \to \mathbb{R}$ for $i = 1, ..., m$.

Proof: See e.g. [9], [10]. ■

Thus, by Lemma 1 the optimal solution to (11) is also an efficient solution to the multiobjective problem:

$$\max\left\{\left(U_1(x_1), ..., U_L(x_L), -p(\sum_{j\in\mathcal{L}} x_j)\right) : \mathbf{x} \in \mathbb{R}_+^L\right\}.$$

Furthermore, by concavity of (11) and the fundamental theorems of welfare economics, the corresponding pair $(p(\bar{y}), \bar{\mathbf{x}})$ constitutes a competitive equilibrium as well. Our first proposition demonstrates that these solutions are substantially different from those that arise at Nash equilibria of the game with payoffs defined by (5). The proposition is technically very useful and will be used to derive our subsequent insights.

Proposition 2: Suppose that $p(y)$ is differentiable and exhibits nondecreasing elasticity $\varepsilon(y)$ for $y \geq 0$. If Assumptions II-A-III-A hold and $\hat{\mathbf{w}}$ is Nash equilibrium of the game defined by $(Q_i(x_i(\mathbf{w}), w_i))_{i\in\mathcal{L}}$, then:

$$p(y(\hat{\mathbf{w}})) < U_i'(x_i(\hat{\mathbf{w}})) < p(y(\hat{\mathbf{w}}))(1 + \varepsilon(y(\hat{\mathbf{w}}))) \qquad (15)$$

for all $i \in \mathcal{L}$ such that $x_i(\hat{\mathbf{w}}) > 0$.

Proof: By the differentiability of p and Proposition 1, if $\hat{\mathbf{w}}$ is a Nash equilibrium, then:

$$\begin{cases} w_i \left[U_i'(x_i(\hat{\mathbf{w}})) \left(1 - \beta(y(\hat{\mathbf{w}})) \frac{x_i(\hat{\mathbf{w}})}{y(\hat{\mathbf{w}})} \right) - p(y(\hat{\mathbf{w}})) \right] = 0 \\ U_i'(0^+) - p(y(\hat{\mathbf{w}})) \le 0. \end{cases} \tag{16}$$

Fix $\hat{\mathbf{w}}_{-i}$ and consider $\hat{w}_i > 0$. Since U_i is strictly increasing, from (7) and the fact that $\beta(y(\hat{\mathbf{w}})) > 0$ for $y(\hat{\mathbf{w}}) > 0$ (observe that $\beta'(y) = \varepsilon'(y)/(1+\varepsilon(y))^2 \ge 0$) it follows, that:

$$U_i'(x_i(\hat{\mathbf{w}})) - p(y(\hat{\mathbf{w}})) = U_i'(x_i(\hat{\mathbf{w}}))\beta(y(\hat{w}))\frac{x_i(\hat{\mathbf{w}})}{y(\hat{\mathbf{w}})} > 0.$$

Now, suppose $U_i'(x_i(\hat{\mathbf{w}})) > p(y(\hat{\mathbf{w}}))(1+\varepsilon(y(\hat{\mathbf{w}})))$. Then, for every $i \in \mathcal{L}$ such that $\hat{w}_i > 0$:

$$p(y(\hat{\mathbf{w}}))(1+\varepsilon(y(\hat{\mathbf{w}}))) \left(1 - \beta(y(\hat{\mathbf{w}})) \frac{x_i(\hat{\mathbf{w}})}{y(\hat{\mathbf{w}})} \right) < p(y(\hat{\mathbf{w}})).$$

This, however, implies that:

$$\varepsilon(y(\hat{\mathbf{w}})) \left(1 - \frac{x_i(\hat{\mathbf{w}})}{y(\hat{\mathbf{w}})} \right) < 0.$$

which is a contradiction, since $\varepsilon(y(\hat{\mathbf{w}})) > 0$ and $y(\hat{\mathbf{w}}) = \sum_{j\in\mathcal{L}} x_j(\hat{\mathbf{w}}) \ge x_i(\hat{\mathbf{w}})$ in equilibrium. As a result, for all $i \in \mathcal{L}$ such that $x_i(\hat{\mathbf{w}}) > 0$:

$$p(y(\hat{\mathbf{w}})) < U_i'(x_i(\hat{\mathbf{w}})) < p(y(\hat{\mathbf{w}}))(1+\varepsilon(y(\hat{\mathbf{w}}))).$$

■

Condition (15), the key result of this paper, shows that the marginal gains from positive allocations assigned to agents in Nash equilibrium of the game induced by the uniform-price mechanism exceed the market-clearing price of resource. As a result they are not Pareto-optimal and do not maximize the efficiency measure (11) applied by the resource manager interested in optimal utilization of system resources. Noticeably, however, from the viewpoint of each agent price-anticipating bidding provides an opportunity to obtain an additional marginal revenue equal to the difference between the marginal utility from the consumption level achieved in equilibrium and the corresponding price equal to the marginal cost of allocation incurred by the system; it improves agents' profits.

