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Optimal inputs for FIR system identification

Roland Hildebrand

Abstract— **In this paper we consider the problem of input design for open loop identification of linear single input single output finite impulse response systems, under the assumption of full-order modeling. We provide analytic formulas for the information matrix of the optimal input when the design is intended to identify a scalar function of the impulse response coefficients. In particular, we characterize the cases when the optimal information matrix is singular and the optimal input is necessarily a multisine signal. Our results have surprising consequences for the robustness properties of optimal input designs. We show that, contrary to intuition, the choice of an optimal regular information matrix might pose robustness problems if there exist also singular solutions.**

I. Introduction

The goal of system identification is to produce or to refine information on the dynamics of an unknown system. In prediction error model identification, this information is contained in the input-output data produced during the experiment. On the one hand, the obtained information depends on the experimental conditions, such as the spectrum of the applied input signal or the controller used during a closed-loop identification. On the other hand, the quality of the information is measured by its utility for the intended model application. The desire to reconcile these conditions leads to an optimal input design problem, namely to optimally meet the requirements posed by the intended application by choice of the experimental conditions.

Before the appearance of powerful numerical optimization methods this was a very challenging task, and the postulated optimality criteria for the input design were of a general nature, such as minimization of the volume of the confidence ellipsoid [10],[3],[14]. However, from the beginning of this decade an increasing number of results on application-tailored optimal input design appeared (e.g. [4],[7]). For an excellent overview see [2]. If the cost criterion of the optimization problem can be chosen to be linear in the design parameters, then the optimal input is often obtained as the solution of a semidefinite program [6]. For more complicated, but still convex optimality criteria it can often also be obtained by a general-purpose convex optimization algorithm [4]. The optimization algorithms usually yield the optimal input in the form of an optimal information matrix. It is a standard task to convert this information matrix to an actual input signal, see e.g. [4],[6].

Such a numerical algorithm is sufficient to obtain reliable optimal designs for concrete problem instances, but does not permit the qualitative and quantitative study of the dependence of the optimal input design on the problem parameters. However, the parameters of the optimal input design problem are in general determined by the unknown system itself. Thus knowledge of this dependence would be extremely helpful, for instance, for the robustification of the solution against uncertainties in these problem parameters. Robust input design is a challenging task which is still at the beginnings of its development.

In [13] a game-theoretic approach to robust input design was initiated, which consists essentially of a worst-case optimization, assuming that the problem parameters can vary in a fixed set. While this approach benefits from the use of tools known from game theory, it bears the usual drawbacks of a worst-case setup. For instance, one cannot account for different likelihoods of different regions of the fixed set to contain the true parameter vector. A related approach was adopted in [1], but here the role of the fixed set is assumed by the uncertainty set resulting from the identification, and depends itself of the input used during the identification.

The most straightforward way to study the dependence of the optimal design on the problem parameters is by means of an analytic formula for the optimal solution. Naturally, such a formula is out of reach except for the most simple modeling setups. In [5, Theorem 3.1], an analytic solution for the optimal input design for full-order open loop identification of single input single output (SISO) finite impulse response (FIR) systems was given in the case of an input power constraint, where the goal of the experiment was to identify a scalar function of the system parameters. The solution was claimed to be valid under the assumption of a special condition [5, Condition 3.1]. This condition yields also the regularity of the information matrix. A regular information matrix can be obtained by a continuum of different input spectra, which leaves room, for instance, for robustification of the input design [5].

Unfortunately, the main result of [5] is not correct. The goal of the present paper is to provide the correct optimality conditions for a large class of input design problems for the identification of SISO FIR systems, which include the problems studied in [5] as a special case. Our conditions have a number of surprising consequences.

We provide a necessary and sufficient condition for the regularity of the optimal information matrix and show that in many cases the optimal solution is defined by a singular information matrix. If this singular solution is unique, then the optimal input is determined to be a unique multi-sine signal (up to phase shifts of the components) [8], which a priori leaves no room for robustification by the methods proposed in [5]. In these cases, other ways to robustify the input design have to be looked for.

Roland Hildebrand is with LJK, Université Grenoble I/CNRS, BP 53, 38041 Grenoble, France roland.hildebrand@imag.fr

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It is a common belief that if the information matrix corresponding to the optimal input design has full rank, then it must be a continuous function of the problem parameters, at least in the neighbourhood of the considered point. However, in this paper we will show that this is false. Moreover, we provide necessary and sufficient conditions under which the regularity itself of the optimal information matrix is unstable. By this is meant that in the neighbourhood of the considered point the set of problem parameters corresponding to a regular optimal information matrix is of zero measure. We illustrate this situation on a concrete example.

The remainder of the paper is organized as follows. In the next section we describe the general setup of the identification procedure, give other necessary definitions and prove some preliminary results. In Section 3 we consider the goal of identifying a scalar function of the system parameters and discuss the results of [5]. In Section 4 we provide examples illustrating the described phenomena. In the last section we draw some conclusions.

