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On the convergence time of distributed quantized averaging algorithms

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Abstract— **We come up with novel quantized averaging algorithms on synchronous and asynchronous communication networks with fixed, switching and random topologies. The implementation of these algorithms is subject to the realistic constraint that the communication rate, the memory capacities of agents and the computation precision are finite. The focus of this paper is on the study of the convergence time of the proposed quantized averaging algorithms. By appealing to random walks on graphs, we derive polynomial bounds on the expected convergence time of all the algorithms presented.**

I. Introduction

Consider a network of N (mobile or immobile) agents. The distributed averaging problem aims to design an algorithm that the agents can utilize to asymptotically reach an agreement by communicating with nearest neighbors. The consensus value is the average of individual initial states.

In real communication networks, the capacities of communication channels and the memory capacities of agents are finite. Furthermore, the computations can only be carried out with finite precision. From a practical point of view, real-valued consensus algorithms are not feasible and these realistic constraints motivate the problem of average consensus via quantized information. Another motivation for distributed quantized averaging is load balancing with indivisible tasks [12]. Prior work on distributed quantized averaging includes [1], [7], [8], [12], [15], [16], on fixed communication graphs and [19] for switching communication graphs. The authors in [15] propose a class of discretized averaging algorithms (quantized gossip algorithms) with a convergence property on fixed graphs. Very recently, [19] studies quantization effects on distributed averaging algorithms over time-varying topologies. As in [15], we focus on quantized averaging algorithms which preserve the sum of the state values at each iteration. This setup has the following properties of interest: the sum cannot be changed in some situations, such as in load balancing [10], [12]; and the constant sum leads to a small steady-state error with respect to the average of individual initial states. This error is equal to either one quantization step size or zero (when the average of the initial states is one of the quantization levels) and thus is independent of N. This is in contrast to the setup in [19] where the sum of the states is not preserved, resulting in a steady-state error of the order $O(N^3 \log N)$. However, high-precision averaging algorithms are essential to many distributed tasks, including distributed estimation[17] and sensor fusion [21] over networks.

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The convergence time is typically utilized to quantify the performance of averaging algorithms. The authors in [4], [5], [22] study the convergence time of real-valued averaging, while the case of quantized averaging is discussed in [15], [19]. The bounds of the expected convergence time on fixed complete and linear graphs are derived in [15]. Very recently, the authors in [19] give a polynomial bound on the convergence time of a class of quantized averaging algorithms over switching topologies.

Organization and Statement of contributions. We now outline the reminder of the paper, and report the main contributions. This paper builds on the results of [15] and proposes novel quantized averaging algorithms on synchronous and asynchronous communication networks with fixed, switching and random topologies. The algorithms can be implemented under the constraint that the capacities of communication channels, the memory capacities of agents and the computation precision are finite. We utilize random walks on graphs to derive polynomial bounds on the expected convergence time of all the algorithms proposed. To the best of our knowledge, this paper is the first step toward the treatment of asynchronous quantized averaging, quantized averaging on random graphs, and random walks on time-varying graphs.

In Section II, we present the problem formulation studied in this paper along with some notation and terminology. In Section III, we study quantized averaging on synchronous communication networks. In Section III-A, we propose a synchronous quantized averaging algorithm on fixed and connected graphs. This case is discussed in [15], and our algorithm improves that of [15] in the sense that we remove the non-local constraint $\sum_{(i,j)\in E} \bar{p}_{ij}(t) = 1$, where $\bar{p}_{ij}(t) > 0$ is the probability that the edge $(i,j) \in E$ is chosen at time $t \geq 0$. In this way, it is unnecessary to have global knowledge of the number of edges in the graph and the algorithm becomes completely distributed. We also derive (1) bounds on the expected convergence time for the cases of linear and complete graphs (improving those in [15] by an order of magnitude in terms of N), and (2) a general polynomial bound $O(N^4)$ that is valid for any fixed graph.

In Section III-B, we propose a synchronous quantized averaging algorithm, and show that this algorithm converges to quantized average consensus over switching graphs which are periodically connected with period B. We provide a polynomial bound $O(BN^5 \log N)$ on the expected convergence time. This case is studied in [19], and our results are different from [19] in the following aspects: (1) we relax the constraint in [19] that the computations are carried out with continuous values, and the computation precision we need is half of one quantization step size; (2) the steady-state error of our algorithm is independent of N and at most one quantization step size as opposed to a polynomial with the order $O(N^3 \log N)$ in [19].

In Section IV, we turn our attention to asynchronous quantized averaging algorithms where we adopt the asynchronous time model in [4]. In Section IV-A, an asynchronous algorithm is proposed to achieve quantized average consensus on fixed and connected topologies. In terms of the number of clock ticks, we obtain a bound $O(N^5)$ on the expected convergence time. In Section IV-B, we provide an asynchronous quantized averaging algorithm for periodically connected graphs with period B. We show this algorithm to be convergent and have an upper bound $O(BN^7 \log N)$ in terms of the number of clock ticks.

In Section V, we briefly discuss the case of quantized averaging on random graph $\mathcal{G}(N,p)$ in synchronous and asynchronous settings. Finally, we provide simulation results for a particular example in Section VI.

