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Input-Output Stability with Input-to-State Stable Protocols for Quantized and Networked Control Systems†

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***Abstract*— This paper introduces the notion of input-to-state stable (ISS) protocols in the context of scheduling and quantization within networked control systems (NCS) and examines conditions under which a class of continuous-time control systems designed ignoring the network achieves input-to-state stability. Verifiable sufficient conditions for robust stability (ISS) are given for a class of nonlinear systems under the constraint of finite data-rate feedback.**

I. Introduction

The premise of a networked control systems (NCS) is that the feedback loop is "closed" over a communications network, thus, freeing the components of the control system from the constraint of physical co-location. The use of a single channel for multiple communication devices implies the use of scheduling protocols [1], [2], [3] to arbitrate communication and, indeed, to remain faithful to the notion of limited data rate control [4], [5], [6], [7], it is scheduling with quantization that must be analyzed. Along those lines, [8] presented an approach for a unified analysis of combined scheduler-quantizers where known scheduling protocols and quantizers are subtly redesigned when used in tandem. This paper introduces an an alternative emulation-based design and analysis framework based on the concept of input-to-state stable *protocols* – a concept that shall shortly be introduced. ISS protocols are pedagogically useful in their own right, facilitating the modeling of scheduling, quantization and combined scheduling-quantization protocols. ISS protocols reduce to the previously introduced class of Lyapunov uniformly globally exponentially stable (UGES) protocols [1] when the ISS "gain" is reduced to zero.

Our results are such that NCS data rate bounds can be explicitly calculated and related to the derived ISS gain – they apply to ISS protocols and the closed-loop stability properties for systems employing these protocols are phrased in terms of ISS and ISS-like notions.[1]

II. Input-to-State Stability and Related Notions

A. Preliminaries

$\mathbb{R}$, $\mathbb{R}_{\geq 0}$ and $\mathbb{N}$ denote, respectively, the sets of real, nonnegative real and natural numbers. Let $\mathcal{K}$ denote the class of continuous functions $f : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ that satisfy $f(0) = 0$ and $f(t_1) < f(t_2)$ for any $0 \leq t_1 < t_2$. We say that $f \in \mathcal{K}$ is of class $\mathcal{K}_\infty$ if it is unbounded. A function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class $\mathcal{KL}$ if for each $s \geq 0$ the function $\beta(s, \cdot)$ is decreasing to zero in the second argument and for each fixed $t \geq 0$, the function $\beta(\cdot, t)$ is of class $\mathcal{K}$. The function β is of class exp-$\mathcal{KL}$ whenever $\beta(s, t)$ can be written in the form $\beta(s, t) = Ks\exp(-\lambda t)$ for $\lambda, K > 0$. We make use of the so-called Kronecker delta defined by

$$\delta_{a,b} = \begin{cases} 1 & a = b \\ 0 & a \neq b \,. \end{cases}$$

Given $t \in \mathbb{R}$ and a piecewise continuous function $f : \mathbb{R} \to \mathbb{R}^n$, we use the notation $f(t^+) = \lim_{s \to t, s > t} f(s)$. All vector (Euclidean) norms are denoted by $|\cdot|$, as is the induced matrix 2-norm. Let $f : \mathbb{R}_{\geq 0} \to A \subset \mathbb{R}^n$ be a (Lebesgue) measurable function and define $\|f[a, b]\|_\infty = \text{ess.}\sup_{t \in [a,b]} |f(t)|$. For brevity, we often write (x, y) in place of $(x^T \; y^T)^T$.

B. Input-to-State Stability & Detectability Notions

We first review the concepts of input-to-output stability (IOS) and input-output-to-state stability (IOSS)[2] for systems of the form

$$\dot{x}(t) = f(t, x, w) \quad t \in [t_{i-1}, t_i] \tag{1}$$

$$x(t_i^+) = h_i(x(t_i), w(t_i)) \tag{2}$$

$$y = H(x, w) \tag{3}$$

where $\epsilon \leq t_i - t_{i-1} \leq \tau < \infty$, where $\epsilon > 0$ for all $i \in \mathbb{N}$.

