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Communication Complexity in the Distributed Design of Linear Quadratic Optimal Controllers

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***Abstract*— We consider a control design situation in which the knowledge of a Linear Time-Invariant (LTI) plant's model is segmented between two parties: one party knows the dynamics of a subsystem within the plant, and how some particular inputs affect the whole system, while the other party knows all the remaining information. We ask: "How much of their partial knowledge of the model should the parties transmit to the control designer in order to enable her to construct an optimal controller?"**

Assuming that models are specified by their state-space representations, we tackle this question within the framework of Real Number Communication Complexity theory and prove that, for certain patterns of segmented model knowledge, the communication complexity of optimal control design is maximal. We also show that satisfactory suboptimal controllers can be constructed with reduced communication complexity.

I. Introduction

When controlling large-scale dynamical systems composed of interconnected subsystems, it is not realistic to assume that the control designer has access to a complete and precise model of the entire plant. There may be several reasons for this fact. First, each subsystem may itself be poorly characterized, resulting in a globally uncertain model. This is the traditional view for the origin of structured uncertainty in control theory, which is consistent with the framework of linear fractional representations, and the interpretation of feedback as a mechanism for "constructing precise systems from imprecise components". However, even when very precise models of individual subsystems are available, it may be impossible for the designer to use them because the resulting global model would be too complicated or impractical. Alternatively, some of the characteristics of a subsystem may have to remain private information. In that case, the control designer has to construct a controller based on reduced representations of the components, which, in turn, limits the quality (i.e., the guaranteed closed loop performance and robustness) of the controller that she can produce. A natural question, then, is

Question 1: "How much information about the subsystems' models should be transmitted to the designer to guarantee that a satisfactory controller can be constructed for the plant?"

Question 1 is a communication problem, since different parties must exchange information to complete a joint task, and the quality of the task (here, control design) depends on what is exchanged. In this paper, we treat a particular instance of this general question in which

- the control design task is to construct the optimal controller for an LTI plant and given, globally known, quadratic closed-loop performance weights Q and R.
- subsystems' models are given by their state-space representation matrices (assuming that subsystems are coupled directly through their states) and can be communicated to the designer entry-wise.
- the quantity of information exchanged is measured by the number of real-valued messages that subsystems send to the designer to describe themselves. This metrics relates indirectly to the complexity of the reduced-order models used to describe each component, since a model of smaller dimension, i.e., a "simpler" model, is specified by less real-valued messages than a model of large dimension.

With these definitions, we can treat this problem within the framework of Communication Complexity introduced in the Computer Science literature by [11], [6], [8]. More precisely, we can determine the minimal number of messages that the control designer needs to receive from each subsystem to compute the desired optimal controller, in the worse case over classes of possible subsystems. *Our results show that, for discrete time dynamics and even for two interconnected subsystems, the number of messages that the designer needs to receive in order to compute the optimal controller is maximal.* This finding complements other recent results

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by one of the authors and coworkers [4], [7], which gave bounds on the best performance achievable by a controller designed using a restricted number of messages, using a slightly different notion of communication protocol than the one considered in the present paper. The present work can also be seen as a conceptual generalization to dynamic control problems of results from the Economics and Computer Science literatures, which investigated the communication complexity of Nash equilibria in finite games [2], [5].

We start by giving a quick review of the main tools and results of (Real Number) Communication Complexity theory of interest to the present work in Section II. We then apply these tools in Section III to compute (bounds on) the communication complexity of optimal linear-quadratic control design in discrete and continuous time, and give more details on how this result pertains to the main question raised in this Introduction. In Section IV, we propose a suboptimal control strategy, which can always be constructed with communication complexity lower than that of the optimal controller, and which achieves a guaranteed level of closed-loop performance. Finally, in Section V, we point to possible extensions of the communication complexity-based approach.