II. Definitions and preliminaries

The SISO system to be identified is modelled by a FIR model structure

$$y(t) = \sum_{k=0}^{n} \theta_k u(t-k) + e(t), \tag{1}$$

where u is the input signal, y the output signal, and e is zero mean white noise of variance σ^2. We assume the true system is represented by the parameter vector $\theta^0 = (\theta_0^0, \dots, \theta_n^0)^T$. The input signal u is assumed to be quasistationary with power spectrum $\Phi_u(\omega)$ and moments

$$\begin{aligned} \mu_k &= \frac{1}{2\pi}\int_{-\pi}^{+\pi} e^{jk\omega}\Phi_u(\omega)\,d\omega \\ &= \frac{1}{2\pi}\int_{-\pi}^{+\pi} \cos(k\omega)\Phi_u(\omega)\,d\omega \in \mathbb{R}. \end{aligned}$$

Denote by $\mu \in \mathbb{R}^{n+1}$ the vector of the first $n+1$ moments and let $T(\mu)$ be the symmetric Töplitz matrix which has μ as its first column. Note that

$$T(\mu) \succeq 0,$$

and this condition is also sufficient for a real vector $\mu \in \mathbb{R}^{n+1}$ to be the moment vector of some power spectrum [8, Chapter VI, Theorem 4.1].

Recall that the cone of positive semidefinite real symmetric Töplitz matrices is the conic convex hull of the moment curve

$$\{Re\,\pi(z)\pi^*(z) = T(\mu_\omega) \,|\, z = e^{j\omega},\ \omega \in [-\pi, +\pi]\},$$

where $\pi(z) = (1, z, z^2, \dots, z^n)^T$ is the vector of powers of z and $\mu_\omega = (1, \cos\omega, \dots, \cos(n\omega))^T$ is the moment vector of the power spectrum concentrating all power at the frequency ω [8]. We will denote this cone by T_+ and its interior, that is the cone of positive definite Töplitz matrices, by T_{++}.

For any vector $v \in \mathbb{R}^{n+1}$, define the linear operator $\mathcal{L}_v$ from the space $\mathcal{T}$ of real symmetric Töplitz matrices of size $(n+1)\times(n+1)$ to $\mathbb{R}^{n+1}$ by

$$\mathcal{L}_v : \ T \mapsto Tv.$$

Note that the dimensions of the initial and target space of $\mathcal{L}_v$ coincide. Hence, after choosing coordinate systems in these spaces, we can define the determinant of $\mathcal{L}_v$. This determinant will be a homogeneous form of degree $n+1$ in the elements of v. If v is the first orthonormal basis vector $e_0 = (1, 0, \dots, 0)^T$, then $\mathcal{L}_v$ maps every Töplitz matrix to its first column and is hence regular. Thus the determinant of $\mathcal{L}_v$ is not identically zero as a function of v, and its zero set

$$\mathcal{Z} = \{v \,|\, |\mathcal{L}_v| = 0\}$$

is of measure zero.

Now let $p(\omega) = \sum_{k=0}^{n} a_k \cos(k\omega)$ be an even trigonometric polynomial. We will say that a matrix M_p is *representing the polynomial* p if $\langle T(\mu), M_p\rangle = \sum_{k=0}^{n} a_k\mu_k$ for all vectors $\mu \in \mathbb{R}^{n+1}$. By $\langle\cdot,\cdot\rangle$ we denote the usual scalar product in matrix spaces, i.e. the one generated by the Frobenius norm. If p is positive (semi-)definite, then there exists a positive (semi-)definite matrix M_p representing p [8]. This matrix is in general not unique. Moreover, if $q(z) = \sum_{k=0}^{n} v_k z^k$ ($v_k \in \mathbb{R}$) is a spectral factor of the non-negative polynomial $p(\omega)$ (i.e. $p(\omega) = |q(e^{j\omega})|^2$ for all ω), then the rank 1 matrix vv^T is representing p. On the other hand, from any positive semidefinite rank 1 matrix vv^T representing p we can obtain a spectral factor.

We identify the system (1) in open loop with a prediction error method to obtain an estimate $\hat\theta$ of the parameter vector. Then the covariance of the estimate is given by [9]

$$E(\hat\theta - \theta^0)(\hat\theta - \theta^0)^T = \sigma^2 N^{-1} T^{-1}(\mu) + o(N^{-1}),$$

where N is the number of input-output data, and μ is the moment vector of the input power spectrum used for the identification. As stated the formula is valid only for regular matrices $T(\mu)$.