II. PROBLEM STATEMENT AND PRELIMINARIES

We will consider a network of N nodes, labeled 1 through N. The state of node i at time t is denoted by $x_i(t) \in \mathbb{R}$ and the network state is denoted by $x(t) = (x_1(t), \cdots, x_N(t))^T \in \mathbb{R}^N$. Suppose $x_i(t) \in [U_{min}, U_{max}]$ for $i \in \{1, \cdots, N\}$ and all $t \geq 0$. Let R denote the number of bits per sample. The total number of quantization levels can be represented by $L = 2^R$ and the step size is $\Delta = (U_{max} - U_{min})/L = (U_{max} - U_{min})/2^R$. The quantization levels, $\{\omega_1, \cdots, \omega_L\}$, are uniformly spaced in the sense that $\omega_{i+1} - \omega_i = \Delta$ for $i \in \{1, \cdots, L-1\}$; i.e., the quantization levels are consecutive multiples of Δ. A quantizer $\mathcal{Q} : [U_{min}, U_{max}] \to \{\omega_1, \cdots, \omega_L\}$ is adopted to quantize the message $u \in [U_{min}, U_{max}]$ in such a way that $\mathcal{Q}(u) = \omega_i$ if $u \in [w_i, w_{i+1})$. We assume that the initial state $x_i(0)$ is a multiple of Δ.

Problem statement. In this paper, the problem of interest is to design distributed averaging algorithms where the nodes update their states by communicating with neighbors via quantized messages in a synchronous or asynchronous setting. Ultimately, quantized average consensus is reached in probability; i.e., for any initial state $x(0)$, there holds that $\lim_{t\to\infty} \mathbb{P}(x(t) \in \mathcal{W}) = 1$. The set $\mathcal{W}$ is dependent on the initial state $x(0)$ and defined as follows. If $\bar{x}(0) = \frac{1}{N}\sum_{i=1}^N x_i(0)$ is not a multiple of Δ, then $\mathcal{W} = \{x \in \mathbb{R}^N \mid x_i \in \{\mathcal{Q}(\bar{x}(0)), \mathcal{Q}(\bar{x}(0)) + \Delta\}\}$. If $\bar{x}(0)$ is a multiple of Δ, then $\mathcal{W} = \{x \in \mathbb{R}^N \mid x_i = \bar{x}(0)\}$. Now it is clear that the steady-state error with respect to $\bar{x}(0)$ is at most Δ after quantized average consensus is reached.

In the sequel, we introduce the notation and terminology of random walks that will be used along the paper. We will employ the undirected graph $\mathcal{G}(t) = (V, E(t))$ to model the network. Here $V = \{1, \cdots, N\}$ is the vertex set, and an edge $(j,i) \in E(t)$ if and only if node j can receive the message from node i (e.g., node j is within the communication range of node i) at time t. The neighbors of node i at time t are denoted by $\mathcal{N}_i(t) = \{j \in V \mid (j,i) \in E(t)$ and $j \neq i\}$.

For $\alpha \in \mathbb{R}$, define $V_\alpha : \mathbb{R}^N \to \mathbb{R}$ as $V_\alpha(x) = \sum_{i=1}^N (x_i - \alpha)^2$. Denote $J = (\max_{i\in V} x_i(0) - \min_{i\in V} x_i(0))/\Delta$. The vector e_i is the i^{th} column of the identity matrix $I_{N\times N}$. Denote Θ^c as the complement of the set $\Theta = \{(k,k) \mid k \in V\}$. We define the *distribution* of a vector $x \in \mathbb{R}^N$ to be the list $\{(q_1,m_1),(q_2,m_2),\cdots,(q_k,m_k)\}$ for some $k \in V$ where $\sum_{\ell=1}^k m_\ell = N$, $q_i \neq q_j$ for $i \neq j$ and m_ℓ is the cardinality of the set $\{i \in V \mid x_i = q_\ell\}$. In this paper, random walks on graphs play an important role in characterizing the convergence properties of our quantized averaging algorithms. The following two definitions are generalized from those defined for fixed graphs in [6] and [9].

Definition 2.1 (Natural and simple random walks): A natural random walk on the graph $\mathcal{G}(t)$ under the transition matrix $P(t) = (p_{ij}(t)) \in \mathbb{R}^{N\times N}$, starting from node v at time s, is a stochastic process $\{X(t)\}_{t\geq s}$ with the state space V such that $X(s) = v$ and $\mathbb{P}(X(t+1) = j \mid X(t) = i) = p_{ij}(t)$. A natural random walk is said to be simple if for any $i \in V$, $p_{ii}(t) = 0$ for all t. •

The hitting time and meeting time are two important notions for random walks on graphs.