II. Notions of Communication Complexity for Real Number Arithmetic

Consider a situation where two parties P_1 and P_2 each privately own a vector $x \in \mathbb{R}^{n_1}$ and $y \in \mathbb{R}^{n_2}$, respectively, and want to compute a known $\mathbb{R}^s$-valued function f of these vectors. We assume that this computation is performed by a fusion center, to which parties must transmit their private information. More precisely, both parties send vector-valued messages $m_1(x) \in \mathbb{R}^{r_1}$ and $m_2(y) \in \mathbb{R}^{r_2}$ to the fusion center, which are in turn used to compute function f according to

$$f(x,y) = h(m_1(x), m_2(y)), \qquad (1)$$

for some appropriate function h. The triple of functions (m_1, m_2, h) constitute a *(one-step communication) protocol*. Assuming that communication is costly, and given a function f and set S (which can be thought of as *a priori* shared information about the parties' otherwise private vectors), it is natural to ask which communication protocol is the "cheapest" to compute f for all elements of S.

To make this question precise and measure the cost of a communication protocol, several authors (e.g., [1], [8]) have proposed to identify cost with the total number $r := r_1 + r_2$ of real scalar messages that are transmitted to the fusion center. For this definition to make sense, however, and ensure that one cannot "smuggle" information in the messages by encoding several real numbers in the decimals of a single scalar, one should require that the message functions m_1 and m_2 are sufficiently regular to avoid this interleaving. In this paper, we follow [8] and require that m_1 and m_2 (along with h) be *analytic functions*. Far less stringent conditions have been introduced on message functions in the mathematical economics literature devoted to message-space complexity [9], which lead to compatible notions of cost of a communication protocol. With these precautions, we can introduce the following

Definition 1 (Communication Complexity): The communication complexity $\mathtt{CC}_\infty(f,S)$ of a function f over a set S is defined as

$$min \left\{ \begin{array}{l} r = r_1 + r_2 \mid \text{there exists a} \\ \text{protocol } (m_1, m_2, h) \text{ for } f \text{ on } S, \\ m_1,\ m_2,\ h \text{ are analytic functions} \end{array} \right\}.$$

For any analytic function f and set S, the communication complexity $\mathtt{CC}_\infty(f,S)$ is clearly upper-bounded by n_1+n_2, since (id, id, f) is a communication protocol for f, corresponding to the case where each party transmits its private information in full to the center. Finding a lower bound is a more delicate task, and several results are available in the literature (some of them for communication protocols that differ slightly from the notion used in this paper) [1], [8], [9] that make use of the differentials of f. For our purposes, the following theorem from [8] will be sufficient.

Theorem 1 (Luo & Tsitsiklis): Let S be the domain of f (i.e., the open subset over which f is finite), S_1, S_2 two subsets of $\mathbb{R}^{n_1}$ and $\mathbb{R}^{n_2}$, respectively, such that $S_1 \times S_2 \subset S$. Let (m_1, m_2, h) be an analytic communication protocol for computing f over $S_1 \times S_2$, with a total of $r_1 + r_2$ messages (i.e., $m_1(x) \in \mathbb{R}^{n_1}$ for all $x \in S_1$ and $m_2(y) \in \mathbb{R}^{n_2}$ for all $y \in S_2$). Then

$$r_1 \geq \max_{x \in S_1}\ dim\ \ span\{\nabla_x\, f_i^\alpha(x,y),\ y \in S_2\}_{i,\alpha},$$

$$r_2 \geq \max_{y \in S_2}\ dim\ \ span\{\nabla_y\, f_i^\beta(x,y),\ x \in S_1\}_{i,\beta},$$

where f_i designates the i^{th} coordinate map of f, and

$$f_i^\alpha(x,y) = \frac{\partial f_i^\alpha}{\partial y_1^{\alpha_1} ... \partial y_{n_1}^{\alpha_{n_1}}}(x,y)$$

for all multi-index vector $\alpha = (\alpha_1, ..., \alpha_{n_1})$,

$$f_i^\beta(x,y) = \frac{\partial f_i^\beta}{\partial x_1^{\beta_1} ... \partial x_{n_2}^{\beta_{n_2}}}(x,y)$$

for all multi-index vector $\beta = (\beta_1, ..., \beta_{n_2})$.