Definition 2.1: Let $\gamma \in \mathcal{K}$ and $\beta \in \mathcal{KL}$ be given. The system (1)-(3) is IOpS (input-to-output *practically* stable) from w to y if for all $t_0 \geq 0$, $x(t_0) \in \mathbb{R}^{n_x}, w \in L_\infty$ and each corresponding solution $x(\cdot)$, we have that for all $t \in [t_0, t_0 + T)$

$$|y(t)| \leq \beta(|x(t_0)|, t - t_0) + \gamma(\|w\|_\infty) + d \,, \tag{4}$$

where $[t_0, t_0 + T)$ is the maximum interval of definition of $x(\cdot)$ and $d \geq 0$. If $y = x$, then we say that (1)-(3) is ISpS (input-to-state *practically* stable). If the respective properties hold with $d = 0$ then the "practical" qualifier may be omitted and (1)-(3) is IOS (resp., ISS). Moreover, if $\gamma(\cdot)$ is a linear

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[1]This is distinct from [9] where the stability properties are indeed ISS but apply to scheduling protocols that are UGAS/UGES, not ISS.

[2]The presentation of the definitions of ISS, IOS and IOSS closely follows that of [1, Section II-B].

function, β is an exp-$\mathcal{KL}$ function and (1)-(3) is IOS (ISS), then we say that (1)-(3) is IOS (ISS) with a linear gain and an exp-$\mathcal{KL}$ function. ISS of system (1)-(3) implies UGAS when $w \equiv 0$ and ISS with a linear gain and an exp-$\mathcal{KL}$ function implies UGES. $\triangleleft$

Definition 2.2: Let $\alpha^0 \in \mathcal{K}$ and $D^0 \geq 0$ be given. The system (1)-(3) has the unboundedness observability (UO) property from (w, y) to x if for all $t_0 \geq 0$, $x(t_0) \in \mathbb{R}^{n_x}, w \in L_\infty$ and each corresponding solution $x(\cdot)$, we have that for all $t \in [t_0, t_0 + T)$ (the maximum interval of definition of $x(\cdot)$)

$$|x(t)| \leq \alpha^0(|x(t_0)| + \|y[t_0, t]\|_\infty + \|w[t_0, t]\|_\infty) + D^0 \,. \quad (5)$$

Definition 2.3: Let $\gamma \in \mathcal{K}$ and $\beta \in \mathcal{KL}$ be given. The system (1)-(3) is IOSS from (w, y) to x if for all $t_0 \geq 0$, $x(t_0) \in \mathbb{R}^{n_x}, w \in L_\infty$ and each corresponding solution $x(\cdot)$, we have that

$$\begin{aligned} |x(t)| \leq \beta(|x(t_0)|, t - t_0) + \gamma(\|y[t_0, t]\|_\infty) \\ + \gamma(\|w[t_0, t]\|_\infty) \quad \forall t \in [t_0, t_0 + T) \,, \end{aligned} \quad (6)$$

where $[t_0, t_0 + T)$ is the maximum interval of definition of $x(\cdot)$. Moreover, if $\gamma(\cdot)$ is a linear function, β is an exp-$\mathcal{KL}$ function and (1)-(3) is IOSS, then we say that (1)-(3) is IOSS with a linear gain and an exp-$\mathcal{KL}$ function. $\triangleleft$

III. Hybrid Systems Model and Input-to-State Stable Protocols

A. Hybrid System Model and ISS Protocol Definition

As outlined in the introduction, this paper examines an emulation-based approach for the design of networked, quantized and combined networked-quantized (NCS, QCS, NQCS) systems. Explicitly, given a nominal plant $\dot{x} = \tilde{f}(t, x, u, w)$ where w denotes an exogenous perturbation, one first designs a stabilizing controller $u = k(t, x)$. In sampled-data, networked and quantized control systems, continuously measured output and continuously applied control must be replaced with an appropriate proxy subject to the constraints of sampling, scheduling from the use of a network protocol or quantization law, respectively. For example, when control can be continuously applied but state-measurements cannot be continuously measured, the controller $u = k(t, \hat{x})$ is one feedback strategy that is admissible, where $\hat{x}$ is an appropriate "estimate" originating from the use of a generalized sampling scheme (NCS, QCS, NQCS).

To model systems (NCS, QCS, NQCS) arising from the use of emulated control and observation strategies, we consider the following class of hybrid systems

$$\dot{x} = f(t, x, z, w) \quad t \in [t_i, t_{i+1}] \quad (7)$$

$$\dot{z} = g(t, x, z, w) \quad t \in [t_i, t_{i+1}] \quad (8)$$

$$z(t_i^+) = h(i, z(t_i), x(t_i), w(t_i)) \quad (9)$$

$$\epsilon \leq t_{i+1} - t_i < \tau < \infty \,, \quad (10)$$

where $x \in \mathbb{R}^{n_x}, z \in \mathbb{R}^{n_z}, w \in \mathbb{R}^{n_z}$ and where $\epsilon > 0$ and the sequence of generalized sampling instants is defined by $\{t_i\}_{i=0}^\infty$. The closed-loop plant (or combined plant, dynamical controller dynamics) is given by (7) whereas z captures emulation-induced error variables as well as auxiliary variables needed to implement the generalized sampling scheme. In analogy with the analysis and design approach adopted in [2], [1], the generalized sampling scheme is described by its effects on the state z at sampling instants (9) that we refer to as the *protocol* and several classes of NCS, QCS and NQCS can be captured by (7)-(9).

