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Practical Stabilization via Relay Delayed Control

A. Polyakov

Abstract— **Two problems of practical stabilization (local and semiglobal) via relay delayed control are considered. Control design algorithms for linear plants operating under uncertainty conditions are proposed. The necessary and sufficient existence conditions of stabilizing relay delayed controls are given. The stabilization problem of an inverted pendulum controlled by flywheel is considered as numerical example.**

I. Introduction

Relay control systems occur in many industrial applications. They are simple in realization, cheap, very effective and sometimes have better dynamic than traditional linear systems [1]. *Relay* nature may be inherent in both sensors and controllers. For example, the HEGO-sensor in the air-to-fuel ratio control system of automobile engine is a relay measurer [2], but the control systems of electric drives [3] have *on-off* "switches" as relay control inputs. The preferable control strategy essentially depends on the device (sensor and/or controller) having the relay nature.

On the other hand, *time delays* that usually take a place in feedback control systems can not be ignored, because they lead to "unmodelled" oscillations (such as "chattering" [3]) and/or system insatiability [4]. This phenomenon is typical for relay control systems [3], [5]. Presence of time delays together with the system uncertainties (such as external disturbances, errors in system parameters estimations, unknown and variable time delay) make the problem of the control design and the stability analysis of the relay control systems essentially complex.

Time delay compensation (or prediction) and the adaptive control of system oscillations are two modern approaches to the problem of control design for time delay systems. In [6], [7], [8] some implementations of prediction method for sliding mode control design can be found. However, the proposed technique *does not allow* to realize sliding mode in the system state space [9]. It ensures the sliding motion *only* in the predictor space [10] and leaves the system state oscillations produced by uncertainties without consideration and estimations.

Methods for relay control of the system oscillations can be found in [11], [12], [13]. They are based on control gain adaptation and need multi-step property of control inputs (i.e. each relay control input may have some finite or discrete set of values). This property does not hold for traditional on-off switching systems. Moreover, all existed relay control algorithms of systems oscillations are presented only for the very restricted class of the plants. In the same time, the methods of the relay control of the system oscillations have two very important practical advantages: they are *robust* with respect to time delay variations and allow *easily* to achieve the estimations of convergence time, control accuracy and attraction domain in the explicit form. So, this study is caused by necessity of extension of the relay oscillation control methods to the general control systems with the on-off relay control inputs.

In the next section the problem formulation, basic definitions and assumptions are presented. Then the necessary existence conditions of the stabilizing control for considered system are given and discussed. After then the control design algorithms and the corresponding theorems of practical stabilization are proposed. Finally, the stabilization problem of an inverted pendulum controlled by flywheel is considered and the simulation results of relay delayed control implementation are graphically illustrated.

II. Problem Formulation and Basic Assumptions

A. Main System

Let us consider the control system with time delay of the form

$$\dot{x}(t) = Ax(t) + Bu(x(t - h(t))) + f(t, x(t)) \tag{1}$$

where $x \in \mathrm{R}^n$ is a state space vector, the system matrix $A \in \mathrm{R}^{n\times n}$ is admitted to be *unstable*, $B \in \mathrm{R}^{n\times m}$ is a control gain matrix, $u \in \mathrm{R}^n$ is a vector of control inputs, $h(t)$ is time delay and the unknown function $f(t, x(t))$ describes system uncertainties.

We suggest that the full state space vector is available for measurement with *unknown but bounded time delay* $h(t)$

$$0 \leq h(t) \leq h_0 \tag{2}$$

where h_0 is known. The function $h(t)$ is supposed to be piece-wise continuous. The system (1) is considered under initial conditions of the form

$$x(t) = \varphi(t), \text{ for } t \in [-h_0, 0]$$

where $\varphi(t)$ is an arbitrary function of time.

The control $u(\cdot)$ in the system (1) is *relay*

$$u(\cdot) = (-p_1 \operatorname{sign}[S_1(\cdot)], ..., -p_m \operatorname{sign}[S_m(\cdot)])^T \tag{3}$$

where the positive parameters $p_i > 0, (i = 1, 2, ..., m)$ and the *linear* mappings $S_i : \mathrm{R}^n \to \mathrm{R}, (i = 1, 2, ..., m)$ should

A. Polyakov is with Department of Applied Mathematics and Mechanics, Voronezh State University, Universitetskaya sq. 1, Voronezh, Russia, 394006, polyakov@amm.vsu.ru (Postdoc position in Automatic Control Department, CINVESTAV-IPN, Av. IPN 2508, Mexico D.F., C.P. 07360, apolyakov@ctrl.cinvestav.mx)