Introducing the partial differential maps $[D_x f(\bar{x},\bar{y})] : \mathbb{R}^{n_1} \to \mathbb{R}^s$ and $[D_y f(\bar{x},\bar{y})] : \mathbb{R}^{n_2} \to \mathbb{R}^s$ of function f at point $(\bar{x},\bar{y}) \in S_1 \times S_2$, Theorem 1 immediately yields the following weaker, yet sometimes sufficient

Corollary 1: With the notations of Theorem 1,

$$r_1 \geq rank \ \ [D_x f(\bar{x},\bar{y})], \ \text{ for all } (\bar{x},\bar{y}),$$

$$r_2 \geq rank \ \ [D_y f(\bar{x},\bar{y})], \ \text{ for all } (\bar{x},\bar{y}).$$

In the remainder of this paper, we will denote $rank \ \ [D_x f(\bar{x},\bar{y})] + rank \ \ [D_y f(\bar{x},\bar{y})]$ by $\bar{\mathtt{CC}}_\infty[\bar{x},\bar{y}](f,S)$ and let

$$\bar{\mathtt{CC}}_\infty(f,S) := \max_{(\bar{x},\bar{y})\in S} \bar{\mathtt{CC}}_\infty[\bar{x},\bar{y}](f,S).$$

The result of Corollary 1 can then be summarized as

$$\mathtt{CC}_\infty(f,S) \geq \bar{\mathtt{CC}}_\infty(f,S),$$

for all analytic f.

III. Communication Complexity of Optimal Control

In this section, we apply the general result expressed in Corollary 1 to optimal control design, in order to answer Question 1 raised in the Introduction.

A. Discrete time systems

We start by focusing on discrete time systems and introduce the map which, to a plant, associates the optimal controller. Let $Q = C^*C \succeq 0$ and $R \succ 0$ be two given $2n \times 2n$ matrices. For any controllable pair of $2n \times 2n$ matrices (A,B) such that (C,A) is observable, define $\mathcal{L}_{Q,R}(A,B)$ to be the optimal controller for linear discrete-time system

$$x(k+1) = Ax(k) + Bu(k) \tag{2}$$

and quadratic cost

$$J(\{u(k)\}_{k=0}^\infty) = \frac{1}{2}\sum_{k=0}^\infty x(k)^*Qx(k) + u(k)^*Ru(k). \tag{3}$$

The resulting map $\mathcal{L}_{Q,R}$ is well-defined over this set of pairs of matrices, which we call $S_{Q,R}$. In particular, for all $(A,B) \in S_{Q,R}$,

$$\mathcal{L}_{Q,R}(A,B) = (B^*XB + R)^{-1}B^*XA,$$

where X is the unique positive-definite solution of Riccati equation

$$A^*XA - X - A^*XB(B^*XB+R)^{-1}B^*XA + Q = 0. \tag{4}$$

The map $\mathcal{L}_{Q,R}$ is known to be analytic in all its variables [3], by virtue of the Implicit Function Theorem and the fact that $\mathcal{L}_{Q,R}(A,B)$ is a stabilizing controller for plant (A,B), which ensures that the partial differential of the right hand side of (4) with respect to X is a bijection.

By identifying $\mathbb{R}^{2n\times 2n}$ with $\mathbb{R}^{4n^2}$, we can consider $\mathcal{L}_{Q,R}$ as a function from $\mathbb{R}^{4n^2} \times \mathbb{R}^{4n^2}$ to $\mathbb{R}^{4n^2}$ and apply the results of Corollary 1 to derive lower-bounds on the communication complexity of this map over the set $S_{Q,R}$. The main idea of our proof is to exhibit, for different partitions of the matrices A and B, specific pairs (A_0,B_0) such that equation (4) can be solved explicitly *and* the rank of the partial differential of $\mathcal{L}_{Q,R}$ at (A_0,B_0) is readily computable.