Definition 2.2 (Hitting time): Consider a random walk on the graph $\mathcal{G}(t)$, beginning from node i at time s and evolving under the transition matrix $P(t)$. The hitting time from node i to the set Θ, denoted as $H_{(\mathcal{G}(t),P(t),s)}(i,\Theta)$, is the expected time it takes this random walk to reach the set Θ for the first time. We denote $H_{(\mathcal{G}(t),P(t))}(\Theta) = \sup_{s\geq 0}\max_{i\in V} H_{(\mathcal{G}(t),P(t),s)}(i,\Theta)$ as the hitting time to reach the set Θ. The hitting time of the pair $i,j \in V$, denoted as $H_{(\mathcal{G}(t),P(t),s)}(i,j)$, is the expected time it takes this random walk to reach node j for the first time. We denote $H_{(\mathcal{G}(t),P(t))} = \sup_{s\geq 0}\max_{i,j\in V} H_{(\mathcal{G}(t),P(t),s)}(i,j)$ as the hitting time of going between any pair of nodes. •

Definition 2.3 (Meeting time): Consider two random walks on the graph $\mathcal{G}(t)$ under the transition matrix $P(t)$, starting at time s from node i and node j respectively. The meeting time $M_{(\mathcal{G}(t),P(t),s)}(i,j)$ of these two random walks is the expected time it takes them to meet at some node for the first time. The meeting time of the graph $\mathcal{G}(t)$ is defined as $M_{(\mathcal{G}(t),P(t))} = \sup_{s\geq 0}\max_{i,j\in V} M_{(\mathcal{G}(t),P(t),s)}(i,j)$. •

For the ease of notation, we will drop the subscript s in the hitting time and meeting time notions for fixed graphs.

III. SYNCHRONOUS QUANTIZED AVERAGING

A. Synchronous quantized averaging on fixed graphs

In this section, we propose an algorithm which can achieve quantized average consensus on a fixed and connected topology. The algorithm is related to that in [15], and to characterize its expected convergence time, we use the upper bound on the hitting time for simple random walks on fixed and connected graphs in [6].

The *synchronous quantized averaging algorithm on a fixed and connected graph* $\mathcal{G}$ (SF, for short) can be described as follows. Initially, a token is assigned to a node in the network. This token will serve to determine which node becomes active. At time t, the active node, say node i, randomly chooses one of its neighbors, say node j, with equal probability. Node i and j then execute the following local computation. If $x_i(t) \geq x_j(t)$, then

$$x_i(t+1) = x_i(t) - \delta, \quad x_j(t+1) = x_j(t) + \delta; \quad (1)$$

otherwise,

$$x_i(t+1) = x_i(t) + \delta, \quad x_j(t+1) = x_j(t) - \delta, \quad (2)$$

where $\delta = \frac{1}{2}|x_i(t) - x_j(t)|$ if $\frac{|x_i(t)-x_j(t)|}{2\Delta}$ is an integer; otherwise, $\delta = \mathcal{Q}(\frac{1}{2}|x_i(t) - x_j(t)|) + \Delta$. Simultaneously, node i passes the token to node j. Every other node $k \in V \setminus \{i,j\}$ preserves its current state; i.e., $x_k(t+1) = x_k(t)$.

Remark 3.1: In the context of load balancing, the quantity δ represents the load which is transferred from a heavy-loaded node to a light-loaded one. The precision $\frac{\Delta}{2}$ is sufficient for the computations of δ and thus the update laws (1) and (2). The sum of the state values is preserved at each iteration, and $x_i(t)$ is a multiple of Δ for all i and t.

If $x_i(t)$ and $x_j(t)$ are located at adjacent quantization levels, the update laws (1) and (2) become $x_i(t+1) = x_j(t)$ and $x_j(t+1) = x_i(t)$. Such update is referred to as a *trivial average*. If $|x_i(t) - x_j(t)| > \Delta$, the update of (1) or (2) is referred to as a *non-trivial average*. Although it does not directly contribute to the averaging, trivial averages help the information flow over the network. •

The convergence time of SF is a random variable defined as follows: $T_{\rm con}(x(0)) = \inf\{t \mid x(t) \in \mathcal{W}\}$, where $x(t)$ evolves under SF, starting from $x(0)$. We define $T_1(x(0))$ as the random variable of the time when the first non-trivial average occurs starting from $x(0)$.

Choose $V_{\bar{x}(0)}(x) = \sum_{i=1}^N (x_i - \bar{x}(0))^2$ as a Lyapunov function candidate for SF. One can readily see that $V_{\bar{x}(0)}(x(t+1)) = V_{\bar{x}(0)}(x(t))$ when a trivial average occurs and $V_{\bar{x}(0)}(x)$ reduces at least $2\Delta^2$ when a non-trivial average occurs. Hence, $V_{\bar{x}(0)}(x)$ is non-increasing along the trajectories starting from $x(0)$, and the number of non-trivial averages is at most $\frac{1}{2\Delta^2}V_{\bar{x}(0)}(x(0))$. Define $\Psi = \{x \in \mathbb{R}^N \mid \text{the distribution of } x \text{ is } \{(0,1),(\Delta, N-2),(2\Delta,1)\}\}$ and denote $\mathbb{E}[T_\Psi] = \max_{x(0)\in\Psi} \mathbb{E}[T_{\rm con}(x(0))]$. It is clear that $\mathbb{E}[T_1(x(0))] \leq \mathbb{E}[T_\Psi]$ and we have the following estimates on $\mathbb{E}[T_{\rm con}(x(0))]$:

$$\begin{aligned}\mathbb{E}[T_{\rm con}(x(0))] &\leq \frac{1}{2\Delta^2}V_{\bar{x}(0)}(x(0))\mathbb{E}[T_1(x(0))] \\ &\leq \frac{1}{2\Delta^2}V_{\bar{x}(0)}(x(0))\mathbb{E}[T_\Psi] \leq \frac{NJ^2}{8}\mathbb{E}[T_\Psi], \quad (3)\end{aligned}$$

where the third inequality uses Lemma 4 in [15].