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be designed. The operator $\mathrm{sign}[\cdot]$ is defined by

$$\mathrm{sign}[z] := \begin{cases} 1 & \text{if} \quad z > 0 \\ -1 & \text{if} \quad z < 0 \\ \in [-1, 1] & \text{if} \quad z = 0 \end{cases} \tag{4}$$

B. Basic Assumptions

We consider the system (1) under the following assumptions:

- the pair $\{A, B\}$ is controllable
$$rank\left[B, AB, ..., A^{n-1}B\right] = n \tag{5}$$
- the matching condition
$$f(t, x(t)) = B\gamma(t, x(t)) \tag{6}$$
holds for the uncertain function $f(t, x(t))$, where the unknown function $\gamma(t, x(t))$ is assumed to be bounded
$$|\gamma_i(t, x)| \le k_i, \text{ for } \forall t \ge 0 \text{ and } \forall x \in \mathrm{R}^n \tag{7}$$
by known nonnegative constants $k_i \ge 0, i = 1, 2, ..., m$;
- the matrix B is full rank matrix with the following representation (after reordering the state space vector components)
$$B = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} \tag{8}$$
where $B_1 \in \mathrm{R}^{(n-m)\times m}, B_2 \in \mathrm{R}^{m\times m}, \det(B_2) \ne 0;$
- the system has sufficiently slow uncontrolled rotational dynamics, i.e.
$$h_0 \,\mathrm{Im}(\mathrm{spec}(A)) < \frac{\pi}{3} \tag{9}$$
where $\mathrm{spec}(A)$ is a spectrum of the matrix A.

C. Definitions of Practical Stability and Main Problem

All existed control algorithm for uncertain relay delayed systems don't guarantee system stability in traditional sense [10], [7], [11], [12]. They give only *practical* stability such as convergence to zone. So, below we introduce two special definitions of practical stability for relay delayed control systems.

Definition 1: The system (1) is called ε - stabilizable, if for some fixed $\varepsilon > 0$ there exist the control $u(\cdot)$ of the form (3) and the number $\delta > 0$, such that any solution $x_\varphi(t)$ of the system (1) with initial function $\varphi(t) : \|\varphi(0)\| < \delta$ is bounded

$$\|x_\varphi(t)\| < \varepsilon \text{ for } \forall t \ge 0$$

In the other words, the system (1) is ε - stable, if the designed control holds any system solution inside the given ε - neighborhood of the origin. Such system motion is typical for relay time delayed systems [5] and completely different from Lyapunov stability, since the control $u(\cdot)$ and number $\delta > 0$ may not exist for all $\varepsilon > 0$.

Definition 2: The system (1) is called $R\varepsilon$ - stabilizable, if for some fixed $\varepsilon > 0$ and fixed $R > \varepsilon$ there exist the control $u(\cdot)$ of the form (3) and time moment $T > 0$, such that any solution $x_\varphi(t)$ of the system (1) with initial function $\varphi(t) : \|\varphi(0)\| < R$ *converges to zone* ε *in a finite time* T

$$\|x_\varphi(t)\| < \varepsilon \text{ for } \forall t \ge T$$

This stability form is similar to semiglobal *stability introduced* by Isidory [14] with only one difference: asymptotic convergence to origin replaced by finite time convergence to zone. Therefore, it can be also called as *practical semiglobal stability* [12].

The main problem is to propose the relay delayed control design algorithms, which guarantee stabilization of the system (1)-(3) in the sense of the given definitions and to find the necessary and sufficient existence conditions of the stabilizing relay delayed control.

D. Notations

Many real control application require a specification of the attraction domain and the control error on each component of the state vector separately. Moreover, usually only one part of the system state vector describes the generalized coordinates. Another part defines their speed, acceleration, etc (see as example any mechanical or electromechanical system).

Therefore, it is preferable to introduce the generalized vector norm of the form $\|x\| := (|x_1|, |x_2|, ..., |x_n|)^T$, where $x \in \mathrm{R}^n$. The space R^n is semiordered in this case (see, for example, [15]), i.e. $x \le y (x < y)$ if and only if $x_i \le y_i (x_i < y_i)$ for all $i = 1, 2, ..., n$. Then all inequalities in the definitions 1,2 should be considered in the presented way. The parameters R, ε and δ in this case are *the positive vectors* from $\mathrm{R}^n : R, \varepsilon, \delta \in \mathrm{R}^n_+ := \{y \in \mathrm{R}^n : y_i > 0\}$.

The corresponding generalization for matrix norm is the modulus of the matrix $|A| := \{|a_{ij}|\}, i = 1, 2, ..., n; j = 1, 2, ..., m$, where $A = \{a_{ij}\} \in \mathrm{R}^{n\times m}$. In the matrix case the ordering signs $\le, <, \ge$ and $>$ will be also considered below *in a component-wise sense*.