Theorem 2: Let $Q \succ 0$ and $R \succ 0$. Assume that the $2n \times 2n$ matrices A and B describing a plant in $S_{Q,R}$ are partitioned between two parties according to

$$A = \left[\begin{array}{c} A_1 \\ \hline A_2 \end{array}\right] \text{ and } B = \left[\begin{array}{c|c} B_1 & B_2 \end{array}\right], \tag{5}$$

i.e., party P_1 knows the first n rows of matrix A and first n columns of matrix B, while party P_2 knows the last n rows of matrix A and last n columns of matrix B. Then

$$\mathtt{CC}_\infty(\mathcal{L}_{Q,R}, S_{Q,R}) = 8n^2.$$

Before proceeding with the proof of Theorem 2, we would like to reformulate its content in relation with the control design problem described in the Introduction. From the assumption on data partition (5), we can think of each party as knowing the A-matrix of a single n dimensional subsystem, and the sensitivity of the entire plant to half of the $2n$ control inputs. Hence, the fact that $\mathtt{CC}_\infty(\mathcal{L}_{Q,R}, S_{Q,R})$ is maximal shows that, even in the case of two interconnected subsystems, it is not possible for components to transmit a simplified description of their dynamics matrix A or their sensitivity matrix to some inputs to the control designer, if one desires to compute the optimal controller for any weights Q, R.

The proof of Theorem 2 will make use of the following simple lemma:

Lemma 1: Let $R \succ 0 \in \mathbb{R}^{2n\times 2n}$. There exist matrices

$$T = \begin{bmatrix} T_{11} & 0 \\ T_{21} & T_{22} \end{bmatrix} \text{ and } S = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & 0 \end{bmatrix}$$

such that $R = TT^* = SS^*$ and T_{11}, T_{22}, S_{21}, S_{12} are invertible $n \times n$ matrices.

Proof: We prove the existence of matrix S. The existence of T is proved similarly. To this end, it is enough to prove that the following matrix equalities have a solution

$$R_{11} = S_{11}S_{11}^* + S_{12}S_{12}^* \tag{6a}$$

$$R_{12} = S_{21}S_{11}^* \tag{6b}$$

$$R_{22} = S_{21}S_{21}^*. \tag{6c}$$

Since $R \succ 0$, $R_{22} \succ 0$ and there thus exists $S_{21} \succ 0$ such that equation (6c) is satisfied. Since it is an invertible matrix, we can let $S_{11} = \left(S_{21}^{-1} R_{12}\right)^*$ to satisfy (6b). With these choices, $R_{11} - S_{11}S_{11}^* = R_{11} - R_{12}^* R_{22}^{-1} R_{12}$, which by the Schur complement formula is a positive definite matrix because $R \succ 0$. Hence, we can always find $S_{12} \succ 0$ such that (6a) is satisfied. ∎

Proof: [of Theorem 2]
Let $Q, R \succ 0$ be given. It is clear that $\mathtt{CC}_\infty(\mathcal{L}_{Q,R}, S_{Q,R}) \leq 8n^2$, since both parties can just pass all their privately known entries to the control designer. To prove the opposite inequality, we use the lower bounds provided by Corollary 1. The idea is to exhibit a pair (A, B) of plant matrices such that the partial differential of map $\mathcal{L}_{Q,R}$, with respect to (A_1, B_1), computed at (A, B), has full rank. By Theorem 1, this will imply that $r_1 \geq 4n^2$. A similar construction can be carried out to exhibit a pair (A', B') at which the partial differential of $\mathcal{L}_{Q,R}$, with respect to (A_2, B_2) has full rank, implying that $r_2 \geq 4n^2$.
By Lemma 1, there exists a lower block-triangular invertible matrix T such that $Q = \frac{1}{2}TT^*$ and matrix S, with the structure indicated in the lemma, such that $R = SS^*$. Let $B := (ST^{-1})^*$, so that $R = B^*TT^*B$ and matrix B is invertible and has the structure

$$B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & 0 \end{bmatrix},$$

because T^{-1} is also block lower triangular. As a result B^{-1} has structure

$$B^{-1} = \begin{bmatrix} 0 & \tilde{B}_{12} \\ \tilde{B}_{21} & \tilde{B}_{22} \end{bmatrix}.$$