III. Identification of a scalar quantity

Suppose the purpose of the identification is to estimate some scalar quantity $J_0 = J(\theta^0)$, which can be expressed as a smooth function of the system parameters. Then the variance of the estimate $\hat J = J(\hat\theta)$ is given by [5]

$$\begin{aligned} &E(\hat J - J_0)^2 = \\ &= \nabla J(\theta^0) E(\hat\theta - \theta^0)(\hat\theta - \theta^0)^T \nabla J(\theta^0)^T + o(N^{-1}) \\ &= \nabla J(\theta^0)\sigma^2 N^{-1} T^{-1}(\mu) \nabla J(\theta^0)^T + o(N^{-1}). \end{aligned}$$

An asymptotically (in the number of input-output data) optimal input design will thus have to minimize the quantity

$$\nabla J(\theta^0) T^{-1}(\mu) \nabla J(\theta^0)^T \tag{2}$$

under given constraints on the input power spectrum Φ_u. In this paper we consider constraints of the form

$$\frac{1}{2\pi}\int_{-\pi}^{+\pi} p(\omega)\Phi_u(\omega)\,d\omega = \sum_{k=0}^{n} a_k\mu_k \le c,$$

where $p(\omega) = \sum_{k=0}^n a_k \cos(k\omega)$ is a positive definite trigonometric polynomial. This condition can be rewritten as

$$\langle T(\mu), M_p \rangle \leq c,$$

where M_p is a positive definite matrix representing p.

Thus the optimal input design reduces to the optimization problem

$$\inf_{\mu} \{g^T T^{-1}(\mu) g : \; T(\mu) \succ 0, \; \langle T(\mu), M_p \rangle \leq c\}, \qquad (3)$$

where $g \in \mathbb{R}^{n+1}$ is a given nonzero vector, $M_p \succ 0$ a given matrix and $c > 0$ a given constant. This problem is equivalent to the problem

$$\min_{\alpha,\mu} \alpha : \qquad \begin{pmatrix} \alpha & g^T \\ g & T(\mu) \end{pmatrix} \succeq 0, \quad \langle T(\mu), M_p \rangle = c. \quad (4)$$

The role of the problem parameters is now assumed by the vector g, which inherits the information on the true parameter vector θ_0 and depends on it. Note that we dropped the condition $T(\mu) \succ 0$, which enlarged the set of feasible moment matrices by including also positive semidefinite ones, but allowed to replace the infimum by a minimum. The condition that the infimum is actually attained is secured by the compactness of the set $\{\mu \,|\, T(\mu) \succeq 0, \; \langle T(\mu), M_p \rangle = c\}$.

Instead of minimizing (2) under a constraint on $\langle T(\mu), M_p \rangle$, we can also minimize $\langle T(\mu), M_p \rangle$ under a constraint on (2). These problems are actually equivalent, as is shown by the following lemma.

Lemma 3.1: Let $X \subset \mathbb{R}^n$ be a set which is invariant with respect to multiplication by nonnegative scalars. Let $f, h : X \to \mathbb{R}$ be homogeneous nonnegative functions, such that h is upper-semicontinuous, the level sets of f are compact, and there exists $x^* \in X$ such that $f(x^*) > 0$, $h(x^*) > 0$. For constants $c, \gamma > 0$, consider the two problems

$$\min\{f(x) \,|\, h(x) \geq \gamma^{-1}\}, \qquad \max\{h(x) \,|\, f(x) \leq c\}.$$

Then the solutions of these problems exist and are equivalent in the sense that there exists a constant $\alpha > 0$ such that the solution set of the first problem is equal to the solution set of the second problem, multiplied by α.

Proof: The set $\{h(x) \,|\, f(x) = 0\}$ is invariant with respect to multiplication by nonnegative scalars and bounded from above, because upper-semicontinous functions attain their maxima on compact sets. It follows that $f(x) = 0$ implies $h(x) = 0$. The set $\{x \,|\, f(x) = 1\}$ is compact and contains the point $x^*/f(x^*)$. Let $r_{\max}$ be the maximum of h on this set. For every $x \in X$ such that $f(x) > 0$ we have $f(\frac{x}{f(x)}) = 1$ and hence $h(x)/f(x) = h(\frac{x}{f(x)}) \leq r_{\max}$. In particular, $r_{\max} \geq h(x^*)/f(x^*) > 0$. Define $X_1 = \{x \in X \,|\, f(x) = 1, \; h(x) = r_{\max}\}$.

Let now $x \in X$ be such that $h(x) \geq \gamma^{-1}$. Then $f(x) > 0$ and hence $f(x) \geq h(x)/r_{\max} \geq \gamma^{-1}/r_{\max}$. Here equality is attained if and only if $h(r_{\max}\gamma x) = r_{\max}$ and $f(r_{\max}\gamma x) = 1$, i.e., $r_{\max}\gamma x \in X_1$. Therefore the solution set of the first problem is exactly $\frac{\gamma^{-1}}{r_{\max}} X_1$.

On the other hand, let $x \in X$ be such that $f(x) \leq c$. Then either $f(x) = 0$, and hence $h(x) = 0$, or $f(x) > 0$ and $h(x) \leq r_{\max} f(x) \leq r_{\max} c$. Equality is attained if and only if $f(x/c) = 1$ and $h(x/c) = r_{\max}$, i.e., $x/c \in X_1$. Therefore the solution set of the second problem is exactly cX_1.