III. Necessary Existence Conditions of the Stabilizing Relay Delayed Control

In [5] it was given the necessary condition

$$\lambda h_0 < \ln(2) \tag{10}$$

of the existence of *nontrivial bounded solutions* for a *scalar* system

$$\dot{x} = \lambda x - p\,\mathrm{sign}[x(t - h_0)], \lambda, p, h_0 > 0 \tag{11}$$

Proposed below theorem presents the generalization of this result to the clase of the relay delayed control systems (the proofs of all propositions below are given in Appendix).

Theorem 1: The condition

$$h_0 \,\mathrm{Re}(\mathrm{spec}(A)) < \ln(2)$$

is necessary for the existence of the control (3) realizing the ε - or $R\varepsilon$ - stabilization of the system (1) under assumptions (5)-(9).

The stabilizing sliding mode controller for the system (1) also has the form (3) in relay case. Therefore, Theorem 1 also presents the restrictions for sliding mode control implementation to time delay systems.

IV. Relay Delayed Control Design

Using transformation $(x_1, x_2)^T = Gx$

$$G = \begin{pmatrix} I & -B_1B_2^{-1} \\ 0 & B_2^{-1} \end{pmatrix}$$

the initial system (1) can be reduced to the *regular* form

$$\begin{cases} \dot{x}_1(t) = A_{11}x_1(t) + A_{12}x_2(t) \\ \dot{x}_2(t) = A_{21}x_1(t) + A_{22}x_2(t) + u_h + \gamma(t, x(t)) \end{cases} \quad (12)$$

where $A_{11} \in \mathrm{R}^{(n-m)\times(n-m)}$, $A_{12} \in \mathrm{R}^{(n-m)\times m}$, $A_{21} \in \mathrm{R}^{m\times(n-m)}$, $A_{22} \in \mathrm{R}^{m\times m}$ are the blocks of the system matrix and $u_h = u(x(t - h(t)))$ has the form (3).

The controllability of the pair $\{A, B\}$ implies that $\{A_{11}, A_{12}\}$ is also controllable [3]. So, we may always choose the matrix $C \in \mathrm{R}^{m\times(n-m)}$, such that $\mathrm{Re}(\mathrm{spec}(A_{11} - A_{12}C)) < 0$.

Let us introduce the number $\mu > 0$ and the matrix with positive elements H, such that

$$\left|e^{t(A_{11}-A_{12}C)}\right| \leq e^{-\mu t}H \text{ for all } t \geq 0 \quad (13)$$

Since $A_{11} - A_{12}C$ is Hurwitz, $\mu > 0$ and $H > 0$ can be always found. Denote

$$D = \mathrm{diag}\{d_1, d_2, ..., d_m\} \quad (14)$$

where $d_i = \max\{0, (A_{11} - A_{12}C)_{ii}\}, i = 1, 2, ..., m$.

Theorem 2: Consider the system (1)-(3) under assumptions (5)-(9). For a given $\varepsilon \in \mathrm{R}_+^n$ let the positive vector ε_2 be

$$\varepsilon_2 \in \mathrm{R}_+^m : \left| \begin{matrix} I - B_1C & B_1 \\ B_2C & B_2 \end{matrix} \right| \begin{pmatrix} E_H|A_{12}| \\ I \end{pmatrix} \varepsilon_2 \leq \varepsilon$$

and the vector $\varepsilon_1 \in \mathrm{R}_+^{n-m}$ is defined as $\varepsilon_1 = E_H|A_{12}|\varepsilon_2$, where $E_H = \mu^{-1}\left[I - \frac{0.5}{\lambda_{\max}(H)}H\right]^{-1}H$ and $\lambda_{\max}(H)$ is "maximal" eigenvalue[1] of the matrix H. Then the conditions

1) $d_ih_0 < \ln(2), i = 1, 2, ..., m$;
2) $k + |Q_1|\varepsilon_1 + |Q_2|\varepsilon_2 < (I - 0.5e^{h_0D})E_D^{-1}\varepsilon_2$;

are sufficient for the ε - stability of the system (1) with the control

$$u_h = -\,\mathrm{diag}(p_1, ..., p_m)\,\mathrm{sign}\left[(C \quad I)\,Gx(t - h(t))\right] \quad (15)$$

for any $\delta \in \mathrm{R}_+^n$:

$$\left| \begin{matrix} I & -B_1B_2^{-1} \\ C & (I - CB_1)B_2^{-1} \end{matrix} \right| \delta \leq \begin{pmatrix} \frac{0.5}{\lambda_{\max}(H)}\varepsilon_1 \\ \varepsilon_2 - 2e^{-h_0D}E_Dp \end{pmatrix} \quad (16)$$

where the vector $k = (k_1, ..., k_m)^T$ is given by (7), $Q_1 = A_{12} - A_{22}C + CA_{11} - CA_{12}C$, $Q_2 = A_{22} + CA_{12} - D$, $E_D = (e^{h_0D} - I)D^{-1}$ and the vector $p = (p_1, ..., p_m)^T$ has the form $p = k + |Q_1|\,\varepsilon_1 + (|Q_2| + D)\varepsilon_2$.