Now, one can just see by inspection that, when taking $A := I_{2n}$, $X = 2Q$ is a solution of Riccati equation (4) and, by uniqueness, the only positive definite solution. Hence $\mathcal{L}_{Q,R}(A, B) = \left[ST^{-1}TT^*(T^{-1})^*S^* + SS^*\right]^{-1} ST^{-1}(TT^*) = \frac{1}{2}B^{-1}$ when $(A, B) = (I_{2n}, (ST^{-1})^*) \in S_{Q,R}$ (the pair is controllable since B is invertible). To apply Corollary 1, we compute the differential of map $\mathcal{L}_{Q,R}$ with respect to (A_1, B_1), at the point $(I_{2n}, (ST^{-1})^*)$, and show that it has full rank.

Because we already know that $\mathcal{L}_{Q,R}$ is a differentiable function of all its variables, it is enough, to compute the partial differential, to compute $\mathcal{L}_{Q,R}(I_{2n} + \epsilon\Delta A_1, (ST^{-1})^* + \epsilon\Delta B_1) - \mathcal{L}_{Q,R}(I_{2n}, (ST^{-1})^*)$, and keep the linear terms in ϵ, where

$$\Delta A_1 = \begin{bmatrix} \Delta A_{11} & \Delta A_{12} \\ 0 & 0 \end{bmatrix}; \; \Delta B_1 = \begin{bmatrix} \Delta B_{11} & 0 \\ \Delta B_{21} & 0 \end{bmatrix}.$$

After some straightforward but lengthy algebra, one finds that

$$\begin{aligned} &[D_{(A_1,B_1)}\mathcal{L}_{Q,R}(I_{2n}, (ST^{-1})^*)](\Delta A_1, \Delta B_1) = \\ &B^{-1}X^{-1}\left[\frac{2}{3}X(\Delta A_1) + \frac{1}{6}(\Delta A_1)^*X + \right. \\ &\left. -\frac{1}{3}X(\Delta B_1)B^{-1} + \frac{1}{6}(B^{-1})^*(\Delta B_1)^*X\right], \end{aligned}$$

where, as before $B = (ST^{-1})^*$ and $X = 2Q = TT^*$. We want to show that the null space of $[D_{(A_1,B_1)}\mathcal{L}_{Q,R}(I_{2n}, (ST^{-1})^*)]$ is trivial, which will imply that its rank is $4n^2$. If $(\Delta A_1, \Delta B_1)$ is in the null space, then

$$\begin{aligned} &X(\Delta A_1) + (\Delta A_1)^*X + X(\Delta C) + (\Delta C)^*X = \\ &-3X(\Delta A_1 - \Delta C), \end{aligned} \tag{7}$$

where we have introduced $\Delta C := (\Delta B)B^{-1}$. Note that ΔC thus has the structure

$$\begin{bmatrix} 0 & \Delta C_1 \\ 0 & \Delta C_2 \end{bmatrix}$$

because of the structure of B^{-1} and ΔB_1. Looking at the (1,1) entry of equation (7), we obtain

$$X_{11}\Delta A_{11} + \Delta A_{11}^* X_{11} = -3X_{11}\Delta A_{11}. \tag{8}$$

Taking the transpose of this equation and combining it with (8) yields $15X_{11}\Delta A_{11} = 0$, which, because X_{11} is invertible by the positive definiteness of X, implies $\Delta A_{11} = 0$. Now, because of (7), $X(\Delta A_1 - \Delta C)$ must be a symmetric matrix. Combining this fact with the (1,2) entry of equation (7) yields $2X_{11}\Delta A_{12} = 0$ and, in turn, $\Delta A_{12} = 0$ because $X_{11} \succ 0$. Finally, knowing that $\Delta A_1 = 0$ turns (7) into

$$X_{11}\Delta C_1 + X_{12}\Delta C_2 = 0 \tag{9a}$$

$$2X_{12}^*\Delta C_1 + \Delta C_1^* X_{12} + 2X_{22}\Delta C_2 + \Delta C_2^* X_{22} = 0 \tag{9b}$$