IV. Resource Bidding

Consider now the classical Cournot competition model [11], [12] and the related resource allocation game with payoffs defined for all $i \in \mathcal{L}$ by:

$$\tilde{Q}_i(\mathbf{x}) = U_i(x_i) - x_i p(\sum_{j\in\mathcal{L}} x_j). \tag{17}$$

Instead of reporting to the mechanism a value of the payment w_i each agent is now required to report the allocation level x_i demanded at the price $p(\sum_{j\in\mathcal{L}} x_j)$. The pointwise characterization of individual demand for the resource is therefore described here directly by the value of x_i rather than indirectly by the value of agent's willingness to pay w_i. The market-clearing equation in this setting is defined as follows:

$$\sum_{i\in\mathcal{L}} x_i = S(p(y)), \tag{18}$$

where $y = \sum_{i\in\mathcal{L}} x_i$ and $S(\mu) = p^{-1}(\mu)$ is a supply correspondence setting the total production level y to the value maximizing the profit $\Pi(\mu, y)$.

We define a Cournot-Nash equilibrium $\tilde{\mathbf{x}} = (\tilde{x}_1, ..., \tilde{x}_L)$ as a vector of the solutions to:

$$\tilde{x}_i \in \arg\max_{s\in\mathbb{R}_+} \tilde{Q}_i((s, \mathbf{x}_{-i})). \tag{19}$$

To see that the vector $\tilde{\mathbf{x}}$ exists, notice that $\tilde{Q}_i$ is concave and continuous in $x_i \in X_i = [0, R_i]$. The strategy space X_i, defined by the rules of resource allocation, is compact and convex for each $i \in \mathcal{L}$; by assumptions II-A and II-B there exists a positive number $R_i > 0$ such that $U_i(R_i) \le R_i p(R_i + \sum_{j\ne i} x_j)$, so it is rational for each agent $i \in \mathcal{L}$ to bid $x_i \in [0, R_i]$. Since agent i's best response $\tilde{x}_i$ to $\mathbf{x}_{-i}$ is a solution to (19), a mapping $\mathbf{H}\colon X \to X$ with $X = [0, R_1]\times\cdots\times[0, R_L]$ and $H_i(\mathbf{x}) \in \arg\max_{s\in X_i} \tilde{Q}_i((x_1, ..., s, ..., x_L))\}$ is upper semicontinuous. Existence of Cournot equilibrium results, as a consequence, from the Kakutani fixed point theorem, as proved by Rosen in [13].

Efficiency Loss

We will now examine the relationship between the allocations arising at the Cournot-Nash equilibrium and at the Nash equilibrium defined by conditions (8)-(9).

Proposition 3: Consider payoffs $(\tilde{Q}_i(\mathbf{x})_{i\in\mathcal{L}})$ defined by (17) and let $\tilde{\mathbf{x}}$ be a vector of the solutions to (19). Suppose also that $\hat{\mathbf{w}}$ is a Nash equilibrium of the game defined by payoffs $(Q_i(x_i(\mathbf{w}), w_i))_{i\in\mathcal{L}}$. If the assumptions of Proposition 2 hold, then for all $i \in \mathcal{L}$:

$$x_i(\hat{\mathbf{w}}) \ge \tilde{x}_i. \tag{20}$$

Proof: Consider first the following system of equations:

$$x_i(\mathbf{w}) - \alpha_i = 0,$$

where $\alpha_i \in [0, \infty)$ for all $i \in \mathcal{L}$. By assumption III-A, the system has a unique trivial solution $\tilde{\mathbf{w}} = 0$ iff $\alpha_i = 0$ for all $i \in \mathcal{L}$. Suppose then that $\alpha_k > 0$ and that $\alpha_i = 0$ for all $i \ne k$. We claim that the system still has a unique solution, but now $\tilde{w} \ne 0$. Indeed, since $x_k(\mathbf{w})$ is continuous, strictly increasing and concave in $w_k \ge 0$, and since $x_k(\mathbf{w}) \to \infty$ as $w_k \to \infty$ (see Proposition 3.3 [4]), there exists a solution $\tilde{\mathbf{w}} \ne 0$, such that $\tilde{w}_i = 0$ for $i \ne k$ and $\tilde{w}_k > 0$. Next, take an arbitrary set $\mathcal{L}_1 \subseteq \mathcal{L}$ such that $\alpha_i > 0$ for all $i \in L_1$ and consider a mapping $\mathbf{H}\colon W \to W$, where $W \equiv [0, \omega_1^{-1}(0) + \epsilon] \times ... \times [0, \omega_L^{-1}(0) + \epsilon]$ with $\omega_i(w_i) \equiv x_i(w_i, \mathbf{w}_{-i}) - \alpha_i$ for both α_i and $\mathbf{w}_{-i}$ fixed, and $\epsilon \ge 0$. Let $H_i(\mathbf{w}) = \omega_i^{-1}(0)$, i.e. for an arbitrarily chosen $i \in \mathcal{L}_1$, given α_i and $\mathbf{w}_{-i}$, let H_i return the value of w_i such that $\omega_i(w_i) = 0$. Observe that $H_i(\mathbf{w}) \in [0, \omega_i^{-1}(0) + \epsilon]$ for all $i \in \mathcal{L}$. Since W is nonempty, compact and convex, and the mapping $\mathbf{H}(\mathbf{w}) = (H_1(\mathbf{w}), ..., H_L(\mathbf{w}))$ is continuous, by the Brouwer's fixed point theorem there exists $\tilde{\mathbf{w}} = \mathbf{H}(\tilde{\mathbf{w}})$, such that $\mathbf{x}(\tilde{\mathbf{w}}) = \boldsymbol{\alpha}$ for all $i \in \mathcal{L}$ with $\boldsymbol{\alpha} = \tilde{\mathbf{x}}$ being solution to (19).