IV. NCS and Sampled-Data

Consider a continuous-time dynamical feedback system with plant state x_P, controller state x_C, controls u and output y:

$$\dot{x}_P = f_P(t, x_P, u, w) \quad (11)$$

$$\dot{x}_C = f_C(t, x_C, y, w) \quad (12)$$

$$u = g_C(t, x_C) \quad y = g_P(t, x_P) \,. \quad (13)$$

We regard the system as consisting of ℓ nodes that are either sensors or controllers that compete for access to the network at transmission instants. That is, at any given transmission instant t_i, only a subset of the components of (u, y) is transmitted, the precise subset being determined by the *scheduling protocol*. Apart from this subset that is determined at each transmission instant, the NCS makes use of appropriate "networked"-version of $(u(t_i), y(t_i))$ denoted by $\hat{u}$ and $\hat{y}$. Define the error induced by these "estimates" as: $e(t) := \begin{pmatrix} \hat{y}(t) - y(t) \\ \hat{u}(t) - u(t) \end{pmatrix}$. The error is governed by continuous-time dynamics dynamics *between* transmission instants $\dot{e} = \begin{pmatrix} d\hat{y}/dt - dy/dt \\ d\hat{u}/dt - du/dt \end{pmatrix}$, where a zero-order hold policy would correspond to $d\hat{y}/dt = 0$ and $d\hat{u}/dt = 0$. At transmission instants, the error is instantly reset according to a jump-map which serves as the initial condition for the subsequent continuous-time evolution interval: $e(t_i^+) = F(e(t_i), x_C(t_i), x_P(t_i), w(t_i))$.

Let $\epsilon > 0$ and define the sequence of transmission instants by $\{t_i\}_{i=0}^\infty$ where $\epsilon \leq t_{i+1} - t_i < \tau < \infty$. As in the analysis and design approach adopted in [2], [1], the scheduling protocol is described by its effects on the induced error at sampling instants i.e., $e(t_i^+) = h(i, e(t_i))$.

In [1], protocols of the following form were considered $h(i, e) = (I - \Psi(s))e$, where $s = s(i, e) : \mathbb{N} \times \mathbb{R}^{n_e} \rightarrow \{1, \ldots, \ell\}$ is a scheduling function, $\Psi(s) = \text{diag}\{\delta_{1,s} I_{n_1}, \ldots, \delta_{\ell,s} I_{n_\ell}\}$, $\delta_{a,b}$ is the Kronecker delta and I_{n_j} are identity matrices of dimension n_j with $\sum_{j=1}^\ell n_j = n_e$. That is, s picks which of the $\{1, \ldots, \ell\}$ nodes is transmitted corresponding to component(s) of $(u(t_i), y(t_i))$ to be transmitted. It is assumed that the transmission results in the respective components of induced error being instantaneously reset to zero, and this is reflected in $\Psi(s)$ and its block-diagonal structure.

Remark 1: The case $\Psi(s) = I_{n_e}$, without s-dependence corresponds to transmitting all of (u, y) at "transmission" instants and demonstrates that sampled-data systems can be viewed as a particular case of NCS. $\triangleleft$

With the addition of a scheduling protocol and a defined sequence of transmission instants, emulation-based NCS version of (11)-(13) is given by (see [1] for details)

$$\dot{x} = f(t, x, e, w) \tag{14}$$
$$\dot{e} = g(t, x, e, w) \tag{15}$$
$$e(t_i^+) = h(i, e(t_i)), \tag{16}$$

where $x = (x_P, x_C)$. By setting $z = e$, this class of NCS can be written in the same form as (7)-(9) and setting $z = e = 0, \dot{z} = \dot{e} = 0$ recovers the nominal network-free system.

V. QCS

This section presents a model of quantized control systems analogous to that of [10] and the preceding section. We restrict our attention to plants and static state-feedbacks of the same form described in the preceding section. The motivation for considering *quantized feedback* is that the feedback path, thought of here as a communications channel, constraints the flow of information to a finite data rate, achievable through the use of sampling together with quantization.