Formally the matrix $E_D = \mathrm{diag}\left\{(e^{d_ih_0} - 1)/d_i\right\}_{i=1}^m$ is undefined if some $d_j \equiv 0, j \in \{1, 2, ..., m\}$. The corresponding term of the matrix E_D in this case can be obtained as $\lim\limits_{d_j\to 0}(e^{d_jh_0} - 1)/d_i = h_0$.

[1]From the Perron's theorem, the matrix with positive elemets has, so-called, "maximal" eigenvalue, which is real, simple, positive and greater then the modulus of all remaining eigenvalues.

It is easy to see that for $h_0 \to 0$ all constraints in the last theorem disappear and we became well-known relay sliding mode controller [3]. The presented theorem does not describe the system motion in the ε - neighborhood. So, in practice it may be periodic, asymptotic or even chaotic one.

The semiglobal case requires more strict stabilization conditions.

Theorem 3: Consider the system (1)-(3) under assumptions (5)-(9). For a given $R \in \mathrm{R}_+^n$ let the positive vectors R_1 and R_2 be

$$\begin{matrix} R_1 \in \mathrm{R}_+^{n-m} \\ R_2 \in \mathrm{R}_+^m \end{matrix} : \begin{pmatrix} R_1 \\ R_2 \end{pmatrix} = \left| \begin{matrix} I & -B_1B_2^{-1} \\ C & (I - CB_1)B_2^{-1} \end{matrix} \right| R$$

For a given $\varepsilon \in \mathrm{R}_+^n : \varepsilon < R$ let the positive vector ε_2 be

$$\varepsilon_2 \in \mathrm{R}_+^m : \left| \begin{matrix} I - B_1C & B_1 \\ B_2C & B_2 \end{matrix} \right| \begin{pmatrix} E_H|A_{12}| \\ I \end{pmatrix} \varepsilon_2 \leq \varepsilon$$

and the vector $\varepsilon_1 \in \mathrm{R}_+^{n-m}$ has the form $\varepsilon_1 = E_H|A_{12}|\varepsilon_2$. Then the conditions

1) $d_ih_0 < 0.5\ln(2), i = 1, 2, ..., m$;
2) $f^{\max} + De^{h_0D}R_2 < (I - 0.5e^{h_0D})E_D^{-1}\varepsilon_2$

are sufficient for the $R\varepsilon$ - stability of the system (1) with the control (15) and the following convergence time estimation holds

$$T \leq T_1 + T_2 \quad (17)$$

$$T_1 = h_0 + \max_{i=1,...,m}\{R_{2i}^{\max}/(p_i - d_iR_{2i}^{\max} - f_i^{\max})\}$$

$$T_2 = \mu^{-1}\max_{i=1,...,n-m}\{\ln(2\lambda_{\max}(H)R_{1i}/\varepsilon_{1i})\}$$

where $f^{\max} = k + |Q_1|R_1^{\max} + |Q_2|R_2^{\max}$, $R_2^{\max} = e^{h_0D}R_2 + \varepsilon_2$, $R_1^{\max} = H(R_1 + \mu^{-1}|A_{12}|R_2^{\max})$, the matrices Q_1, Q_2, E_H, E_D are defined as in Theorem 2 and the control gain vector $p = (p_1, ..., p_m)^T$ is taken from the interval

$$DR_2^{\max} + f^{\max} < p < E_D^{-1}\varepsilon_2 - f^{\max} \quad (18)$$

As in ε - stabilization case, the conditions of the last theorem vanish for $h_0 \to 0$ and semiglobal stabilization is possible for any $R > \varepsilon > 0$ with convergence time formed of two intervals T_1 and T_2 corresponded to *finite reaching time [3]* and *finite time convergence to zone* ε, respectively.

The constraint $d_ih_0 < 0.5\ln(2)$ of Theorem 3 means that time delay, in the case of semiglobal stabilization, should be half the local one. In simplest scalar case (11), this restriction is needed to guarantee the inclusion of steady motion domain (ε - neighborhood) into an attraction domain (R - neighborhood).

Theorems 2 and 3 can be easily extended to the class of the *saturated linear* delayed feedback controls. Unfortunately, they do not allow to guarantee the asymptotic convergence of the system (1) to the equilibrium point. To show this an additional research will be needed.