Next, from the first order optimality conditions for (19) it follows that:

$$\tilde{x}_i p'(\tilde{y}) = U_i'(\tilde{x}_i) - p(\tilde{y})$$

whenever $\tilde{x}_i > 0$ and $\tilde{y} = \sum_{i\in\mathcal{L}} \tilde{x}_i$. Substituting right hand side of the above equation to the partial derivative of $Q_i(x_i(\tilde{\mathbf{w}}), \tilde{w}_i)$ yields:

$$\begin{aligned}\frac{\partial Q_i(x_i(\tilde{\mathbf{w}}), \tilde{w}_i)}{\partial w_i} &= \frac{U_i'(\tilde{x}_i)}{p(\tilde{y})}\left(1 - \beta(\tilde{y})\frac{\tilde{x}_i}{\tilde{y}}\right) - 1 \\ &= \frac{U_i'(\tilde{x}_i)}{p(\tilde{y})}\left(1 - \frac{U_i'(\tilde{x}_i) - p(\tilde{y})}{p(\tilde{y})(1+\varepsilon(\tilde{y}))}\right) - 1.\end{aligned}$$

Suppose now that $0 < x_i(\hat{\mathbf{w}}) < \tilde{x}_i$ for some $i \in \mathcal{L}_1 \subseteq \mathcal{L}$ and $x_k(\hat{\mathbf{w}}) \le \tilde{x}_k$ for $k \in \mathcal{L} \setminus \mathcal{L}_1$. This immediately implies that $p(y(\hat{\mathbf{w}})) < p(\tilde{y})$, $\tilde{w}_i > \hat{w}_i$ and $U_i'(x_i(\hat{\mathbf{w}})) \ge U_i'(\tilde{x}_i)$ for $i \in \mathcal{L}_1$. Since p is assumed to be differentiable and exhibits nondecreasing elasticity, from the fact that Q_i is strictly concave in w_i (see Proposition 3.7 [4]) and $\partial Q_i(x_i(\hat{\mathbf{w}}), \hat{w}_i)/\partial w_i = 0$ for $\hat{w}_i > 0$, it follows that:

$$\frac{\partial Q_i(x_i(\tilde{\mathbf{w}}), \tilde{w}_i)}{\partial w_i} = \frac{U_i'(\tilde{x}_i)}{p(\tilde{y})}\left(1 - \frac{U_i'(\tilde{x}_i) - p(\tilde{y})}{p(\tilde{y})(1+\varepsilon(\tilde{y}))}\right) - 1 < 0.$$

From this we obtain:

$$\frac{U_i'(\tilde{x}_i) - p(\tilde{y})}{U_i'(\tilde{x}_i)} < \frac{U_i'(\tilde{x}_i) - p(\tilde{y})}{p(\tilde{y})(1+\varepsilon(\tilde{y}))},$$

which implies that: $U_i'(x_i(\hat{\mathbf{w}})) \ge U_i'(\tilde{x}_i) > p(\tilde{y})(1+\varepsilon(\tilde{y})) > p(y)(1+\varepsilon(y(\hat{\mathbf{w}})))$. By Proposition 2 this is a contradiction, since in equilibrium $U_i'(x_i(\hat{\mathbf{w}})) < p(y(\hat{\mathbf{w}}))(1+\varepsilon(y(\hat{\mathbf{w}})))$ whenever $x_i(\hat{\mathbf{w}}) > 0$. We have, therefore, demonstrated that $\partial Q_i(x_i(\tilde{\mathbf{w}}), \tilde{w}_i)/\partial w_i \ge 0$ for $\tilde{\mathbf{w}}$ such that $\tilde{\mathbf{x}} = \mathbf{x}(\tilde{\mathbf{w}})$. This proves that $x_i(\hat{\mathbf{w}}) \ge \tilde{x}_i$ for all $i \in \mathcal{L}$. ■

V. Quantitative Comparison of Market Strategies

In this section we compare the mechanisms studied above with respect to the generated efficiency and allocation levels.