Definition 5.1 (Quantizer): Let $|\cdot|_P$ denote a fixed norm on $\mathbb{R}^{n_x}$. Let $\Delta \in (0, \infty)$ and suppose $|\zeta|_P \le M$. A static map $q : \mathbb{R}^{n_x} \to \mathbb{R}^{n_x}$ is a quantizer if it satisfies the following properties:

1) $q(0) = 0$
2) $|q(\zeta) - \zeta|_P \le \Delta$.

Definition 5.2 (Box quantizer): In the case of a quantizer q_s defined on the $\mathbb{R}$, a vector quantizer can be realized by defining $q(\zeta) := (q_s(\zeta_1), \dots, q_s(\zeta_{n_x}))$ and, hence, satisfies the properties:

1) $q(0) = 0$
2) $|\zeta|_\infty \le M \Rightarrow |q(\zeta) - \zeta|_\infty \le \Delta$.

This renders q a quantizer when using the $|\cdot|_\infty$ norm. Geometrically, the quantization regions are the Cartesian products of $[-M, M]$ intervals, hence they are often referred to as "box" quantizer. Box quantizers have a closed-form expression for the number of bits required to encode the quantization region that the state x lies in:

$$\text{number of regions per dimension} = \frac{M}{\Delta}$$
$$\Rightarrow \text{no. bits to encode } [-M, M]^{n_x} = n_x \log_2\left(\frac{M}{\Delta}\right) \text{ bits.}$$

Although the quantizers above are written as maps from $\mathbb{R}^{n_x}$ to $\mathbb{R}^{n_x}$, the image of the quantizer is finite – there are only finitely many quantization regions and the image typically consists of the *centroids* of these regions – and, hence, it is enough to transmit a finite-length string that identifies the particular region the state lies in. The key requirement for these quantizers is that they do not *saturate*, that is, the state to be encoded ζ should lie in the compact region $|\zeta| \le M$. ◁

We now consider a linear-time invariant plant with additive disturbance $w \in L_\infty$: $\dot{x} = Ax + Bu + w$ and assume that a static state-feedback control has been designed $u = Kx$ such that $(A + BK)$ is stable. For QCS, the first departure from continuous-time feedback is sampling at the sequence of sampling times $\{t_i\}_{i\in\mathbb{N}}$ that satisfy $\epsilon \le t_{i+1} - t_i \le \tau < \infty$, for $\epsilon > 0$. The second is quantization of the state x at these sampling instants to yield the "estimated" state $\hat{x}$ that is available to the feedback law. In view of these modifications, the emulated system takes the form:

$$\dot{x} = Ax + BK\hat{x} + w \qquad t \in [t_i, t_{i+1}] \tag{17}$$
$$\dot{\hat{x}} = f_E(\hat{x}) \qquad t \in [t_i, t_{i+1}] \tag{18}$$
$$\hat{x}(t_i^+) = \text{quantized}(x(t_i)), \tag{19}$$

where (18) denotes arbitrary $\hat{x}$ dynamics[3] *between* sampling instants maintained by the receiving end of the communication channel, restricted to depend on only $\hat{x}$; and (19) refers to an as yet unknown quantization scheme. As in [8], use of the class of quantizers defined earlier will be facilitated through the appropriate scaling of x to ensure that $|\text{scaled}(x(t))| \le M \quad$ for all $t \ge t_0$. Although it is not necessary, the choice of f_E is fixed to "match" the plant dynamics and the scaling variable μ is introduced to yield the QCS

$$\dot{x} = Ax + BK\hat{x} + w \qquad t \in [t_i, t_{i+1}] \tag{20}$$
$$\dot{\hat{x}} = (A + BK)\hat{x} \qquad t \in [t_i, t_{i+1}] \tag{21}$$
$$\dot{\mu} = g(\mu, \hat{x}) \qquad t \in [t_i, t_{i+1}] \tag{22}$$
$$\hat{x}(t_i^+) = \mu(t_i) q\left(\frac{x(t_i)}{\mu(t_i)}\right) \tag{23}$$
$$\mu(t_i^+) = h(i, \mu(t_i), \hat{x}(t_i)), \tag{24}$$

where the quantization *protocol* is defined through (24) and (23) and, in particular, μ dynamics and jump-map are chosen in such a way so that the quantizer q is never saturated. We can now make this assumption precise:

Assumption 1: Consider (20)-(24) initialized at $(t_0, x(t_0), \hat{x}(t_0), \mu(t_0))$ using a quantizer defined earlier. We assume that $\left|\frac{x(t)}{\mu(t)}\right|_P \le M$ for all $t \ge t_0$. ◁

The assumption can be made true by appropriate definitions of the rhs (22), (24) and assuming that $\|w\|_\infty$ is a known quantity. As for NCS, we define an emulation-induced error $e = \hat{x} - x$ and the QCS can be rewritten:

$$\dot{x} = (A + BK)x + BKe + w \qquad t \in [t_i, t_{i+1}] \tag{25}$$
$$\dot{e} = Ae - w \qquad t \in [t_i, t_{i+1}] \tag{26}$$
$$\dot{\mu} = g(\mu, x + e) \qquad t \in [t_i, t_{i+1}] \tag{27}$$
$$e(t_i^+) = \mu(t_i)\left[q\left(\frac{x(t_i)}{\mu(t_i)}\right) - \frac{x(t_i)}{\mu(t_i)}\right] \tag{28}$$
$$\mu(t_i^+) = h(i, \mu(t_i), x(t_i) + e(t_i)). \tag{29}$$

Defining a protocol for this class of QCS becomes a matter of specifying h in (29), since (28) is holds by definition of induced error.

Naturally, scheduling (NCS) can be combined with quantization (QCS) to yield networked quantized control systems (NQCS), as discussed in [8] but extended here to include

[3] Nonzero choices of $d\hat{y}/dt, d\hat{u}/dt$ may lead to improved performance in an appropriate sense. See [8], for instance.

disturbances and consideration of nonzero μ continuous-time dynamics:

$$\dot{x} = (A+BK)x + BKe + w \quad t \in [t_i, t_{i+1}] \tag{30}$$
$$\dot{e} = Ae - w \quad t \in [t_i, t_{i+1}] \tag{31}$$
$$\dot{\mu} = g(\mu, x+e) \quad t \in [t_i, t_{i+1}] \tag{32}$$
$$e(t_i^+) = [I - \Psi(i, e(t_i))]e(t_i) + \mu(t_i)\Psi(i, e(t_i)) \times \left[q\left(\frac{\Psi(i, e(t_i))x(t_i)}{\mu(t_i)}\right) - \frac{\Psi(i, e(t_i))x(t_i)}{\mu(t_i)}\right] \tag{33}$$
$$\mu(t_i^+) = h_\mu(i, \mu(t_i), x(t_i) + e(t_i)), \tag{34}$$

A. *ISS Protocols*

We now introduce the class of ISS scheduling protocols, where, as indicated earlier, a protocol is (a representation of) the jump-map (9). This definition formalizes the interpretation in [11, Section 6] and allows the modeling of protocols in the presence of exogenous disturbances.

Definition 5.3: Consider the discrete-time system

$$z^+ = h(i, z, x, w). \tag{35}$$

We say that (35) is an input-to-state stable (ISS) protocol if there exists a function $W : \mathbb{N} \times \mathbb{R}^{n_z} \to \mathbb{R}_{\geq 0}$ such that there exist constants $a_1 > 0, a_2 > 0, \rho \in [0, 1)$ and $\gamma_p \geq 0$ such that, for all $z \in \mathbb{R}^{n_z}$, $i \in \mathbb{N}$, $x \in \mathbb{R}^{n_x}$ and $w \in \mathbb{R}^{n_w}$

$$a_1|z| \leq W(i, z) \leq a_2|z| \tag{36}$$
$$W(i+1, h(i, z, x, w)) \leq \rho W(i, z) + \gamma_p|\tilde{y}|, \tag{37}$$

where $\tilde{y} : \mathbb{R}^{n_x} \times \mathbb{R}^{n_w} \to \mathbb{R}$ is a continuous function of (x, w). ◁

The ISS protocol definition captures properties of several important protocol classes that arise in the study of hybrid systems and particularly networked and quantized control systems.