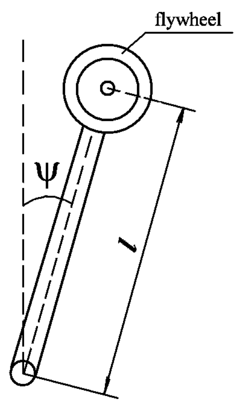

Fig. 1. Inverted pendulum with flywheel

V. Numerical Example

Consider the stabilization problem of an inverted pendulum controlled by flywheel (see, Fig.1). The flywheel is fixed on the rotor of DC motor and located on the end of pendulum link. The corresponding linearized dynamitic system has the form [16]

$$\left\{ \begin{array}{l} J\ddot{\psi} + (J_r + J_f)\dot{\omega} = (Md + ml)g\psi \\ (J_r + J_f)(\ddot{\psi} + \dot{\omega}) = c_1 u - c_2\omega + \gamma \end{array} \right. \tag{19}$$

where ψ is a pendulum inclination angle, ω is a flywheel angular speed, $J = J_p + ml^2 + J_r + J_f$, $J_p = 1.2 \times 10^{-3}$ [kg m^2] - inertia of the pendulum, $J_r = 1.2 \times 10^{-6}$ [kg m^2] - inertia of the rotor, $J_f = 7.65 \times 10^{-5}$ [kg m^2] - inertia of the flywheel, $M = 0.04$ [kg] - pendulum mass, $m = 0.13$ [kg] is sum of the flywheel and DC motor masses, $l = 0.3$ [m] - pendulum length, $d = 0.15$ [m] - distance from fixing point to the center mass of the pendulum, $c_1 = 6.9 \times 10^{-3}$ [N m / V] and $c_2 = 1.0 \times 10^{-4}$ [N m sec] are the parameters of DC motor rotating the flywheel and $u = \pm 19$ V is control voltage, $g = 9.8$ [m / sec^2]. Here it is used the simplified model of the DC motor [16] and $\gamma : |\gamma| < 0.01$ [N m] describes the system uncertainties produced by this simplification.

The initial system (19) can be rewritten in the form (1) with $x = (x_1, x_2, x_3)^T = (\psi, \dot{\psi}, \omega)^T$ and

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 34.1860 & 0 & 0.0078 \\ -34.1860 & 0 & -1.3098 \end{pmatrix}, B = \begin{pmatrix} 0 \\ -0.5349 \\ 90.3786 \end{pmatrix}$$

Since, the eigenvalues of the matrix A are $\lambda_1 = 5.8437, \lambda_2 = -5.8519, \lambda_3 = -1.3017$, from Theorem 1 it follows that the system (19) is ε - or $R\varepsilon$ - stabilizable *only* in the case $h_0 < 0.118$ [sec]. As it was remarked above the $R\varepsilon$ - stabilization of the considered system requires at least twice less time delay: $h_0 < 0.059$. Select $C = (-13.374, -2.2886)$ and define the relay delayed feedback control in the form

$$u(x(t-h(t))) = 19\,\mathrm{sign}[(13.374, 2.2886, 0.0025)x(t-h(t))]$$

Fig. 2-4 show the converging process and Fig.5 illustrates the control function for the case $R = (0.15, 0.4, 80), \varepsilon = (0.01, 0.23, 45)^T$, $h(t) = 0.015 + 0.005\sin(0.1t)$ and $\gamma(t) = 0.01\cos(10t)$.

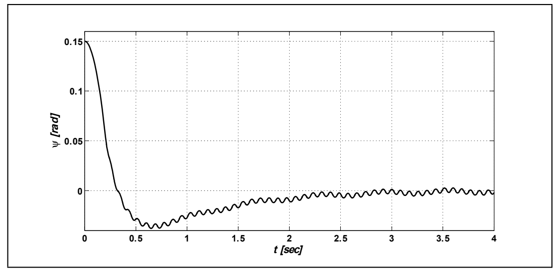

Fig. 2. Pendulum inclination angle $\psi(t)$

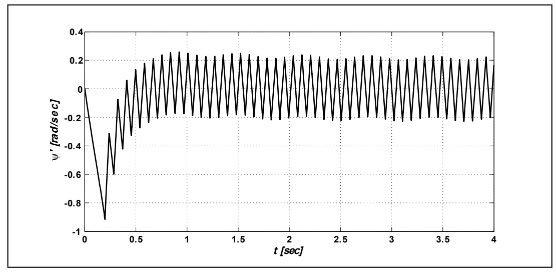

Fig. 3. Pendulum angular speed $\dot{\psi}(t)$

VI. Conclusion

In this paper the stabilization problem via the relay delayed control is considered. The necessary control existence condition is presented. This condition also gives the restrictions for sliding mode control realization for time delayed systems. Two control design algorithms guaranteeing the practical stabilization of the linear uncertain plants are proposed. The robustness conditions of considered controls with respect to time delay and small external disturbances are given. The estimation of the time delay permitting the stabilization of the inverted pendulum controlled by flywheel is obtained for the first time. The pendulum stabilization process via relay delayed control is graphically illustrated.