Observe that an immediate corollary to condition (15) of Proposition 2 is that $y(\hat{\mathbf{w}}) \le \bar{y} = \sum_{i\in\mathcal{L}} \bar{x}_i$, where $\bar{x}$ is a vector of Pareto-optimal allocations satisfying conditions (12). Furthermore, concavity and monotonicity of U_i, together with convexity and monotonicity of p imply that $p(y(\hat{\mathbf{w}})) \le p(\bar{y})$. As a consequence, profits of resource manager are reduced in comparison to those of competitive equilibrium. To see this assume, to the contrary, that:

$$p(y(\hat{\mathbf{w}}))y(\hat{\mathbf{w}}) - C(y(\hat{\mathbf{w}})) > p(\bar{y})\bar{y} - C(\bar{y}).$$

This implies that:

$$C(\bar{y}) > C(y(\hat{\mathbf{w}})) + C'(\bar{y})(\bar{y} - y(\hat{\mathbf{w}})),$$

which is a contradiction, since C is strictly convex. (From the viewpoint of mechanism design it is important to notice, though, that condition (6) guarantees that allocation costs are covered. Indeed, by convexity $p(y)y \ge \int_0^y p(a)da$. This condition need not hold for other classes of mechanisms.) This, therefore, gives the following result:

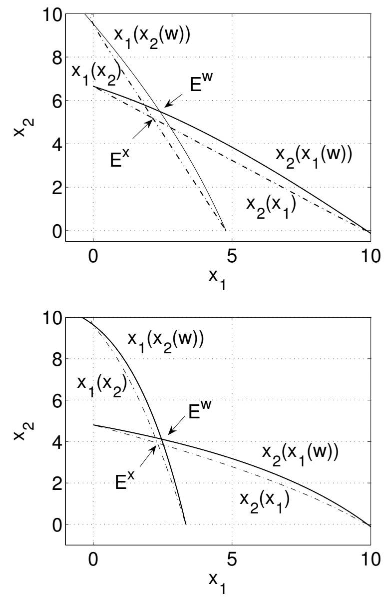

Fig. 1. Optimal response curves for the setting with two agents: $p(\mathbf{x}) = (10-y)^{-1}$, $U_i(x_i) = \gamma_i \log(x_i + 1)$ (left) and $U_i(x_i) = \gamma_i \arctan(x_i)$ (right). Optimal response curves are tangent to the iso-payoff curves at the points being solution to (10) and (19). In equilibrium E^w of the game with payments w_i as strategies each agent is assigned more resources then in equilibrium E^x of the game with allocations x_i as strategies.

Corollary 1: Let $(\bar{\mu}, \bar{\mathbf{x}})$ be a competitive equilibrium defined by conditions (12) and let $\hat{\mathbf{w}}$ be a Nash equilibrium of the game with payoffs defined by (5). Then:

$$\Pi(p(y(\hat{\mathbf{w}})), y(\hat{\mathbf{w}})) \le \Pi(p(\bar{y}), \bar{y}). \quad (21)$$

The results of Proposition 2 and Corollary 1 stem from the fact that the goals of resource manager, expressed by conditions (12), are not compatible with the myopic goals of agents maximizing their payoffs, expressed by conditions (8)-(9). Equivalently, the rules of a uniform-price mechanism, applied to the setting of strategic agents capable of exerting their market power, are characterized by the undesirable property of creating *incentives* to misrepresent preferences. The observation is straightforward because for $x_i(\hat{\mathbf{w}}) > 0$:

$$0 < 1 - \beta(y(\hat{\mathbf{w}}))\frac{x_i(\hat{\mathbf{w}})}{y(\hat{\mathbf{w}})} < 1,$$

which implies that:

$$U_i'(x_i(\hat{\mathbf{w}}))\left(1 - \beta(y(\hat{\mathbf{w}}))\frac{x_i(\hat{\mathbf{w}})}{y(\hat{\mathbf{w}})}\right) < U_i'(x_i(\hat{\mathbf{w}})).$$

Thus, from the perspective of the mechanism processing messages submitted by agents, it is as though agent $i \in \mathcal{L}$ were characterized by the marginal gains described by the left hand side of the equation above, which are lower then the real ones defined by U_i'. This is precisely the effect of *demand reduction* discussed by Ausubel and Cramton in [5].

Each agent's optimal strategy is to shade bids in order to reduce the market-clearing price of the resource and achieve profits that are higher in comparison to those that would be achieved with price-taking behavior in competitive equilibrium. This brings us to the conclusion that in the considered setting of strategically interdependent agents and elastic supply the uniform-price mechanism optimizes utilization of resources for the manipulatively declared preference model.

The next result demonstrates that the positive allocations arising in the Nash equilibrium of the game where agents report their willingness to pay are at least as high as those arising in the Cournot-Nash equilibrium of the game where agents declare the demanded amount of resource.