Example 5.4 (UGES Protocols): Recall that a protocol

$$e^+ = p(i, e) \tag{38}$$

is Lyapunov UGES (uniformly globally exponentially stable) if there exists a function $W : \mathbb{R}^{n_e} \times \mathbb{N} \to \mathbb{R}_{\geq 0}$ such that

$$a_1|e| \leq W(i, e) \leq a_2|e| \tag{39}$$
$$W(i+1, p(i, e)) \leq \rho W(i, e), \tag{40}$$

where $\rho \in [0, 1)$. These protocols – scheduling protocols, were defined in [1] to analyze stability properties of networked control system. When we define $z = e$ and $h(i, z, x, w) = p(i, z)$, these protocols are ISS with $\gamma_p = 0$, one example of producing ISS protocols. ◁

Example 5.5: The e-protocol (33) in NQCS has a particular form where it is essentially an NCS protocol with an additive disturbance:

$$e^+ = h(i, e) + v, \tag{41}$$

where in the case of NQCS, the additive disturbance v is due to the use of quantization. In the case of TOD [13] and conceivably many other UGES protocols, the same Lyapunov function that demonstrates Lyapunov UGES can be used to show that the protocol with additive disturbance is ISS. For example, TOD is Lyapunov UGES with Lyapunov function $V(e) = |e|$, hence, for (41),

$$V(e^+) \leq \sqrt{\frac{\ell - 1}{\ell}} V(e) + |v|,$$

where ℓ denotes the number of nodes.

Before introducing examples for specific quantization schemes, the following propositions establish that it is enough to consider definitions of the quantization protocol (29) in systems of the form (25)-(29) and not the (e, μ) system as a whole. ISS of the (e, μ) subsystems follows if Assumption 1 holds.

Proposition 5.6: Consider the QCS (25)-(29) and suppose that:

1) the system satisfies Assumption 1;
2) the quantization protocol (29) is ISS from $(x + e)$ to μ with ISS-Lyapunov function V_μ satisfying $a_1|\mu| \leq V_\mu(\mu) \leq a_2|\mu|$; and
3) Δ, M and ρ, γ_p come from the quantization protocol and ISS-Lyapunov function of the protocol, respectively and satisfy
$$\frac{\gamma_p}{k_1}\frac{\Delta}{a_1} < 1, \tag{42}$$
where $k_1 > 0, k_2 > 0$ are constants such that $k_1|e| \leq |e|_P \leq k_2|e|$.

Then the (e, μ) (discrete-time) subsystem is ISS from x to (e, μ), thus the discrete-time system induced by the (e, μ) is an ISS protocol in the usual sense. The proof is a straightforward application of a discrete-time small-gain argument e.g., see [12].

Similarly, many scheduling protocols (Example 5.4) combined with ISS quantization scheme yield ISS protocols for (e, μ) in NQCS.

Example 5.7 (Zoom-Quantization Scaling): The following protocol is an example of the quantization scaling factor evolution typical in QCS and is an example of a protocol that can be used in (29) and (34):

$$\mu^+ = \begin{cases} \alpha\mu, & |x+e| \leq C\mu \\ \mu, & |x+e| > C\mu \end{cases}, \tag{43}$$

where $\alpha \in (0, 1)$. Let $V_\mu(\mu) = |\mu|$, hence

$$V_\mu(\mu^+) \leq \alpha V_\mu(\mu) + C^{-1}(|x+e|),$$

and the protocol is ISS from $(x+e)$ to μ with ISS-Lyapunov function V_μ. This protocol can only "zoom-in", that is, the scaling factor can only decrease in discrete-time. We later show that an appropriate choice of continuous-time dynamics for μ, the key difference between the zoom protocol of this paper and that of [11], eliminates the need for discrete-time zoom-outs. ◁

VI. Main Results

A. Input-to-State Stability Properties of Error Subsystem with ISS Protocols

In this section we show that ISS protocols with ISS-Lyapunov function W lead to ISS/IOS for the error (e) dynamics in (7)-(9) under relatively mild assumptions on regularity of W and assumptions on the error dynamics.

Proposition 6.1: Consider the system

$$\dot{z} = g(t, z, x, w) \quad t \in [t_{i+1}, t_i] \tag{44}$$

$$z(t_i^+) = h(i, z(t_i), x(t_i)) \tag{45}$$

and suppose that

1) the (discrete-time map induced by) protocol h is ISS with ISS-Lyapunov function W in the sense of Definition 5.2;
2) ISS-Lyapunov function W is locally Lipschitz, uniformly in i and satisfies for almost all $z \in \mathbb{R}^{n_z}$
$$\left\langle \frac{\partial W(i,z)}{\partial z}, g(t,z,x,w) \right\rangle \leq LW(i,z) + |\tilde{y}|, \tag{46}$$
where $\tilde{y}$ comes from (37);
3) there exists $\epsilon > 0$ such that the monotonically increasing sequence of time instants $\{t_i\}_{i=0}^{\infty}$ satisfies $\epsilon \leq t_{i+1} - t_i < \tau$;
4) τ satisfies $\tau < \frac{1}{L} \ln\left(\frac{1}{\rho}\right)$, where ρ is from (37) .