VII. Appendix

A. Proof of Theorem 1

Suppose a contradiction(i.e. there exists an eigenvalue $\lambda_0 : h_0 \operatorname{Re}(\lambda_0) \geq \ln(2)$) and show that in this case for any $\delta > 0$, any finite $p_i > 0$ and any linear mappings $S_i(\cdot)$, $(i =$

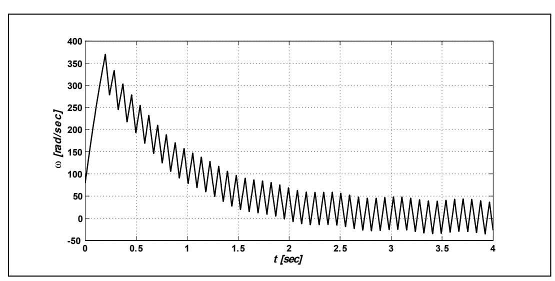

Fig. 4. Rotor(Flywheel) angular speed $\omega(t)$

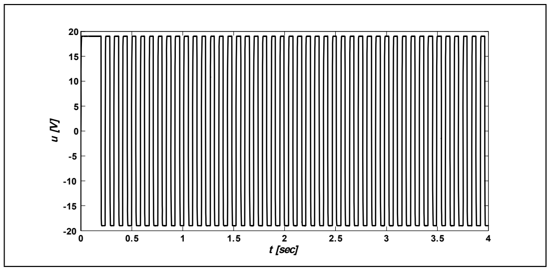

Fig. 5. Control law $u(t)$

$1, 2, ..., m)$ there exists an initial function $\varphi(t) : \|\varphi(0)\| < \delta$, such that the corresponding solution $x_\varphi(t)$of the system (1) is unbounded on the interval $[0, +\infty)$ even in the non-perturbed case: $f(t, x(t)) \equiv 0$ and $h(t) \equiv h_0$.

Then, using a real Jordan transformation $x = Gy$, the system (1) can be rewritten in the form

$$\dot{y}(t) = Jy(t) + \tilde{B}u(x(t - h_0)) \tag{20}$$

where $J = G^{-1}AG$ is the Jordan matrix and $\tilde{B} = G^{-1}B$.

Consider two possible cases.

I. Let the eigenvalue λ_0 is real. The corresponding equation from the system (20) has the form

$$\dot{y}_0 = \lambda_0 y_0 + \tilde{b}_0 u(x(t - h_0)) \tag{21}$$

where $y_0 \in \mathrm{R}$ is component of the vector y with number i_0, which corresponds to the considered eigenvalue λ_0, $\tilde{b}_0$ is the corresponding row of the matrix $\tilde{B}$. Since pair $\{A, B\}$ is controllable, $\tilde{b}_0 \neq 0$. Moreover, due to linearity of $S_i(\cdot)$ and the relay form of the control we have

$$\tilde{b}_0 u(\alpha x) = \mathrm{sign}[\alpha]\tilde{b}_0 u(x) \text{ for } \forall x \in \mathrm{R}^n \text{ and } \forall \alpha \in \mathrm{R}$$

Hence, obviously, for any finite $p_i > 0$ and any linear mappings $S_i(\cdot)$, $(i = 1, 2, ..., m)$ it can be found the number $p_0 > 0$ and the vector $x_{\max} \in \mathrm{R}^n$, such that

$$-p_0 = \tilde{b}_0 u(-x_{\max}) \leq \tilde{b}_0 u(x) \leq \tilde{b}_0 u(x_{\max}) = p_0$$

for $\forall x \in \mathrm{R}^n$.

Now, let us define the vector $x_0 = G(0, ..., 0, \frac{\delta}{2}, 0, ..., 0)^T$ and select the initial function $\varphi(t)$ as

$$\varphi(t) = \begin{cases} x_{\max} & \text{if} \quad t \in [-h_0, 0) \\ x_0 & \text{if} \quad t = 0 \end{cases}$$