From (9), $\Delta C_1 = -X_{11}^{-1}X_{12}\Delta C_2$ and

$$\begin{aligned} &2(X_{22} - X_{12}^*X_{11}^{-1}X_{12})\Delta C_2 \\ &+ \Delta C_2^*(X_{22} - X_{12}^*X_{11}^{-1}X_{12}) = 0. \end{aligned}$$

Combining this latter equation with its transpose yields $(X_{22} - X_{12}^*X_{11}^{-1}X_{12})\Delta C_2 = 0$. Noting that $(X_{22} - X_{12}^*X_{11}^{-1}X_{12})$ is positive definite by the Schur complement formula and the fact that $X \succ 0$ finally implies $\Delta C_1 = \Delta C_2 = 0$. We have thus proved that

$$Ker\;[D_{(A_1,B_1)}\mathcal{L}_{Q,R}(I_{2n}, (ST^{-1})^*)] = \{0\},$$

which concludes the proof. ∎

B. Continuous time systems

So far, we have focused on LTI discrete time systems. In the remainder of this section, we consider the communication complexity of the optimal control design map $\mathcal{L}^c_{Q,R}$ for continuous time LTI plants. In this case, the lower bound of Corollary 1 does not match the trivial upper bound of $8n^2$, and can thus not be used to compute the communication complexity of the map. However, as we will see in Section IV, this lower bound implies that good *suboptimal* controllers can be designed with only partial revelation of the plant model.

Let $\mathcal{L}^c_{Q,R}$ be the map that, to the LTI plant

$$\dot{x}(t) = Ax(t) + Bu(t), \tag{10}$$

associates the optimal controller for the quadratic cost

$$J^c(u) = \frac{1}{2}\int_0^\infty x(t)^*Qx(t) + u(t)^*Ru(t)dt. \tag{11}$$

Proposition 1: Let $Q, R \succ 0$ and the plant matrices (A, B) in $S_{Q,R}$ be partitioned according to the pattern of Equation (5). Then, for all pairs (A, B)

$$\bar{\mathtt{CC}}_\infty[A, B](\mathcal{L}^c_{Q,R}, S_{Q,R}) \leq 7n^2 + n. \tag{12}$$

Proof: As before, we are interested in computing $Ker\ [D_{(A_1,B_1)}\mathcal{L}^c_{Q,R}(A, B)]$. To this end, recall that for all (A, B), $\mathcal{L}^c_{Q,R}(A, B) = -R^{-1}B^*X$, where X is the unique positive definite solution of the continuous time Riccati equation

$$A^*X + XA - XBR^{-1}B^*X + Q = 0. \tag{13}$$

Now, let us partition X into four $n \times n$ matrices and consider $\bar{\Delta}A_1 = \begin{bmatrix} \bar{\Delta}A_{11} & \bar{\Delta}A_{12} \\ 0 & 0 \end{bmatrix}$ with

$$\begin{aligned} \bar{\Delta}A_{11} &= X_{11}^{-1}S \\ \bar{\Delta}A_{12} &= X_{11}^{-1}SX_{11}^{-1}X_{12} \end{aligned} \tag{14}$$

for some skew-symmetric $n \times n$ matrix S. It is easy to see that $X\bar{\Delta}A_1 = 0$ and, thus, that X satisfies (13) with A replaced by $A + \bar{\Delta}A_1$. Hence, for all (A, B), $\mathcal{L}^c_{Q,R}(A + \bar{\Delta}A_1, B) = \mathcal{L}^c_{Q,R}(A, B)$, which means that $(\bar{\Delta}A_1, 0)$ belongs to $Ker\ [D_{(A_1,B_1)}\mathcal{L}^c_{Q,R}(A, B)]$ for every choice of skew symmetric matrix S. Since the set of all matrices $\bar{\Delta}A_1$ parameterized by (14) is a subspace of $\mathbb{R}^{2n\times 2n}$, this implies that