Lemma 3.1: For any fixed connected graph $\mathcal{G}$, $\mathbb{E}[T_{\rm con}(x(0))]$ of SF is upper bounded by $\frac{1}{27}J^2N^4$.

Proof: Due to the space limitations, we omit the proof. ■

Theorem 3.1: Let $x(0) \in \mathbb{R}^N$ and suppose $x(0) \notin \mathcal{W}$. Under SF, any evolution $x(t)$ starting from $x(0)$ reaches quantized average consensus.

Proof: Denote $\tilde{T} = \frac{2}{27}J^2N^4$, and consider the first $\tilde{T}$ time units of evolution of SF starting from $x(0)$. It follows from Markov's inequality that

$$\mathbb{P}(T_{\rm con}(x(0)) > \tilde{T} | x(0) \notin \mathcal{W}) \leq \frac{\mathbb{E}[T_{\rm con}(x(0))]}{\tilde{T}} \leq \frac{1}{2},$$

that is, the probability that after $\tilde{T}$ time units SF has not reached quantized average consensus is less than $\frac{1}{2}$. Starting from $x(\tilde{T})$, let us consider the posterior evolution of $x(t)$ in the next $\tilde{T}$ time units. We have

$$\mathbb{P}(T_{\rm con}(x(\tilde{T})) > \tilde{T} | x(\tilde{T}) \notin \mathcal{W}) \leq \frac{\mathbb{E}[T_{\rm con}(x(\tilde{T}))]}{\tilde{T}} \leq \frac{1}{2}.$$

That is, the probability that after $2\tilde{T}$ time units $x(t)$ has not reached quantized average consensus is at most $(\frac{1}{2})^2$. By induction, it follows that after $n\tilde{T}$ time units the probability $x(t)$ not reaching quantized average consensus is at most $(\frac{1}{2})^n$. Letting $n \to \infty$, we obtain $\lim_{t\to\infty} \mathbb{P}(x(t) \notin \mathcal{W}) = 0$ and this completes the proof. ■

The bound obtained in Lemma 3.1 is valid for any fixed and connected graph. If we restrict our attention to some specific graphs (e.g., the complete graph and the linear network in [15]), a tighter bound can be obtained by using the properties of each graph. Denote the complete graph with N nodes as $\mathcal{G}_{\rm com} = (V, E_{\rm com})$ with $E_{\rm com} = \{(i,j) \mid i \neq j\}$. The linear network with N nodes is denoted as $\mathcal{G}_{\rm lin} = (V, E_{\rm lin})$ with $E_{\rm lin} = \{(i,j) \mid |i-j| = 1\}$.

Lemma 3.2 (Convergence over the complete graph): Suppose that SF be implemented on $\mathcal{G}_{\rm com}$. We have that $\mathbb{E}[T_{\rm con}(x(0))]$ of SF is upper bounded by $\frac{1}{4}N(N-1)J^2$.

Proof: The proof is omitted due to the space limit. ■

Lemma 3.3 (Convergence over the linear network): Suppose that SF be implemented on $\mathcal{G}_{\rm lin}$. We have that $\mathbb{E}[T_{\rm con}(x(0))]$ of SF is upper bounded by $\frac{1}{4}N(N-1)^2J^2$.

Proof: Due to the space limit, we omit the proof. ■

B. Synchronous quantized averaging on switching graphs

This section introduces a *synchronous quantized averaging algorithm for switching graphs* (SS, for short). The convergence rate of real-valued averaging algorithms on switching graphs in [19] will be applied to characterize the hitting time of random walks on switching graphs.

The main steps of SS can be summarized as follows. Initially, a token is assigned to a node in the network. This token will serve to determine which node becomes active. Assume that node i be active at time t. If $|\mathcal{N}_i(t)| \neq 0$, node i randomly chooses one of its neighbors, say node j, with probability $p_{ij}(t) = \frac{1}{2\max\{|\mathcal{N}_i(t)|,|\mathcal{N}_j(t)|\}}$. Then node i and j execute the computation (1) or (2); and simultaneously, node i passes the token to node j. Every other node $k \in V \setminus \{i,j\}$ preserves its current state. If $|\mathcal{N}_i(t)| = 0$, all nodes preserve their current states.

In what follows, we assume that the communication graph satisfies the following connectivity assumption also used in [2], [14], [19], [22].

Assumption 3.1 (Periodical connectivity): There exists some $B \in \mathbb{N}_{>0}$ such that, for all $t \geq 0$, the undirected graph $(V, E(t) \cup E(t+1) \cup \cdots \cup E(t+B-1))$ is connected.

The movement of the token on $\mathcal{G}(t)$ under SS is a natural random walk. This is in contrast to the simple random walk describing the token's motion on fixed graphs under SF. The natural random walk has an associated transition matrix $P_{\rm SS}(t)$ defined as follows. If $|\mathcal{N}_i(t)| \neq 0$, then $p_{ij}(t) = \frac{1}{2\max\{|\mathcal{N}_i(t)|,|\mathcal{N}_j(t)|\}}$ for $(i,j) \in E(t)$, and $p_{ii}(t) = 1 - \sum_{j\in\mathcal{N}_i(t)} p_{ij}(t) \geq \frac{1}{2}$; otherwise, $p_{ii}(t) = 1$.