Corollary 2: If the assumptions of Proposition 2 hold, then:

$$\sum_{i\in\mathcal{L}} U_i(x_i(\hat{\mathbf{w}})) - C(y(\hat{\mathbf{w}})) \geq \sum_{i\in\mathcal{L}} U_i(\tilde{\mathbf{x}}) - C(\tilde{y}), \qquad (22)$$

where $y(\hat{\mathbf{w}}) = \sum_{j\in\mathcal{L}} x_j(\hat{\mathbf{w}})$ and $\tilde{y} = \sum_{j\in\mathcal{L}} \tilde{x}_j$.

Proof: First, notice that from Proposition 2 we have:

$$U_i'(\tilde{x}_i) \geq p(\tilde{y}), \quad U_i'(x_i(\hat{\mathbf{w}})) \geq p(y(\hat{\mathbf{w}})).$$

Furthermore, from the Proposition 2 and Proposition 3 we obtain: $U_i'(\tilde{x}_i) \geq p(y(\hat{\mathbf{w}}))$. As a result, by convexity of p:

$$\sum_{i\in\mathcal{L}} \int_0^{x_i(\hat{\mathbf{w}})} U_i'(a)da - \sum_{i\in\mathcal{L}} \int_0^{\tilde{x}_i} U_i'(a)da =$$

$$\geq \int_{\tilde{y}}^{y(\hat{\mathbf{w}})} p(s)ds = C(y(\hat{\mathbf{w}})) - C(\tilde{y}),$$

which completes the proof. ■

Corollary 2 shows that the total welfare of the allocations attainable in the case of payment bidding is at least as high as the welfare of the allocations attainable in the case of resource bidding. Notice, though, that both signals w_i and x_i provide the unique characterization of the agent i's demand for resource. This characterization is given by:

$$p(y(\mathbf{w}))x_i(\mathbf{w}) = w_i.$$

However, different strategy spaces make agents maximize different payoff functions, which in turn leads to different solutions.

Clearly, by Corollary 2, from the viewpoint of resource manager it is more reasonable to apply the mechanism where payments w_i are reported. Our next result shows that it is not what agents would prefer.

Corollary 3: Suppose that assumptions of Proposition 2 hold. Define $\bar{Q}_i(x_i,\mu) \equiv U_i(x_i) - \mu x_i$ and let $(\bar{\mu}, \bar{\mathbf{x}})$ denote competitive equilibrium being solution to (11). Then for all $i \in \mathcal{L}$:

$$\bar{Q}_i(\bar{x}_i, \bar{\mu}) \leq Q_i(x_i(\hat{\mathbf{w}}), \hat{w}_i) \leq \tilde{Q}_i(\tilde{\mathbf{x}}). \qquad (23)$$

Proof: Suppose first, to the contrary, that we have $Q_i(x_i(\hat{\mathbf{w}}), \hat{w}_i) > \tilde{Q}_i(\tilde{\mathbf{x}})$. As a consequence, from Proposition 2, we obtain for $x_i(\hat{\mathbf{w}}) > 0$ and $\tilde{x}_i > 0$:

$$U_i(x_i(\hat{\mathbf{w}})) - U_i(\tilde{x}_i) > x_i(\hat{\mathbf{w}})p(y) - \tilde{x}_i p(\tilde{y})$$

$$\geq U_i'(\tilde{x}_i)(x_i(\hat{\mathbf{w}}) - \tilde{x}_i) - \tilde{x}_i p'(\tilde{y})(x_i(\hat{\mathbf{w}}) - \tilde{x}_i),$$

where the last inequality follows from the fact that $p(\tilde{y}) = U_i'(\tilde{x}_i) - \tilde{x}_i p'(\tilde{y})$. We, therefore, conclude that:

$$U_i(x_i(\hat{\mathbf{w}})) + \tilde{x}_i p'(\tilde{y})(x_i(\hat{\mathbf{w}}) - \tilde{x}_i)$$

$$> U_i(\tilde{x}_i) + U_i'(\tilde{x}_i)(x_i(\hat{\mathbf{w}}) - \tilde{x}_i),$$

which is a contradiction. Indeed, by concavity of U_i, we must have: $U_i(x_i(\hat{\mathbf{w}})) \leq U_i(\tilde{x}_i) + U_i'(\tilde{x}_i)(x_i(\hat{\mathbf{w}}) - \tilde{x}_i)$. However, since $\tilde{x}_i p'(\tilde{y}) + p(\tilde{y}) = U_i'(\tilde{x}_i)$ in equilibrium, it follows that $\tilde{x}_i p'(\tilde{y}) < U_i'(\tilde{x}_i)$.