This proves the assertion of the lemma with $\alpha = \frac{\gamma^{-1}}{r_{\max} c}$. ∎

In our case X is the cone of possible moment vectors, $\langle T(\mu), M_p \rangle$ plays the role of the function f, and the inverse of (2) plays the role of the function h. It is not hard to see that this inverse can be expressed as $h(\mu) = \max\{\beta \,|\, T(\mu) - \beta g g^T \succeq 0\}$ and hence fulfills the conditions of the lemma.

In [5], the input power was minimized under a constraint on the quantity (2). In the above sense this is equivalent to the minimization of (2) under a constraint $\mu_0 \leq c$ on the input power, where $c > 0$ is a given constant. In this case $p(\omega) \equiv 1$ and the matrix M_p can be chosen as $\frac{1}{n+1} I_{n+1}$, where I_n is the $n \times n$ identity matrix. Taking above equivalence into account, we could use Theorem 3.1 in [5] to provide an analytical solution to the input design problem (3) for $M_p = \frac{1}{n+1} I_{n+1}$. The theorem requires a certain condition ([5, Condition 3.1]) on the vector g (λ in the notation of [5]), and under this condition guarantees regularity of the optimal information matrix. The condition amounts to the membership of g in the image of a Hankel matrix constructed of the first $2n + 1$ moments of a positive definite power spectrum (Φ_λ in the notation of [5]). Theorem 3.1 in [5] then states that this power spectrum is proportional to an optimal input power spectrum. In particular, regardless of g, any positive definite power spectrum producing a regular Hankel matrix will be proportional to an optimal input power spectrum. As a consequence, Theorem 3.1 in [5] implies that every properly normalized positive definite input power spectrum with a regular Hankel moment matrix of size $(n + 1) \times (n + 1)$ is optimal for the identification of every scalar quantity, which is certainly not true.

Next we give the correct answer to the questions posed in [5]. Moreover, we generalize this result in several directions. Firstly, we show that the correct version of [5, Condition 3.1] is not only sufficient for regularity of the optimal information matrix, but also necessary, and give a simple interpretation of this condition. Secondly, we elaborate when exactly the regularity of the optimal information matrix is stable under perturbations of the system parameters. Our results hold not only for the case $M_p = \frac{1}{n+1} I_{n+1}$, but for the case of general positive definite matrices M_p. In the following we stick to our formulation (4) of the problem, which is equivalent to the formulation in [5] in the above sense.

Theorem 3.2: Consider optimization problem (4). A moment vector $\hat{\mu}$ is an optimal solution to this problem with optimal cost $\hat{\alpha}$ if and only if there exist a vector $\hat{v} \in \mathbb{R}^{n+1}$ and a real symmetric $(n+1) \times (n+1)$ matrix $\hat{M}$ such that the following conditions hold.

$$P(\hat{\alpha}, \hat{\mu}) = \begin{pmatrix} \hat{\alpha} & g^T \\ g & T(\hat{\mu}) \end{pmatrix} \succeq 0,$$

$$D(\hat{v}, \hat{M}) = \begin{pmatrix} 1 & \hat{v}^T \\ \hat{v} & \hat{M} \end{pmatrix} \succeq 0,$$

$$\langle P(\hat{\alpha}, \hat{\mu}), D(\hat{v}, \hat{M}) \rangle = 0, \qquad (5)$$

and there exists a constant $\hat{\kappa}$ such that $\hat{\kappa}\langle T(\mu), M_p\rangle = \langle T(\mu), \hat{M}\rangle$ for all $\mu \in \mathbb{R}^{n+1}$.

Moreover, conditions (5) ensure that the quantities $\hat{v}, \hat{\kappa}, \hat{M}$ are an optimal solution to the optimization problem

$$\min\ 2v^T g + \kappa c : \quad \begin{pmatrix} 1 & v^T \\ v & M \end{pmatrix} \succeq 0,$$
$$\kappa\langle T(\mu), M_p\rangle = \langle T(\mu), M\rangle \quad \forall\ \mu \in \mathbb{R}^{n+1}. \tag{6}$$

Proof: The theorem is a simple consequence of convex programming duality theory [12, Section 30].

First note that (4) is a semi-definite program and its dual is given by (6). Both primal and dual program are strictly feasible. Namely, choosing $\mu_0 = c/tr\, M_p$, $\mu_k = 0$ for $k > 0$, and $\alpha > ||g||^2/\mu_0$ ensures $P(\alpha, \mu) \succ 0$, and choosing $v = 0$, $M = M_p$, $\kappa = 1$ ensures $D(v, M) \succ 0$. Hence [11, Section 4.2] the optimal values of both the primal and the dual problem are attained, and a primal-dual pair of feasible solutions is an optimal pair of solutions if and only if the sum of the optimal values of (4) and (6) is zero. This is equivalent to the complementarity condition (5). ■