The following lemma considers two random walks: X with transition matrix $P_{\rm SS}(t)$, and X_M with a single absorbing state j and transition matrix $\bar{P}_{\rm SS}(t)$ obtained by replacing the j^{th} row of $P_{\rm SS}(t)$ with e_j^T. Define $\vartheta_i(t) = \mathbb{P}(X(t) = i)$, $\bar{\vartheta}_i(t) = \mathbb{P}(X_M(t) = i)$, and $\vartheta(t) = (\vartheta_1(t), \cdots, \vartheta_N(t))^T \in \mathbb{R}^N$, $\bar{\vartheta}(t) = (\bar{\vartheta}_1(t), \cdots, \bar{\vartheta}_N(t))^T \in \mathbb{R}^N$.

Lemma 3.4: Consider a network of N nodes whose communication graph $\mathcal{G}(t)$ satisfies Assumption 3.1. Let $i \in V$ be a given node and suppose that X and X_M start from node i at time 0. Then for any $j \neq i$, we have $\bar{\vartheta}_j(t) \geq \frac{1}{2N}$ for $t \geq t_1$ where $t_1 = B(8N^2(N-1)\log(2N) + 1)$.

Proof: We omit the proof due to the space limit. ■

Define the quantities $T_{\rm con}(x(0))$ and T_Ψ for SS in a similar way to those for SF in Section III-A.

Theorem 3.2: Suppose the communication graph $\mathcal{G}(t)$ satisfies Assumption 3.1. Let $x(0) \in \mathbb{R}^N$ and suppose $x(0) \notin \mathcal{W}$. Under SS, any evolution $x(t)$ starting from $x(0)$ reaches quantized average consensus. Furthermore, the expected convergence time $\mathbb{E}[T_{\rm con}(x(0))]$ of SS is upper bounded by $BJ^2N^2(8N^2(N-1)\log(2N) + 1)$.

Proof: Keep in mind that the movement of the token is a natural random walk on $\mathcal{G}(t)$ under transition matrix $P_{\rm SS}(t)$ defined in Lemma 3.4. Similarly to Lemma 3.1, we have $\mathbb{E}[T_\Psi(x(0))] < 2H_{(\mathcal{G}(t),P_{\rm SS}(t))}$. Since the inequality (3) also works for SS, it holds that

$$\mathbb{E}[T_{\rm con}(x(0))] \leq \frac{1}{4}NJ^2 H_{(\mathcal{G}(t),P_{\rm SS}(t))}.$$

We first need to find an upper bound on $H_{(\mathcal{G}(t),P_{\rm SS}(t))}$. To do this, construct random walk $X_M^{[i]}$ in such a way that $X_M^{[i]}$ starts from node i at time 0 and the state j ($j \neq i$) is the single absorbing state of $X_M^{[i]}$. The transition matrix of $X_M^{[i]}$ is $\bar{P}_{\rm SS}(t)$ defined in Lemma 3.4. Define $\vartheta_k^{[i]}(t) = \mathbb{P}(X_M^{[i]}(t) = k)$, and $\vartheta^{[i]}(t) = (\vartheta_1^{[i]}(t), \cdots, \vartheta_N^{[i]}(t))^T \in \mathbb{R}^N$. The dynamics of $\vartheta^{[i]}(t)$ is given by $\vartheta^{[i]}(t+1) = \bar{P}_{\rm SS}^T(t)\vartheta^{[i]}(t)$, with the initial state $\vartheta^{[i]}(0) = e_i$.

Define the function $\mu_\ell^{[i]} : \mathbb{N}_0 \to \{0,1\}$ in such a way that $\mu_\ell^{[i]}(t) = 1$ if $X_M^{[i]}(t) = \ell$ and $\mu_\ell^{[i]}(t) = 0$ if $X_M^{[i]}(t) \neq \ell$. Define $n_\ell^{[i]} = \sum_{\tau=0}^{+\infty}\mu_\ell^{[i]}(\tau)$ which is the total times that $X_M^{[i]}$ is at node ℓ. Hence, the hitting time $H_{(\mathcal{G}(t),P_{\rm SS}(t),0)}(i,j)$ of $X_M^{[i]}$ equals the expected time before $X_M^{[i]}$ reaching the absorbing state j for the first time, that is,

$$\begin{aligned} H_{(\mathcal{G}(t),P_{\rm SS}(t),0)}(i,j) &= \sum_{\ell\neq j}\mathbb{E}[n_\ell^{[i]}] = \sum_{\ell\neq j}\mathbb{E}[\sum_{\tau=0}^{+\infty}\mu_\ell^{[i]}(\tau)] \\ &= \sum_{\ell\neq j}\sum_{\tau=0}^{+\infty}\mathbb{E}[\mu_\ell^{[i]}(\tau)] = \sum_{\tau=0}^{+\infty}\sum_{\ell\neq j}\vartheta_\ell^{[i]}(\tau). \end{aligned} \quad (4)$$