Then, with respect to (44)-(45), there exists $\beta(\cdot, \cdot) \in \mathcal{KL}$ such that

$$W(t) \leq \beta(W_0, t - t_0) + \gamma_w \|\tilde{y}[t_0, t]\|_\infty, \tag{47}$$

where

$$\gamma_w = \frac{(1 + \gamma_p)\exp(L\tau) - 1}{L(1 - \rho \exp(L\tau))}, \tag{48}$$

where γ_p is from (37). That is, (44)-(45) is IOS from $\tilde{y}$ to W with linear gain. The proof follows from similar arguments to those found in [1]. ◁

B. Input-to-State Stability Properties of Hybrid Systems with ISS Protocols

This section considers hybrid systems of the form (7)-(9) and establishes conditions on the x, e subsystems as well as the protocols under which the system is IOS with respect to certain inputs and outputs and, further conditions under which this property specializes to ISS and UGES/UGAS.

Theorem 6.2: Consider system (7)-(9) and suppose that

1) the (discrete-time map induced by) protocol h in (9) is ISS with ISS-Lyapunov function W in the sense of Definition 5.2;
2) the x-subsystem in (7) is IOS from (W, w) to $\tilde{y}$ with gain γ_x satisfying $\gamma_x \gamma_p < L(1 - \rho)$, where ρ, γ_p comes from (37); the x-subsystem of (7) regarded as having state x, input (W, w) and output $\tilde{y}$ has the UO property with couple $(\alpha_x, 0)$, for some $\alpha_x \geq 0$;
3) ISS-Lyapunov function W is locally Lipschitz, uniformly in i and satisfies for almost all $z \in \mathbb{R}^{n_z}$
$$\left\langle \frac{\partial W(i,z)}{\partial z}, g(t,z,x,w) \right\rangle \leq LW(i,z) + |\tilde{y}|, \tag{49}$$
where $\tilde{y}$ comes from (37);
4) the z-subsystem of (7) regarded as having state z, input $\tilde{y}$ and output W has the UO property with couple $(\alpha_z, 0)$, for some $\alpha_x \geq 0$;
5) (ρ as in (37)) τ satisfies
$$\tau < \frac{1}{L} \ln\left(\frac{L + \gamma_x}{L\rho + \gamma_x(1 + \gamma_p)}\right) \tag{50}$$

Then (7)-(9) is IOS from w to (W, x) with linear gain. Theorem 6.2 follows from a small-gain argument, using a small-gain theorem for hybrid systems such as [1, Theorem 2]: From Proposition 6.1, conditions 1) and 3)-5) render the e-subsystem of (7) IOS from $\tilde{y}$ to e with linear gain $\gamma_w(\tau)$ in (48) and $\gamma_x \gamma_w < 1$ is satisfied for τ defined in (50). ◁

VII. Quantization, Scheduling & Robustness

Common to all ISS protocols and the view of the emulated system as a particular class of hybrid systems is the notion of the system as a generalized sampled-data system with an associated maximum sampling interval τ, as in Theorem 6.2. Once the protocol is selected, the primary mechanism for recovering the qualitative behavior of the nominal system is sufficient reduction of τ. For NCS, our results reduce to those found in [1] for UGES protocol.

A. Quantized Control Systems

This section considers the class of QCS introduced in Section V. The following proposition developed for the class of quantizers considered in this paper demonstrates that a particular choice of μ continuous-time dynamics in (27) and additional constraints on the generalized sampling rate allow Assumption 1 to be proved rather than assumed.

Proposition 7.1: Consider QCS (25)-(29) and suppose that

1) the μ-subsystem of the quantization protocol (29) is ISS from $x + e$ to μ with ISS-Lyapunov function satisfying:
$$V_\mu^+ \leq \rho_\mu V_\mu + \gamma_p |x + e| \tag{51}$$
$$V_\mu^+ \geq \tilde{\rho}_\mu V_\mu + \tilde{\gamma}_p |x + e| \tag{52}$$
where $0 < \tilde{\rho}_\mu \leq \rho_\mu$ and $0 < \tilde{\gamma}_p \leq \gamma_p$;
2) the QCS is sampled at a sequence of time instants $\{t_i\}_{i \in \mathbb{N}}$ satisfying
$$0 < \epsilon \leq t_{i+1} - t_i \leq \tau < \infty; \tag{53}$$
3) the quantizer q is such that
$$|x(t_i)| \leq MV_\mu(\mu(t_i)) \Rightarrow \left| q\left(\frac{x(t_i)}{\mu(t_i)}\right) - \frac{x(t_i)}{\mu(t_i)} \right| \leq \Delta \tag{54}$$
4) μ dynamics (27) are chosen such that for each $i \in \mathbb{N}$
$$\frac{dV_\mu(t)}{dt} = \tilde{\lambda}_x V_\mu(t) + \frac{\theta}{M}\left(\hat{x} + \|w\|_\infty\right), \tag{55}$$