Similarly, suppose that $\bar{Q}_i(\bar{x}_i, \bar{\mu}) > Q_i(x_i(\hat{\mathbf{w}}), \hat{w}_i)$. By Proposition 1 and 2, since $\bar{\mu} > p(y)$, for $x_i(\hat{\mathbf{w}}) > 0$ and $\bar{x}_i > 0$:

$$U_i(\bar{x}_i) + U_i'(x_i(\hat{\mathbf{w}}))\beta(y(\hat{\mathbf{w}}))\frac{x_i(\hat{\mathbf{w}})}{y(\hat{\mathbf{w}})}(\bar{x}_i - x_i(\hat{\mathbf{w}}))$$

$$> U_i(x_i(\hat{\mathbf{w}})) + U_i'(x_i(\hat{\mathbf{w}}))(\bar{x}_i - x_i(\hat{\mathbf{w}})).$$

By the concavity argument, this constitutes a contradiction as well. Therefore, $\bar{Q}_i(\bar{x}_i, \bar{\mu}) \leq Q_i(x_i(\hat{\mathbf{w}}), \hat{w}_i)$ and $Q_i(x_i(\hat{\mathbf{w}}), \hat{w}_i) \leq \tilde{Q}_i(\tilde{\mathbf{x}})$, which completes the proof. ■

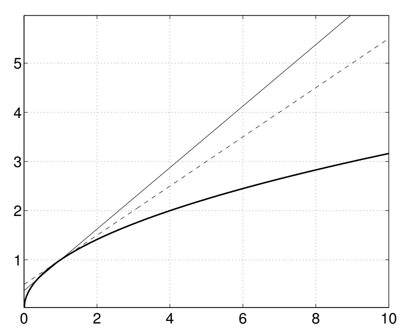

Fig. 2. Proof of Corollary 3. Solid line is a tangent to $U_i(x_i)$ at $x_i = \tilde{x}_i$. Dashed line represents the function $\alpha(x_i) = U_i(x_i) + \lambda(x_i - \tilde{x}_i)$, where $\lambda < U_i'(\tilde{x}_i)$.

Corollary 3 provides a ranking of the uniform-pricing schemes for three different settings. From the viewpoint of resource manager the best scenario is the one where agents act as price-takers and competitive equilibrium is achieved; resulting allocations are then Pareto-optimal and market-clearing price is equal to marginal cost of production, which maximizes manager's profit. Unfortunately, the manager cannot guarantee that the price-taking assumption will hold. In fact, Corollary 3 shows that there is a rationale for agents to view the resource allocation problem as a game with payoffs defined by (5) or (17). Price-anticipating bidding, resulting in demand reduction, is a way to improve profits and therefore is a reasonable strategy. Thus, from the perspective of the agents the most desirable setting is the one described by the Cournot competition model leading to the highest individual payoffs. However, it is the mechanism designer, not agents, that defines the model by the rules of resource allocation. Corollary 2 demonstrates that the potential inefficiency of outcomes makes the Cournot setting the most undesirable one from the resource manager viewpoint. As a result, agents should not expect from the rational resource manager to apply it. In the considered setting of uniform-pricing scheme designer's best response to the agents' expected price-anticipating behavior is to apply the solution where agents report their willingness to pay as it reduces the adverse effects of agents' misrepresentation of preferences.

VI. Bidding algorithm

To finalize the paper we will now investigate the stability of Nash equilibria and the convergence conditions of bidding process in the uniform-price share auction with price-anticipating agents reporting their willingness to pay. Auction rules correspond to the competition model defined by the market-clearing equation (6), where allocation rule satisfies (7) and payoffs are defined by (5). We focus our attention on the family of smooth price functions characterized by nondecreasing elasticity, i.e. $\beta'(y) \geq 0$.

Lemma 2: If the assumptions of Proposition 2 hold, then:

$$(p(\tilde{y}) - p(\bar{y}))\,(\bar{y} - \tilde{y}) \leq 0, \tag{24}$$

where $\tilde{y} = \sum_{i\in\mathcal{L}} x_i(\tilde{\mathbf{w}})$ and $\bar{y} = \sum_{i\in\mathcal{L}} x_i(\bar{\mathbf{w}})$.

Proof: Suppose that $p(\tilde{y}) \leq p(\bar{y})$. Then, by assumptions II-B and III-A, $\tilde{y} \leq \bar{y}$. This implies that $p(\tilde{y})\tilde{y} \leq p(\tilde{y})\bar{y}$ and $p(\bar{y})\tilde{y} \leq p(\bar{y})\bar{y}$. Subtracting the first inequality from the latter yields (24). The same follows for $p(\tilde{y}) \geq p(\bar{y})$. Finally, (24) follows from the fact that p is is strictly increasing, i.e. $-(p(\bar{y}) - p(\tilde{y}))/(\bar{y} - \tilde{y}) < 0$. ∎