It follows from Lemma 3.4 that $\vartheta_j^{[i]}(t) \geq \frac{1}{2N}$ for $t \geq t_1$. Since $\sum_{\ell\in V}\vartheta_\ell^{[i]}(t) = 1$ for any $t \geq 0$, we have

$$\sum_{\ell\neq j}\vartheta_\ell^{[i]}(t_1) \leq 1 - \frac{1}{2N}. \quad (5)$$

For each $k \neq j$, construct random walk $\tilde{X}_M^{[k]}$ in such a way that $\tilde{X}_M^{[k]}$ starts from node k at time t_1 and the state j is the single absorbing state of $\tilde{X}_M^{[k]}$. The transition matrix of $\tilde{X}_M^{[k]}$ is $\bar{P}_{\rm SS}(t)$. Define $\tilde{\vartheta}_\ell^{[k]}(t) = \mathbb{P}(\tilde{X}_M^{[k]}(t) = \ell)$, and $\tilde{\vartheta}^{[k]}(t) = (\tilde{\vartheta}_1^{[k]}(t), \cdots, \tilde{\vartheta}_N^{[k]}(t))^T \in \mathbb{R}^N$. Following the same arguments above for $X_M^{[i]}$, we have

$$\sum_{\ell\neq j}\tilde{\vartheta}_\ell^{[k]}(2t_1) \leq 1 - \frac{1}{2N}. \quad (6)$$

Combining (5) and (6) gives that

$$\begin{aligned} \sum_{\ell\neq j}\vartheta_\ell^{[i]}(2t_1) &= \sum_{\ell\neq j}\sum_{k\neq j}\vartheta_k^{[i]}(t_1)\tilde{\vartheta}_\ell^{[k]}(2t_1) \\ &= \sum_{k\neq j}\vartheta_k^{[i]}(t_1)\sum_{\ell\neq j}\tilde{\vartheta}_\ell^{[k]}(2t_1) \leq (1 - \frac{1}{2N})^2. \end{aligned} \quad (7)$$

Repeatedly applying the arguments for (7) yields $\sum_{\ell\neq j}\vartheta_\ell^{[i]}(nt_1) \leq (1 - \frac{1}{2N})^n$ and we obtain a strictly decreasing sequence $\sum_{\ell\neq j}\vartheta_\ell^{[i]}(nt_1)$ with respect to $n \in \mathbb{N}_0$.

Since node j is the single absorbing state, then $\sum_{\ell\neq j}\vartheta_\ell^{[i]}(t)$ is non-increasing with respect to $t \in \mathbb{N}_0$.

Combining the strictly decreasing property of the sequence $\sum_{\ell\neq j}\vartheta_\ell^{[i]}(nt_1)$ and the non-increasing property of $\sum_{\ell\neq j}\vartheta_\ell^{[i]}(t)$, we have the following estimate

$$\sum_{\ell\neq j}\vartheta_\ell^{[i]}(t) \leq \sum_{\ell\neq j}\vartheta_\ell^{[i]}(0)(1 - \frac{1}{2N})^{\frac{t}{t_1}-1} = (1 - \frac{1}{2N})^{\frac{t}{t_1}-1}. \quad (8)$$

Substituting (8) into (4) gives that

$$\begin{aligned} H_{(\mathcal{G}(t),P_{\rm SS}(t),0)}(i,j) &\leq \sum_{\tau=0}^{+\infty}(1 - \frac{1}{2N})^{\frac{\tau}{t_1}-1} \\ &= (1 - \frac{1}{2N})^{-\frac{1}{t_1}} \cdot \frac{1}{1 - (1 - \frac{1}{2N})^{\frac{1}{t_1}}}. \end{aligned} \quad (9)$$

Since $t_1 > 1$, it holds that $(1 - \frac{1}{2N})^{-\frac{1}{t_1}} \leq 2^{\frac{1}{t_1}} < 2$. From Bernoulli's inequality in [18] it follows that $(1 - \frac{1}{2N})^{\frac{1}{t_1}} \leq 1 - \frac{1}{2Nt_1}$, and thus $\frac{1}{1-(1-\frac{1}{2N})^{\frac{1}{t_1}}} \leq 2Nt_1$. Inequality (9) becomes

$$H_{(\mathcal{G}(t),P_{\rm SS}(t),0)}(i,j) \leq 4Nt_1. \quad (10)$$

Since the inequality (10) holds for any starting time, any starting node i and any end node j, we have $H_{(\mathcal{G}(t),P_{\mathrm{SS}}(t))} \leq 4Nt_1$ and thus

$$\mathbb{E}[T_{\mathrm{con}}(x(0))] \leq \frac{1}{4}NJ^2H_{(\mathcal{G}(t),P_{\mathrm{SS}}(t))} \leq N^2J^2t_1$$
$$= BJ^2N^2(8N^2(N-1)\log(2N)+1).$$

Since $\mathbb{E}[T_{\mathrm{con}}(x(0))]$ is bounded, the proof on the convergence property of SS is analogous to Theorem 3.1. ∎

IV. Asynchronous quantized averaging

In this section, we will employ the asynchronous time model proposed in [4] and also used in [11]. This model matches well the decentralized nature of peer-to-peer, sensor, ad hoc networks. More precisely, each node has a clock which ticks according to a rate 1 Poisson process. Hence, the inter-tick times at each node are random variables with rate 1 exponential distribution, independent across nodes and independent over time.

By the superposition theorem for Poisson processes, this set up is equivalent to a single global clock modeled as a rate N Poisson process ticking at times $\{Z_k\}_{k\geq 0}$. By the orderliness property of Poisson process, the clock ticks do not occur simultaneously. The inter-agent communication and the update of consensus states only occur at $\{Z_k\}_{k\geq 0}$. Hence, in what follows the time instant t will be discretized according to $\{Z_k\}_{k\geq 0}$ and defined in terms of the number of clock ticks; i.e., $t \in \{Z_k\}_{k\geq 0}$. The interval $[Z_k, Z_{k+1})$ corresponds to the k^{th} time-slot.