$$dim\ Ker\ [D_{(A_1,B_1)}\mathcal{L}^c_{Q,R}(A, B)] \geq \frac{n(n-1)}{2},$$

i.e., $rank\ [D_{(A_1,B_1)}\mathcal{L}^c_{Q,R}(A, B)] \leq \frac{7}{2}n^2 + \frac{n}{2}$. A similar construction shows that

$$dim\ Ker\ [D_{(A_2,B_2)}\mathcal{L}^c_{Q,R}(A, B)] \geq \frac{n(n-1)}{2}$$

as well. All in all, this thus proves that

$$\bar{\mathtt{CC}}_\infty[A, B](\mathcal{L}^c_{Q,R}, S_{Q,R}) \leq 7n^2 + n.$$

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IV. Sub-optimality and communication complexity

In the previous section, we computed (bounds on) the worst case number of real-valued messages needed to construct the optimal controller for sets of plants. When this number is unacceptably large, it is natural to try and construct a suboptimal controller that requires less communication between the parties and the designer, while guaranteeing some satisfactory level of closed-loop performance for every plant.

The following theorem asserts that it is possible to construct such controllers and provide a good level of closed-loop performance for a ball of plants in $S_{Q,R}$.

Theorem 3: Let $(\bar{A}, \bar{B}) \in S_{Q,R}$. There exists a control design method $\mathcal{K}$ such that

(i) $\mathtt{CC}_\infty(\mathcal{K}, S_{Q,R}) = \bar{\mathtt{CC}}_\infty[\bar{A}, \bar{B}](\mathcal{L}_{Q,R}, S_{Q,R})$, if matrices are partitioned according to the pattern of Equation (5).

(ii) For all (A, B) in the ball of center $(\bar{A}, \bar{B})$ and radius ϵ (for the Frobenius norm),

$$|J(\mathcal{K}(A, B)) - J(\mathcal{L}_{Q,R}(A, B))| = O(\epsilon^4).$$

(Note that performance is only degraded by a factor of ϵ^4 over a ball of radius ϵ).

These results are also valid in continuous time, when replacing $\mathcal{L}_{Q,R}$ by $\mathcal{L}^c_{Q,R}$ and the closed-loop performance criterion J by J^c.

Theorem 3 is a simple consequence of the two following propositions, which also provide a constructive proof of the existence of the control design method $\mathcal{K}$. From here on, we will restrict ourselves to the discrete time case, but similar results can be derived for continuous time systems as well.

Proposition 2: Let $(\bar{A}, \bar{B}) \in S_{Q,R}$ and consider the control design strategy $\mathcal{K}_{(\bar{A},\bar{B})}$ defined by

$$\begin{aligned} \mathcal{K}_{(\bar{A},\bar{B})}(A, B) &:= \mathcal{L}_{Q,R}(\bar{A}, \bar{B}) + \\ &[D\mathcal{L}_{Q,R}(\bar{A}, \bar{B})](A - \bar{A}, B - \bar{B}) \end{aligned} \tag{15}$$

for all $(A, B) \in S_{Q,R}$, i.e., $\mathcal{K}_{(\bar{A},\bar{B})}$ is the linear approximation of map $\mathcal{L}_{Q,R}$ at $(\bar{A}, \bar{B})$. Then,

(i) $\|\mathcal{K}_{(\bar{A},\bar{B})}(A, B) - \mathcal{L}_{Q,R}(A, B)\|_F = O(\|A - \bar{A}\|_F^2, \|B - \bar{B}\|_F^2)$ in a neighborhood of $(\bar{A}, \bar{B})$,

(ii) $|J(\mathcal{K}_{(\bar{A},\bar{B})}(A,B)) - J(\mathcal{L}_{Q,R}(A,B))| = O(\|A-\bar{A}\|_F^4, \|B-\bar{B}\|_F^4)$ in a neighborhood of $(\bar{A},\bar{B})$

where $\|.\|_F$ designates the Frobenius norm.