Theorem 3.3: Consider optimization problem (4). This problem has an optimal solution with regular information matrix if and only if there exists a spectral factor $q(z) = \sum_{k=0}^n v_k z^k$ of the polynomial $p(\omega)$ such that the vector $v = (v_0, \dots, v_n)^T$ satisfies the condition

$$g \in \mathcal{L}_v[T_{++}].$$

Proof: Let $\hat{\mu}$ be an optimal moment vector satisfying $T(\hat{\mu}) \succ 0$. Then the optimal cost is given by $\hat{\alpha} = g^T T^{-1}(\hat{\mu})g$ and the matrix $P(\hat{\alpha}, \hat{\mu})$ has rank n. It follows by Theorem 3.2 that the dual optimal solution $\hat{v}, \hat{\kappa}, \hat{M}$ yields a rank 1 matrix $D(\hat{v}, \hat{M})$. In particular, $\hat{M} = \hat{v}\hat{v}^T$, and the complementarity condition yields $g + T(\hat{\mu})\hat{v} = 0$, which is equivalent to $g = \mathcal{L}_{-\hat{v}}(T(\hat{\mu}))$. This proves one direction of the equivalence relation.

Let now v be the vector of coefficients of a spectral factor of p such that $g = \mathcal{L}_v(T(\mu))$ for some moment vector μ with positive definite matrix $T(\mu)$. Let further $\kappa = c/\langle T(\mu), M_p\rangle > 0$ and $\alpha = \kappa^{-1} g^T T^{-1}(\mu) g$. Then $(\alpha, \kappa\mu)$, $(-\kappa^{-1}v, \kappa^{-2}vv^T)$ is a primal-dual feasible pair satisfying the complementarity condition, and hence must be optimal. ■

Theorem 3.4: Let $v \in \mathbb{R}^{n+1}$ be a spectral factor of a strictly positive polynomial $p(\omega)$ and let k be the rank of $\mathcal{L}_v$. Then $\mathcal{L}_v[T_+] \subset \mathbb{R}^{n+1}$ is a pointed (that means, containing no lines) closed convex cone of dimension k. For any point $x \in \mathcal{L}_v[T_{++}]$ there exists a subset of dimension $n+1-k$ of positive definite Töplitz matrices T such that $x = \mathcal{L}_v(T)$.

If v, v' are two different spectral factors of p, then the cones $\mathcal{L}_v[T_{++}], \mathcal{L}_{v'}[T_+]$ are disjoint.

Proof: The assertions of the first paragraph of the theorem, with exception of the pointedness of $\mathcal{L}_v[T_+]$, are obtained by application of elementary linear algebra. Let us prove the pointedness by assuming the contrary. Namely, let there exist positive semidefinite Töplitz matrices T_1, T_2 such that $\mathcal{L}_v(T_1) = -\mathcal{L}_v(T_2) \neq 0$. Then $T_1 + T_2 \succeq 0$ must lie in the kernel of $\mathcal{L}_v$. It follows that $\langle T_1 + T_2, vv^T\rangle = \langle T_1 + T_2, M_p\rangle = 0$, where M_p is any positive definite matrix representing p. Hence $T_1 + T_2 = 0$, which leads to a contradiction.

Let v, v' be two different spectral factors of p. Assume that there exist Töplitz matrices $T \succ 0, T' \succeq 0$ such that $\mathcal{L}_v(T) = \mathcal{L}_{v'}(T')$, i.e. $Tv = T'v'$. Since vv^T, $v'(v')^T$ represent the same polynomial, we have also $v^T T v = (v')^T T v'$ and $v^T T' v = (v')^T T' v'$. Hence we have

$$\begin{aligned} & (v - v')^T T (v - v') \\ = \ & (v')^T T v' + v^T T v - (v')^T T v - v^T T v' \\ = \ & 2(v')^T T v' - 2(v')^T T' v' \\ = \ & 2(v')^T (T - T') v' = 2v^T (T - T') v. \end{aligned}$$

Likewise we obtain $(v' - v)^T T' (v' - v) = 2v^T (T' - T) v$. Adding these equalities, we obtain $(v - v')^T (T + T')(v - v') = 0$, which leads to a contradiction. ■

Note that the number of spectral factors of a positive polynomial is finite. Hence above theorems state that problem (4) has a regular optimal information matrix if and only if the vector g is contained in the union of a finite number of some disjoint pointed convex cones with open relative interior. The dimension of the cone $\mathcal{L}_v[T_{++}]$ corresponding to the spectral factor v depends on the rank of the operator $\mathcal{L}_v$.

If g happens to lie in a cone $\mathcal{L}_v[T_{++}]$ such that $\mathcal{L}_v$ is rank deficient, then the optimal regular information matrix is not unique, and there exist also singular solutions. Moreover, in this situation for any spectral factor v' other than v we have $g \notin \mathcal{L}_{v'}[T_+]$. Hence there exists a neighbourhood U of g such that $U \cap \mathcal{L}_{v'}[T_+] = \emptyset$ for all spectral factors $v' \neq v$ of p. It follows that for almost all points of $g' \in U$ the corresponding optimal input design problem (4) has no regular solution. In other words, the property of existence of a regular solution is not stable in g.