A. Asynchronous quantized averaging on fixed graphs

In this section, we propose and analyze an asynchronous quantized averaging algorithm on fixed and connected communication graph $\mathcal{G}$. Main reference for this section is [6] on the meeting time of two simple random walks.

The *asynchronous quantized averaging algorithm on a fixed connected graph* $\mathcal{G}$ (AF, for short) is described as follows. Suppose node i's clock ticks in the k^{th} time-slot. Node i randomly chooses one of its neighbors, say node j, with equal probability. This pair of nodes executes the computation (1) or (2). Every other node preserves its current state. To study the convergence of AF, we first consider the following problem which is a variation of that in [9].

The meeting time of two natural random walks on $\mathcal{G}$: Initially, two tokens are placed on $\mathcal{G}$; at each clock tick, one of the tokens is chosen with probability $\frac{1}{N}$ and the chosen token moves to one of the neighboring nodes with equal probability. What is the meeting time for these two tokens?

The tokens move as two natural random walks with transition matrix P_{AF} on graph $\mathcal{G}$. The matrix $P_{\mathrm{AF}} = (\tilde{p}_{ij}) \in \mathbb{R}^{N\times N}$ is given by $\tilde{p}_{ii} = 1 - \frac{1}{N}$ for $i \in V$, $\tilde{p}_{ij} = \frac{1}{N|\mathcal{N}_i|}$ for $(i,j) \in E$. Denote any of these two natural random walks as $X_{\mathcal{N}}$. Correspondingly, we define a simple random walk, say $X_{\mathcal{S}}$, with transition matrix P_{SF} on the graph $\mathcal{G}$ where P_{SF} is defined in Lemma 3.1.

Lemma 4.1: For the problem of the meeting time of two natural random walks on $\mathcal{G}$, there holds that $M_{(\mathcal{G},P_{\mathrm{AF}})} < 2NH_{(\mathcal{G},P_{\mathrm{SF}})} - N$.

Proof: Since $\mathcal{G}$ is connected and fixed, $X_{\mathcal{N}}$ and $X_{\mathcal{S}}$ are irreducible. The reminder of the proof is based on the following claims, and we omit the details of the proof due to the space limit.

(i) It holds that $H_{(\mathcal{G},P_{\mathrm{AF}})}(i,j) \geq N$ for any $i,j \in V$.

(ii) For any pair $i,j \in V$, we have $H_{(\mathcal{G},P_{\mathrm{AF}})}(i,j) = NH_{(\mathcal{G},P_{\mathrm{SF}})}(i,j)$.

(iii) For any $i,j,k \in V$, the following equality holds: $H_{(\mathcal{G},P_{\mathrm{AF}})}(i,j) + H_{(\mathcal{G},P_{\mathrm{AF}})}(j,k) + H_{(\mathcal{G},P_{\mathrm{AF}})}(k,i) = H_{(\mathcal{G},P_{\mathrm{AF}})}(i,k) + H_{(\mathcal{G},P_{\mathrm{AF}})}(k,j) + H_{(\mathcal{G},P_{\mathrm{AF}})}(j,i)$.

(iv) There holds that $M_{(\mathcal{G},P_{\mathrm{AF}})} \leq 2H_{(\mathcal{G},P_{\mathrm{AF}})} - N$.

∎

The quantities $T_{\mathrm{con}}(x(0))$ and T_Ψ for AF are defined in terms of the number of clock ticks in a similar way to those for SF in Section III-A.

Theorem 4.1: Let $x(0) \in \mathbb{R}^N$ and suppose $x(0) \notin \mathcal{W}$. Under AF, any evolution $x(t)$ starting from $x(0)$ reaches quantized average consensus. Furthermore, $\mathbb{E}[T_{\mathrm{con}}(x(0))]$ is upper bounded by $\frac{N^2J^2}{8}(\frac{8}{27}N^3 - 1)$ in terms of the number of clock ticks.

Proof: Note that $\mathbb{E}[T_\Psi] = M_{(\mathcal{G},P_{\mathrm{AF}})}$ and we have

$$\mathbb{E}[T_{\mathrm{con}}(x(0))] \leq \frac{NJ^2}{8}M_{(\mathcal{G},P_{\mathrm{AF}})} \leq \frac{NJ^2}{8}(2NH_{(\mathcal{G},P_{\mathrm{SF}})} - N),$$

where we use Lemma 4.1 in the last inequality. Substituting the bound on $H_{(\mathcal{G},P_{\mathrm{SF}})}$ in [6] in the Appendix into the above inequality gives the desired upper bound on $\mathbb{E}[T_{\mathrm{con}}(x(0))]$. The reminder of the proof on the convergence property of AF is analogous to Theorem 3.1. ∎

B. Asynchronous quantized averaging on switching graphs

In this section, we will study an *asynchronous quantized averaging algorithm on switching graphs* (AS, for short). As in Section IV-A, we will adopt the asynchronous time model and discretize the time instant t according to $\{Z_k\}_{k\geq 0}$.