where

$$\lambda_x = \sup\left(\sigma^+(A)\right) \tag{56}$$

$$\tilde{\lambda} \geq \frac{1}{\epsilon}\ln\left(\frac{1}{\tilde{\rho}_\mu}\right) + \lambda_x \tag{57}$$

$$\theta \geq \max\left\{\exp((\tilde{\lambda}_x - \lambda_x)\tau), \frac{1}{\tilde{\rho}_\mu}\right\}, \tag{58}$$

and A, B, K are from (25); and

5) the quantizer is initially not saturated:

$$V_\mu(\mu(t_0)) \geq \frac{1}{M}|x(t_0)|. \tag{59}$$

Then the quantizer never saturates at sampling/quantization instants:

$$V_\mu(\mu(t_i)) \geq \frac{1}{M}|x(t_i)| \qquad \forall i \in \mathbb{N}. \tag{60}$$

The proof follows by definition of the constants and an induction argument.

Remark 2: Knowledge of the magnitude of w is analogous to knowledge of the statistics (various moments) of stochastic perturbations – a standard assumption in the analysis stochastic systems and an assumption made in [6]. ◁

The use of Theorem 6.2 together with Proposition 7.1 for QCS (25)-(29) is illustrated with the following result for a particular quantization protocol.

Proposition 7.2: Consider QCS (25)-(29) and suppose that

1) (29) is given by Example 5.6;
2) the quantizer q is a box quantizer, as described in Definition 5.1;
3) all conditions of Proposition 7.1 are satisfied with $V_\mu(\mu) = |\mu|$;
4) and (25) is ISS from (μ, e, w) to x with gain γ_x.
5) in addition to the constraints imposed on τ by Proposition 7.1, τ satisfies

$$\tau < \frac{1}{L}\ln\left(\frac{L + \tilde{\gamma}_x}{L\rho + \tilde{\gamma}_x(1+\gamma_p)}\right),$$

where

$$\tilde{\gamma}_x = \frac{n_x^{1/2}\theta}{M}\gamma_x \tag{61}$$

$$\lambda = \frac{C^{-1}\sqrt{n_x}\Delta + 1}{2C^{-1}\sqrt{n_x}} \tag{62}$$

$$L = 2\max\left\{|A|n_x^{1/2} + \frac{\theta n_x^{1/2}}{M}, \lambda\tilde{\lambda}_x\right\} \tag{63}$$

$$\rho = \max\left\{\frac{C^{-1}\Delta + n_x^{-1/2}}{2n_x^{-1/2}}, \frac{2C^{-1}\Delta}{C^{-1}\Delta + n_x^{-1/2}}, \alpha\right\} \tag{64}$$

and

$$\gamma_p = \frac{C^{-1}\Delta + n_x^{-1/2}}{2n_x^{-1/2}}, \tag{65}$$

and α is from (43).

Then (25)-(29) is ISS with linear gain and a finite data rate given by $\frac{n_x}{\tau}\log_2\left(\frac{M}{\Delta}\right)$ bps.

Proof: (Sketch) First note that by application of Proposition 5.5 and a particular choice of ISS-Lyapunov functions for the e and μ subsystems we have

$$V_e(e) = |e|_\infty \Rightarrow |e|\frac{1}{\sqrt{n_x}} \leq V_e(e) \leq |e| \tag{66}$$

$$V_\mu(\mu) = |\mu|. \tag{67}$$

The proof follows from using the ISS-Lyapunov function $V(e,\mu) = \max\{V_e(e), \lambda V_\mu(\mu)\}$ with λ given in (62) and, hence, $V^+ \leq \rho V + \gamma_p|x|$, where ρ, γ_p are given by (64) and (65), respectively. ■

VIII. Conclusion

ISS protocols provide new insights into the analysis and design of networked and quantized control system as well as combinations thereof. In particular, with appropriate modifications of "zoom" protocols the analysis framework presented quantitatively characterizes the relationship between data rates and robustness in a straightforward manner.

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