Proposition 4: Let $\mathcal{F}(\mathbf{w}(t)) \equiv p(y(\mathbf{w}(t)))\hat{\mathbf{x}}(t) - \mathbf{w}(t)$, where $\hat{x}_i(t) \in \{[x_i]^+ \colon\ U_i'(x_i)(1 - \beta(y(\mathbf{w}(t)))x_i/y(\mathbf{w}(t))) = p(y(\mathbf{w}(t)))\}$ for every $i \in \mathcal{L}$. If the assumptions of Proposition 2 hold, then every Nash equilibrium of the game with payoffs defined by (5) is an asymptotically stable fixed point of the system:

$$\dot{\mathbf{w}}(t) = \mathcal{F}(\mathbf{w}(t)). \tag{25}$$

Proof: We will show that $p(y(\mathbf{w}(t)))$ converges to the equilibrium price $\hat{p}$ when agents modify their bids according to $\mathcal{F}(\mathbf{w}(t))$. Since for $\mathbf{w} \neq 0$:

$$\frac{\partial y(\mathbf{w})}{\partial w_i} = \left(p(y(\mathbf{w})) + y(\mathbf{w})\frac{\partial p(y(\mathbf{w}))}{\partial w_i}\right)^{-1},$$

we conclude that $\partial y(\mathbf{w})/\partial w_i \leq p(y(\mathbf{w}))^{-1}$. This implies that:

$$\dot{p}(y(\mathbf{w}(t))) = p'(y(\mathbf{w}(t)))\sum_{i\in\mathcal{L}}\frac{\partial y(\mathbf{w}(t))}{\partial w_i}\dot{w}_i(t)$$

$$\leq p'(y(\mathbf{w}(t)))\sum_{i\in\mathcal{L}}\frac{p(y(\mathbf{w}(t)))\hat{x}_i(t) - w_i(t)}{p(y(\mathbf{w}(t)))}$$

$$= p'(y(\mathbf{w}(t)))\sum_{i\in\mathcal{L}}(\hat{x}_i(t) - x_i(\mathbf{w}(t))).$$

Now, define $V(t) \equiv (p(y(\mathbf{w}(t))) - \hat{p})^2/2$ and observe that:

$$\dot{V}(t) = (p(y(\mathbf{w}(t))) - \hat{p})\dot{p}(y(\mathbf{w}(t))) \leq$$

$$p'(y(\mathbf{w}(t)))(p(y(\mathbf{w}(t))) - \hat{p})(\hat{y} - y(\mathbf{w}(t))),$$

for $\hat{y} \equiv \sum_{i\in\mathcal{L}}\hat{x}_i(t)$ and $y(\mathbf{w}(t)) \equiv \sum_{i\in\mathcal{L}} x_i(\mathbf{w}(t)))$. Since $\hat{y}$ satisfies the equilibrium condition $\hat{y}\hat{p} = \sum_{i\in\mathcal{L}}\hat{w}_i$, from Lemma 2 it follows that $\dot{V}(t) \leq 0$, with equality only for $p(y(\mathbf{w}(t))) = \hat{p}$ defining a unique stationary point of $\mathcal{F}(\mathbf{w}(t))$. Thus, V is a Lyapunov function for system $\mathcal{F}(\mathbf{w}(t))$. Under assumptions of the proposition this implies that $\mathbf{w}(t)$ converges to $\hat{\mathbf{w}}$ for $\mathbf{w}(0) = \mathbf{w}_0 \neq 0$. ∎

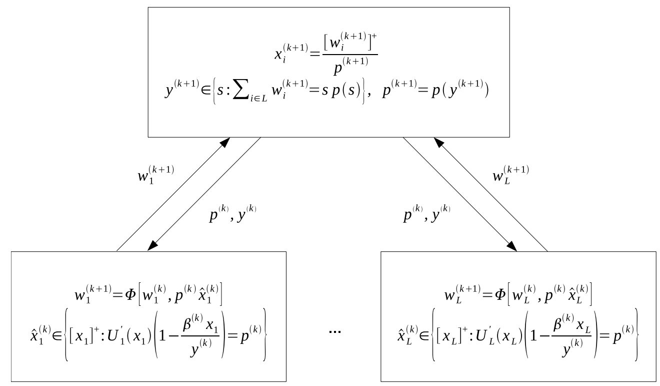

Fig. 3. Communication scheme of mechanism (26)-(29).

Proposition 4 proves the convergence of a bidding process in the uniform-price share auction under the assumptions of Proposition 2. Figure VI explains the procedure. Resource manager begins by setting an initial value of the price $p^{(0)} > 0$ and the corresponding total production level $y^{(0)}$. These two values are then communicated to all bidders eligible to participate in the auction. In response to the received signals each price-anticipating bidder calculates a profit-maximizing consumption level $\hat{x}_i$ and translates the value to the signal required by auction rules:

$$w_i^{(k+1)} = \Phi\left[w_i^{(k)}, p^{(k)}\hat{x}_i^{(k)}\right], \tag{26}$$

where $\Phi(a, b) \equiv a + \rho(b - a)$ with $\rho \in (0, 1)$ and:

$$\hat{x}_i^{(k)} \in \left\{[x_i]^+ \colon U_i'(x_i)\left(1 - \frac{\beta^{(k)}x_i}{y^{(k)}}\right) = p^{(k)}\right\}. \tag{27}$$

It is assumed that each agent i can approximate the price elasticity measure $\beta^{(k)}$ (corresponding to the total resource utilization $y^{(k)}$) to calculate $\hat{x}_i^{(k)}$. The amount of resource $x_i^{(k)}$ assigned to the agent i in the iteration $k = 1, .., T-1$ can be either communicated to each agent or calculated by each agent individually: $x_i^{(k)} = w_i^{(k)}/p^{(k)}$.