This behaviour is common for convex optimization problems over cones or bodies with non-trivial facial structure.

A. Input power constraint

Let us consider the special case of an input power constraint, i.e. when $p(\omega) \equiv 1$ and M_p can be chosen equal to $\frac{1}{n+1} I_{n+1}$. This is the case studied in [5]. In this case we have exactly $2(n+1)$ different spectral factors, corresponding to the vectors $v = \pm e_k$, where e_k, $k = 0, \dots, n$ are the orthonormal basis vectors of $\mathbb{R}^{n+1}$. The rank of the operator $\mathcal{L}_{\pm e_k}$ is equal to the number of different elements in the $(k+1)$-th column of a generic real symmetric Töplitz matrix, that is $n/2 + 1 + |n/2 - k|$. We obtain the following result.

Theorem 3.5: Consider optimization problem (4) with an input power constraint. Then there exists a regular optimal information matrix if and only if g is collinear with a column of some positive definite Töplitz matrix. Some multiple of this Töplitz matrix is an optimal moment matrix. If g is collinear with the first or the last column, then the optimal information matrix is unique and is a continuous function of the problem parameters in some neighbourhood of g. If this column is not the first or the last one, then the optimal information matrix is not unique and the property of existence of a regular solution is not stable under perturbations of g. □

In [5] a special case of above theorem is proven, namely when the spectral factor corresponds to the basis vector e_0.

Above theorem does not provide the optimal solution if g is not a column of a positive definite Töplitz matrix. However, from Theorem 3.2 it follows that if g is proportional to a column of a positive semidefinite Töplitz matrix, then some multiple of this matrix must also be optimal.

Let us analyze the set of optimal solutions of problem (4) independently of the conditions of Theorem 3.5. Since the cone T_+ is the convex conic hull of the moment curve, we can factorize any matrix $T \in T_+$ as $T = \Pi D \Pi^*$, where $D = \mathrm{diag}(d_1, \dots, d_m)$ is a positive definite diagonal matrix, and $\Pi = (\pi(e^{j\omega_1}), \dots, \pi(e^{j\omega_m}))$ for some number m and some frequencies $\omega_1, \dots, \omega_m$. Note that since T is real, for any frequency ω appearing in the list the frequency $-\omega$ must also appear, and its weight in the matrix D must be identical to the weight of ω (except for the case $\omega = \pm\pi$). The optimal solution to (4) must then factorize as

$$\begin{pmatrix} \alpha & g^T \\ g & T(\mu) \end{pmatrix} = \begin{pmatrix} a^* D^{-1/2} \\ \Pi D^{1/2} \end{pmatrix} \begin{pmatrix} a^* D^{-1/2} \\ \Pi D^{1/2} \end{pmatrix}^*$$

for some complex vector a. Here again, elements of a corresponding to a frequency pair $\pm\omega$ must be complex conjugates of each other. Hence problem (4) transforms to

$$\min\, a^* D^{-1} a : \; g = \Pi a, \; tr\,(\Pi D \Pi^*) = (n+1) tr\, D = (n+1)c,$$

and subject to above restrictions on the pairwise appearance of the elements of D, a, and ω_l. Now note that we can determine D as a function of a by partial minimization over the group of variables d_l. It is not hard to see that $\mathrm{diag}\, D$ has to be proportional to the vector of absolute values of the elements of a, and $a^* D^{-1} a = (\sum_{k=1}^m |a_k|)^2/c$. Note also that for any complex number $w = re^{j\varphi}$ and any frequency $\omega \in (0, \pi)$ we have

$$\begin{aligned} & \pi(e^{j\omega})w + \pi(e^{-j\omega})\bar{w} = 2Re(\pi(e^{j\omega})w) \\ = \; & 2r(\cos\varphi, \cos(\omega+\varphi), \dots, \cos(n\omega+\varphi))^T \\ = \; & 2r\pi_{\varphi,\omega} = r\pi_{\varphi,\omega} + r\pi_{-\varphi,-\omega}, \end{aligned}$$

where the vector $\pi_{\varphi,\omega}$ is defined accordingly. Let now $a_k = r_k e^{j\varphi_k}$. Then the problem is equivalent to

$$\min\, c^{-1} \left(\sum_{k=1}^m r_k \right)^2 : \quad g = \sum_{k=0}^m r_k \pi_{\varphi_k,\omega_k}.$$

Finally, let $\mathbf{C} \subset \mathbb{R}^{n+1}$ be the convex hull of the set $C = \{\pi_{\varphi,\omega} \,|\, \omega \in [0,\pi], \; \varphi \in (-\pi, \pi]\}$. Then we finally obtain the formulation

$$\min \frac{R^2}{c} : \quad g \in R\mathbf{C}.$$

In other words, the optimal value of program (4) is determined by the largest constant β such that βg is still in the compact set $\mathbf{C}$. Any representation of the corresponding boundary point of $\mathbf{C}$ as convex combination of points π_{φ_k,ω_k}