Proof:

(i) Since $\mathcal{L}_{Q,R}$ is analytic,

$$\begin{aligned}\mathcal{L}_{Q,R}(A,B) &= \mathcal{L}_{Q,R}(\bar{A},\bar{B})\\ &+ [D\mathcal{L}_{Q,R}(\bar{A},\bar{B})](A-\bar{A},B-\bar{B})\\ &+ O(\|A-\bar{A}\|_F^2, \|B-\bar{B}\|_F^2),\end{aligned}$$

for all (A,B) in a neighborhood of $(\bar{A},\bar{B})$. Hence, maps $\mathcal{K}_{(\bar{A},\bar{B})}$ and $\mathcal{L}_{Q,R}$ agree up to linear order.

(ii) By definition, the control signal induced by the control matrix gain $\mathcal{L}_{Q,R}(A,B)$ minimizes criterion J for the plant (A,B) among all control signals. If we abuse notation and write $J(K)$ for the performance of static feedback control signal defined as

$$u(k) = Kx(k), \text{ for all } k$$

then

$$J(\mathcal{L}_{Q,R}(A,B)) \leq J(K),$$

for every matrix $K \in \mathbb{R}^{2n\times 2n}$ and every plant (A,B). As a result, the derivative $D_K J$ of the closed-loop cost function with respect to the control matrix gain K must vanish at $\mathcal{L}_{Q,R}(A,B)$. In turn, $|J(\mathcal{L}_{Q,R}(A,B)) - J(K)| = O(\|K-\mathcal{L}_{Q,R}(A,B)\|_F^2)$ in the neighborhood of $\mathcal{L}_{Q,R}(A,B)$, for all (A,B). The result of (i) then concludes the proof.

■

Proposition 3: Let f be a *linear*, $\mathbb{R}^s-$valued function of two privately owned vectors x and y, with domain S, i.e.,

$$f(x,y) = M_1 x + M_2 y,$$

for all $(x,y) \in S$ and some matrices M_1, M_2 of appropriate dimensions. Then,

$$\mathtt{CC}_\infty(f,S) = \bar{\mathtt{CC}}_\infty(f,S) = rank\ M_1 + rank\ M_2.$$

A proof can be found in [8].

If we assume that the plant $(\bar{A},\bar{B})$ and its optimal controller $\mathcal{L}_{Q,R}(\bar{A},\bar{B})$ are known to both parties and the fusion center, Proposition 3 can be used to prove item (i) of Theorem 3 since we can write

$$\begin{aligned}&\mathcal{K}_{(\bar{A},\bar{B})}(A,B) := \mathcal{L}_{Q,R}(\bar{A},\bar{B})\\ &+ [D_{(A_1,B_1)}\mathcal{L}_{Q,R}(\bar{A},\bar{B})]((A-\bar{A})_1,(B-\bar{B})_1)\\ &+ [D_{(A_2,B_2)}\mathcal{L}_{Q,R}(\bar{A},\bar{B})]((A-\bar{A})_2,(B-\bar{B})_2),\end{aligned}$$

with $((A-\bar{A})_1,(B-\bar{B})_1)$ and $((A-\bar{A})_2,(B-\bar{B})_2)$ being privately known and the constant term $\mathcal{L}_{Q,R}(\bar{A},\bar{B})$ known to the control designer.

V. Conclusion and Perspectives

We have shown that, for some partitions of a large-scale plant's state-space model between different parties, the communication complexity of calculating the optimal controller is maximal. This can be interpreted as saying that, under such assumptions on the segmentation of knowledge and the communication protocol, it is not possible to compute an optimal controller with a reduced or "compressed" representation of the plant's components. We have also shown how to construct linear protocols, with lower communication complexity, which result in suboptimal controllers with guaranteed good performance for a ball of plants.

In the future, we plan to study other partition patterns, investigate the communication complexity of optimal control design in a *multiparty* environment (with strictly more than two parties involved), or under more general types of communication protocols. In particular, it might be interesting to consider protocols which, in addition to using a low number of messages, can guarantee that some *a priori* specified set of subsystems' characteristics remain private.

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