Next, when optimized bids are set, the vector $(w_1^{(k+1)}, ..., w_L^{(k+1)})$ is submitted to the manager to clear the market, i.e.:

$$x_i^{(k+1)} = \frac{[w_i^{(k+1)}]^+}{p^{(k+1)}}, \tag{28}$$

where $p^{(k+1)} = p(y^{(k+1)})$ and:

$$y^{(k+1)} \in \left\{s \colon \sum_{i\in\mathcal{L}} w_i^{(k+1)} = sp(s)\right\}. \tag{29}$$

Having calculated the price and the allocations, resource manager sends the values to the agents and waits for their response. The bidding procedure stops when predefined stopping conditions are satisfied. Final outcomes of the auction are communicated to all participants.

Example 1: Consider a divisible resource characterized by the following price function $p(y) = (C-y)^{-1}$ with elasticity $\varepsilon(y) = y(C-y)$. Suppose that there are $L = 4$ agents competing for the resource and let $U_1(x_1) = \gamma_1 x_1$, $U_2(x_2) = \gamma_2 \log(1+x_2)$, $U_3(x_3) = \gamma_3 \arctan(x_3)$ and $U_4(x_4) = \gamma_4\sqrt{x_4}$. Figure 1 demonstrates the bidding process. Observe that allocations $x_i^{(k)}$ decrease as price $p^{(k)}$ increases with $k \to T = 20$. △

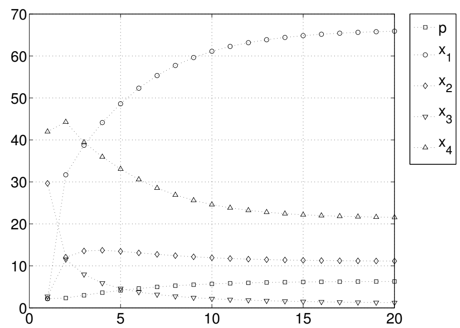

Fig. 4. Convergence of the uniform-price mechanism (26)-(29).

The sequences of bids $\{\mathbf{w}^{(k)}\}_{k=0}^{\infty}$, allocations $\{\mathbf{x}^{(k)}\}_{k=0}^{\infty}$ and prices $\{p^{(k)}\}_{k=0}^{\infty}$ are proved to converge to a Nash equilibrium point if the conditions of Proposition 4 are met. It is important to notice that the proof of convergence relates on the property (24), which can be interpreted as the law of demand—agents' aggregate demand y and resource price move in opposite directions [12]. Furthermore, the allocations are entirely determined by a pointwise characterization of agents' preferences. Messages reported to the mechanism contain *scalar* values, which reveal enough information to calculate the resource assignments and the corresponding payments. This is an extremely desirable property of a mechanism, especially in the network environments of large scale systems with communication constraints and limited computational power.

On the other hand, the model assumes synchronous communication pattern, i.e. allocations are determined based on signals received from all agents within a predefined time period. This is a restrictive constraint which should be relaxed in many real-life settings—asynchronous communication should be admitted. We have also considered all agents' continuous dynamics evolving at the same rate, which may be a simplified and potentially harmful assumption as well; see Shenker [14] for an interesting comment. Furthermore, the model ignores the problem of agents dropping off when payoff values become negative (individual rationality constraint). These and other issues related to the best response dynamics are the subject of further investigations.

VII. SUMMARY

The above discussion illustrates the consequences of incompatibility of the overall system goals and the individual objectives of the agents—the uniform-price market-clearing mechanism creates incentives to reveal to the system a reduced value of demand. On one hand, this improves agents' profits, but on the other it gives rise to allocations that do not maximize utilization of the resource in the system. Theoretical results established in the paper can therefore serve as a guideline for limiting the adverse effects of strategic behavioral patterns within the uniform-pricing framework—it relates to the choice of strategies agents are allowed to play. For other guidelines, resulting from the exploration of the relationship between conditions (12) and (16), see [15], [16], [17]. Since both the allocations and the payments are arguments of the payoff functions, the structure of the mechanism's rules determines the optimality conditions a rational agent will try to achieve. As a result, an appropriate choice of pricing functions and allocation rules can make both the resource manager and agents approach the similar goals.

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