The algorithm AS is described as follows. In the k^{th} time-slot, let node i's clock tick. If $|\mathcal{N}_i(k)| \neq 0$, node i randomly chooses one of its neighbors, say node j, with probability $\frac{1}{\max\{|\mathcal{N}_i(k)|,|\mathcal{N}_j(k)|\}}$, and node i and j execute the computation (1) or (2). If $|\mathcal{N}_i(k)| = 0$, node i does nothing at this time. Before showing the convergence of AS, we consider the following problem.

The meeting time of two natural random walks on $\mathcal{G}(t)$. Initially, two tokens are placed on $\mathcal{G}(t)$; at each clock tick, one of the tokens is chosen with probability $\frac{1}{N}$. The chosen token at some node, say i, moves to one of its neighbors, say node j, with probability $\frac{1}{\max\{|\mathcal{N}_i(k)|,|\mathcal{N}_j(k)|\}}$ if $|\mathcal{N}_i(k)| \neq 0$; otherwise, it will not move. What is the meeting time for these two tokens?

Clearly, the movements of two tokens are two natural random walks, say X_1 and X_2, on the switching graph $\mathcal{G}(t)$. Their meeting time is denoted as $M_{(\mathcal{G}(t),P_{\mathrm{AS}}(t))}$ where the transition matrix $P_{\mathrm{AS}}(t)$ is given as follows: if $|\mathcal{N}_i(t)| \neq 0$, then $p_{ij}(t) = \frac{1}{N\max\{|\mathcal{N}_i(t)|,|\mathcal{N}_j(t)|\}}$ for $(i,j) \in E(t)$ and $p_{ii}(t) = 1 - \sum_{(i,j)\in E(t)} \frac{1}{N\max\{|\mathcal{N}_i(t)|,|\mathcal{N}_j(t)|\}}$; if $|\mathcal{N}_i(t)| = 0$, then $p_{ii}(t) = 1$.

Lemma 4.2: Suppose the communication graph $\mathcal{G}(t)$ satisfies Assumption 3.1. For the problem of the meeting time of two natural random walks on $\mathcal{G}(t)$, we have that $M_{(\mathcal{G}(t),P_{\mathrm{AS}}(t))} \leq 4Nt_2$ where $t_2 = \frac{1}{2}B(8N^5(N-1)\log(\sqrt{2}N)+1)$

Proof: Due to space limitations, we omit the proof. ■

The quantities $T_{\mathrm{con}}(x(0))$ and T_Ψ in terms of the number of clock ticks for AS are defined in a similar way to those for SF in Section III-A.

Theorem 4.2: Let $x(0) \in \mathbb{R}^N$ and suppose $x(0) \notin \mathcal{W}$. Assume that $\mathcal{G}(t)$ satisfies Assumption 3.1. Under AS, any evolution $x(t)$ starting from $x(0)$ reaches the quantized average consensus. Furthermore, $\mathbb{E}[T_{\mathrm{con}}(x(0))]$ is upper bounded by $\frac{1}{2}BJ^2N(8N^6\log(\sqrt{2}N)+1)$ in terms of the number of clock ticks.

Proof: Due to space limitations, we omit the proof. ■

V. Discussion on Quantized Averaging over Random Graphs

The graph $\mathcal{G}(N,p)$, $0 < p < 1$, is one of two most frequently occurring random graph models [3]. The algorithm SF can also be implemented on $\mathcal{G}(N,p)$. The corresponding upper bound on expected convergence time is $\frac{J^2}{4p}\frac{N(N-1)^2}{N-2}$. The algorithm AF has an upper bound $\frac{J^2}{8p}\frac{N^3(N-1)^2}{N(N-1)-2p}$ on expected convergence time when it is implemented on $\mathcal{G}(N,p)$. Due to the space limit, we omit the proof.

VI. Simulation Example

This section presents a simulation of AS. Consider a network of 10 nodes. Assume that the quantization step size $\Delta = 1$ and graph $\mathcal{G}(t)$ satisfies Assumption 3.1 with $B = 3$. Suppose the initial state $x(0) = (5,6,14,17,0,11,10,21,10,6)^T$ with average $\bar{x}(0) = 10$. The worst-case upper bound on $\mathbb{E}[T_{\mathrm{con}}(x(0))]$ in Theorem 4.2 is 10^{10} clock ticks. Figure 1 shows that all the consensus states agree on $\bar{x}(0)$ after about 65 clock ticks.

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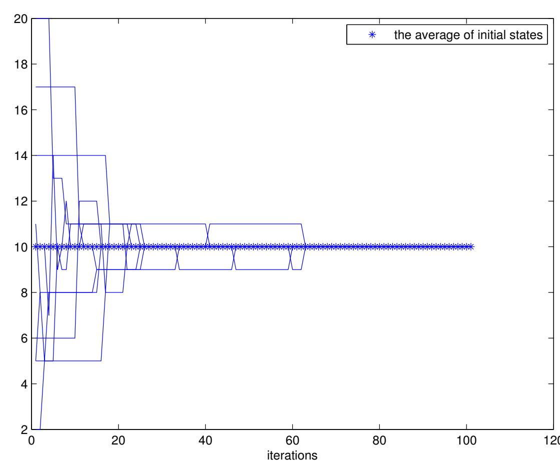

Fig. 1. The states of asynchronous quantized averaging algorithm on switching graphs