Then the equation (21) on the interval $[0, h_0)$ has the form

$$\dot{y}_0 = \lambda_0 y_0 + p_0, y_0(0) = \frac{\delta}{2}$$

Hence, $y_0(h_0) = (\frac{\delta}{2} + \frac{p_0}{\lambda_0})e^{\lambda_0 h_0} - \frac{p_0}{\lambda_0}$ and from $\lambda_0 h_0 \geq \ln(2)$ we have

$$\begin{gathered} \dot{y}_0(h_0) = (\lambda_0 \tfrac{\delta}{2} + p_0)e^{\lambda_0 h_0} - p_0 + \tilde{b}_0 u(x(t - h_0)) \\ \geq \lambda_0 \tfrac{\delta}{2} e^{\lambda_0 h_0} + p_0(e^{\lambda_0 h_0} - 2) \geq \lambda_0 \tfrac{\delta}{2} e^{\lambda_0 h_0} > 0 \\ \text{for } \forall p_0 > 0 \text{ and } \forall \delta > 0 \end{gathered}$$

Since $\tilde{b}_0 u(x(t - h_0)) \geq -p_0$, the last inequality also holds for all $t > h_0$ and $y_0(t)$ is unbounded increasing function.

II. In the case of the complex eigenvalue $\lambda_0 = \alpha + i\beta$ with $\alpha h_0 \geq \ln(2)$ we come to the system

$$\begin{cases} \dot{y}_1(t) = \alpha y_1(t) - \beta y_2(t) + \tilde{b}_1 u(x(t - h_0)) \\ \dot{y}_2(t) = \beta y_1(t) + \alpha y_2(t) + \tilde{b}_2 u(x(t - h_0)) \end{cases} \tag{22}$$

Let the vector $x_{\max}$ and the number $\psi_{\max}$ be defined from

$$\max_{\forall \psi \in [0, 2\pi], \forall x \in R^n} g(\psi, x) = g(\psi_{\max}, x_{\max})$$

where $g(\psi, x) = \cos(\psi)\tilde{b}_1 u(x) + \sin(\psi)\tilde{b}_2 u(x)$. It is easy to see that

$$\min_{\forall \psi \in [0, 2\pi], \forall x \in R^n} g(\psi, x) = -g(\psi_{\max}, x_{\max}) = g(\psi_{\max}, -x_{\max})$$

Let us show that for the initial function

$$\varphi(t) = \begin{cases} x_{\max} & \text{if} \quad t \in [-h_0, 0) \\ 0 & \text{if} \quad t = 0 \end{cases}$$

the corresponding solution of the subsystem (22) is unbounded. The solution on the interval $t \in [0, h_0]$ has the form

$$\begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} = \int_0^t e^{\alpha\tau} \begin{pmatrix} \cos(\beta\tau)p_1 - \sin(\beta\tau)p_2 \\ \sin(\beta\tau)p_1 + \cos(\beta\tau)p_2 \end{pmatrix} d\tau$$

where $p_1 = \tilde{b}_1 u(x_{\max})$ and $p_2 = \tilde{b}_2 u(x_{\max})$. Hence,

$$y_1^2(h_0) + y_2^2(h_0) = \frac{p_1^2 + p_2^2}{\alpha^2 + \beta^2} \left(e^{2\alpha h_0} - e^{\alpha h_0}\cos(\beta h_0) + 1\right)$$

On the over hand, we may rewrite the system (22) in polar coordinates using the following transformations: $y_1(t) = \rho(t)\cos(\psi(t))$ and $y_2(t) = \rho(t)\sin(\psi(t))$

$$\begin{cases} \dot{\rho}(t) = \alpha\rho(t) + \cos(\psi(t))\tilde{b}_1 u + \sin(\psi(t))\tilde{b}_2 u \\ \dot{\psi}(t) = \beta + \left(\sin(\psi(t))\tilde{b}_1 u - \cos(\psi(t))\tilde{b}_2 u\right)/\rho(t) \end{cases}$$

Since,

$$\rho^2(h_0) = y_1^2(h_0) + y_2^2(h_0) > \frac{p_1^2 + p_2^2}{\alpha^2}$$

then

$$\dot{\rho}(h_0) \geq \alpha\rho(h_0) - g(\psi_{\max}, x_{\max}) \geq \alpha\rho(h_0) - \sqrt{p_1^2 + p_2^2} > 0$$

and the solution of the subsystem is unbounded on the interval $[0, +\infty)$.

B. Proof of Theorem 2

First of all, remark that the matrix $I - \frac{0.5}{\lambda_{\max}(H)}H$ is invertible and from the Banach inverse mapping theorem

$$\left[I - \frac{0.5}{\lambda_{\max}(H)}H\right]^{-1} = \sum_{i=0}^{\infty} \left(\frac{0.5}{\lambda_{\max}(H)}H\right)^i \geq 0$$

I. Rewrite the system (12) in variables x_1 and $y = Cx_1 + x_2$

$$\begin{cases} \dot{x}_1 = (A_{11} - A_{12}C)x_1 + A_{12}y \\ \dot{y} = Dy + u_\varepsilon + \gamma + Q_1 x_1 + Q_2 y \end{cases} \tag{23}$$

and show that $\|x_1(t)\| < \varepsilon_1$ and $\|y(t)\| < \varepsilon_2$.