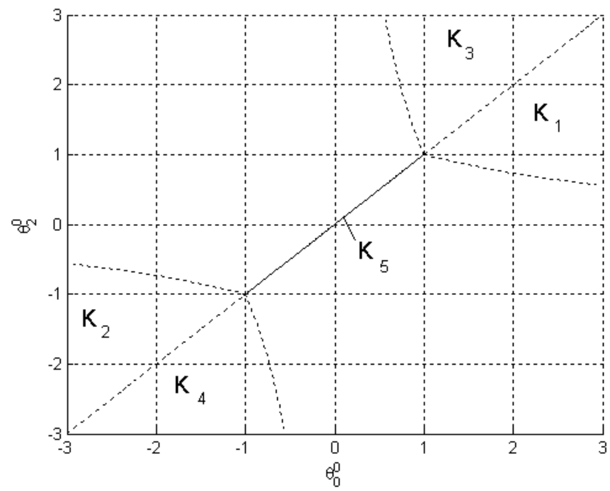

Fig. 1. Intersections of cones $K_1 - K_5$ with hyperplane $\theta_1^0 = 1$

yields an optimal information matrix. Let us formalize this result.

Theorem 3.6: Define the sets $C, \mathbf{C}$ and the constant β as above and consider problem (4). Then the optimal cost is given by $\hat{\alpha} = \frac{1}{\beta^2 c}$. Let now $\beta g = \sum_{k=1}^m \lambda_k \pi_{\varphi_k,\omega_k}$, $\lambda_k > 0$, $\sum_{k=1}^m \lambda_k = 1$ be a representation of the boundary point $\beta g \in \partial\mathbf{C}$ as a convex combination of points of the surface C. Then the Töplitz matrix $T = c\sum_{k=1}^m \lambda_k T(\mu_{\omega_k})$ is an optimal moment matrix and $\mu = c\sum_{k=1}^m \lambda_k \mu_{\omega_k}$ is an optimal moment vector. □

IV. Example

Let us illustrate above results on the example of L_2-gain estimation for a 2nd order FIR system. In this case the derivative $g = \nabla J(\theta^0)$ is proportional to the true parameter vector θ^0. The set of parameter vectors θ^0 for which a regular optimal information matrix exists is the union of 6 convex cones, namely

$$\begin{aligned} K_1 &= \{\theta^0 \,|\, |\theta_1^0| < \theta_0^0, \; \theta_2^0 \in (-(\theta_0^0 - 2(\theta_1^0)^2/\theta_0^0), \theta_0^0)\}, \\ K_2 &= -K_1, \\ K_3 &= \{\theta^0 \,|\, |\theta_1^0| < \theta_2^0, \; \theta_0^0 \in (-(\theta_2^0 - 2(\theta_1^0)^2/\theta_2^0), \theta_2^0)\}, \\ K_4 &= -K_3, \\ K_5 &= \{\theta^0 \,|\, \theta_0^0 = \theta_2^0, \; \theta_1^0 > |\theta_0^0|\}, \\ K_6 &= -K_5. \end{aligned}$$

The first two cones correspond to the situation when θ_0^0 is proportional to the first column of a positive definite 3×3 Töplitz matrix, the next two cones to proportionality to the last column and the last two cones to proportionality to the second column. In [5] it was shown that for $\theta^0 \in K_1$ there exists a regular solution. Figure 1 shows the intersections of the cones K_1 to K_5 with the hyperplane $\theta_1^0 = 1$. The cone K_6 has an empty intersection with this hyperplane.

If θ^0 is in one of the cones K_1 to K_4, then the optimal solution is regular and unique. Let us now consider a point in the cone K_5. Let $a > |b|$ and $\theta^0 = (b, a, b)^T$. Then the moment vector μ is optimal if and only if

$$T(\mu) = \begin{pmatrix} c & bc/a & x \\ bc/a & c & bc/a \\ x & bc/a & c \end{pmatrix} \qquad (7)$$

and x is chosen such that $T(\mu) \succeq 0$, i.e. $x/c \in [-(1 - 2(b/a)^2), 1]$. For x in the interior of this interval the matrix $T(\mu)$ is regular, for x at an endpoint $T(\mu)$ is singular. A similar situation holds if $\theta^0 \in K_6$.

If θ^0 is not contained in one of the cones K_1 to K_6, then the set of optimal solutions can be described by Theorem 3.6. It is not hard to check that in this case the one-dimensional linear subspace generated by the vector θ^0 in $\mathbb{R}^3$ intersects the boundary of the set $\mathbf{C}$ in two opposite points of the generating surface C, and there exist supporting planes to $\mathbf{C}$ at these points that intersect C only at the points themselves. Therefore there exists a unique representation of these boundary points as a convex combination of points of C, namely that putting all weight in the point itself. As a consequence, the optimal information matrix is unique and singular. The optimal moment vector is proportional to μ_ω, where the frequency ω is determined by the condition

$$\theta^0 = \beta \begin{pmatrix} \cos\varphi \\ \cos(\varphi + \omega) \\ \cos(\varphi + 2\omega) \end{pmatrix}$$

for some constants $\beta > 0$ and φ. It is not hard to check that this yields $\cos\omega = \frac{\theta_0^0 + \theta_2^0}{2\theta_1^0}$ if the generic case $\theta_1^0 \neq 0$, $\theta_0^0 \neq \theta_2^0$ holds.

Let us now return to the case $\theta^0 = (b, a, b)^T \in K_5$. From the above it can be seen that if the parameter vector approaches this particular value from outside of the cone K_5, then the optimal solution will tend to expression (7) with $x/c = -(1 - 2(b/a)^2)$, that is to a singular matrix, which is an endpoint of the interval of optimal solutions for θ^0. This means that the choice of an input signal based on a regular optimal information matrix is not robust. We have to stress that we mean robustness here in the sense of stability of the optimal solution to problem (4) with respect to perturbations of the true parameter values. A choice of a singular information matrix might of course have other undesirable consequences, for instance due to second order terms in the Taylor expansion of the quantity to be identified.

V. Conclusions

In this contribution we considered the problem of optimal input design for FIR system identification. The input design is tailored to identify a scalar function of the system parameters under the constraint that the convolution of the input power spectrum with some positive trigonometric polynomial is bounded. Constraints on the total energy of the input signal fall into this category. We provided a necessary and sufficient optimality condition (Theorem 3.2), which generalizes the optimality condition in [5] and can be interpreted as the complementarity condition of convex duality. Using this optimality condition, we provided descriptions of the sets of optimal solutions in dependence of the problem parameters. In particular, we characterized the cases when a regular optimal information matrix exists (Theorem 3.3), when it is unique, and computed the dimension of the solution set (Theorem 3.4). These results were specialized to the case of an input power constraint in Theorem 3.5. For this special case we gave a description of the set of optimal solutions for arbitrary parameter vectors (Theorem 3.5).

Our results revealed a number of surprising phenomena. Namely, if for some instance of the input design problem there exists a regular optimal information matrix, then it might cease to exist if the problem parameters are perturbed. This happens if and only if the regular solution is not unique. This has consequences for the robustness properties of the optimal input design. So we showed in an example that the choice of an optimal regular information matrix might pose robustness problems if there exist also singular solutions, however paradox this might seem.

References

[1] Xavier Bombois, Gerard Scorletti, Michel Gevers, Roland Hildebrand and Paul Van Den Hof. Least costly identification experiment for control. Automatica, 42(10):1651-1662, 2006.

[2] Michel Gevers, Xavier Bombois. Input design: From open-loop to control-oriented design System Identification. Proc. of the 14th IFAC Symposium on System Identification, Newcastle, Australia, 2006.

[3] Graham Goodwin and Robert Payne. Design and Characterization of Optimal Test Signals for Linear SISO Parameter Estimation. Preprints of 3rd IFAC Symposium, Paper TT-1, The Hague, 1973.

[4] Roland Hildebrand and Michel Gevers. Identification for control: Input design with respect to the worst-case ν-gap. SIAM Journal on Control and. Optimization, 41(5):15861605, 2003.

[5] Hakan Hjalmarsson, Jonas Martensson, Bo Wahlberg. On some robustness issues in input design. Proc. of the 14th IFAC Symposium on System Identification, Newcastle, Australia, 2006.

[6] Henrik Jansson. Experiment design with applications in identification for control. Doctoral Thesis, KTH, Dept. of signals, sensors and systems, Stockholm, Sweden, 2004.

[7] Henrik Jansson and Hakan Hjalmarsson. Input design via LMIs admitting frequency-wise model specifications in confidence regions. IEEE Transact. for Automatic Contr. 50(10):1534-1549, 2005.

[8] Samuel Karlin and William Studden. Tchebycheff systems: with applications in analysis and statistics. Pure Appl. Math., vol. XV. Interscience Publishers, 1966.

[9] Lennart Ljung. System identification: theory for the user. Prentice Hall Information and system science series, Prentice Hall, sec. ed., 1999.

[10] Raman Mehra. Optimal Inputs for Linear System Identification. IEEE Trans. Automat. Control 19(3):192-200, 1974.

[11] Yuri Nesterov and Arkadi Nemirovski. Interior-point polynomial methods in convex programming. Studies in Applied Mathematics, vol. 13, SIAM, Philadelphia 1994.

[12] R. Tyrell Rockafellar. Convex analysis. PUP, Princeton, second edition, 1970.

[13] Cristian R. Rojas, James S. Welsh, Graham C. Goodwin and Arie Feuer. Robust optimal experiment design for system identification. Automatica, 43(6):993-1008, 2007.

[14] Martin Zarrop. Optimal experiment design for dynamic system identification. Lecture Notes in Control and Inform. Sci. vol. 21, Springer, Berlin, New York, 1979.