Using the Cauchy formula we obtain the inequality

$$\|x_1(t)\| \leq \left|e^{t(A_{11}-A_{12}C)}\right| \|x_1(0)\| + \int_0^t \left|e^{(t-\tau)(A_{11}-A_{12}C)}\right| \cdot |A_{12}| \cdot \|y(\tau)\| \, d\tau$$

which implies $\|x_1(t)\| < \varepsilon_1$ in the case $\|y(t)\| < \varepsilon_2, \forall t > 0$. Really, from

$$\begin{pmatrix} x_1 \\ y \end{pmatrix} = \begin{pmatrix} I & -B_1 B_2^{-1} \\ C & (I - CB_1)B_2^{-1} \end{pmatrix} x \tag{24}$$

and the inequalities (13),(16) it follows

$$\|x_1(t)\| \leq e^{-\mu t} H \|x_1(0)\| + \mu^{-1}(1 - e^{-\mu t}) H |A_{12}| \varepsilon_2 < \varepsilon_1$$

So, it is enough to show that $\|y(t)\| < \varepsilon_2$ for all $t > 0$ under assumption $\|x_1(t)\| < \varepsilon_1$. Suppose a contradiction, i. e. there exists the first time moment $T_\varepsilon > 0$ and the natural number $j \in \{1, 2, ..., n - m\}$, such that $y_j(T_\varepsilon) = \varepsilon_{2j}$ and $\|y(t)\| < \varepsilon_2$ for all $t \in [0, T_\varepsilon]$(the case $y_j(T_\varepsilon) = -\varepsilon_{2j}$ can be proven analogously). Then, from (24) and (16) it follows

$$y_j(0) < \delta_j^y = \varepsilon_{2j} + 2p_j(e^{-d_j h_0} - 1)/d_j \leq \varepsilon_{2j} \tag{25}$$

and there exists the time moment $T_\delta > 0$, such that $y_j(T_\delta) = \delta_j^y$ and $\delta_j^y < y_j(T_\delta) < \varepsilon_{2j}$ for $t \in [T_\delta, T_\varepsilon]$. Consider the corresponding equation from the system (23) on the interval $[T_\delta, T_\varepsilon]$

$$\dot{y}_j = d_j y_j - p_j \operatorname{sign}[y_j(t - h(t))] + g_j$$

where $g = (g_1, ..., g_m) = \gamma + Q_1 x_1 + Q_2 y$ and $\|g\| \leq k + |Q_1| \varepsilon_1 + |Q_2| \varepsilon_2$. Using the upper estimation of the function $y_j(t)$ on the considered interval $\dot{y}_j \leq d_j y_j + p_j + |g_j|, y_j(T_\delta) = \delta_j^y$ we obtain

$$y_j(T_\varepsilon) = \varepsilon_{2j} \leq \left(\delta_j^y + \frac{p_j + |g_j|}{d_j}\right) e^{d_j(T_\varepsilon - T_\delta)} - \frac{p_j + |g_j|}{d_j}$$

Hence, from the conditions 1),2) of the theorem we have $T_\varepsilon - T_\delta > h_0$, $\operatorname{sign}[y_j(t - h(t))] = 1$ for $t \in (T_\delta + h_0, T_\varepsilon]$ and

$$\dot{y}_j(t) \leq \varepsilon_{2j} d_j - p_j + |g_j| < 0$$

Consequently, the function $y_j(t)$ is decreasing on the interval $(T_\delta + h_0, T_\varepsilon]$ and could not achieve the bound ε_{2j}.

II. Finally, remark that the inequalities $\|x_1(t)\| < \varepsilon_1$ and $\|y(t)\| < \varepsilon_2$ imply $\|x(t)\| < \varepsilon$

$$\|x\| = \left\| \begin{pmatrix} I - B_1 C & B_1 \\ B_2 C & B_2 \end{pmatrix} \begin{pmatrix} x_1 \\ y \end{pmatrix} \right\| \leq \varepsilon$$

C. Proof of Theorem 3

I. By analogy with the proof of Theorem 2 it can be easily shown that $\|x_1(t)\| < R_1^{\max}$ and $\|y(t)\| < R_2^{\max}$ for all $t > 0$.

II. Prove that for any $i \in \{1, ..., m\}$ there exist time moment $t_i^0 \in [0, T_1] : \left|y(t_i^0)\right| = 0$. Suppose a contradiction, i.e. $\exists j \in \{1, ..., m\} : y_j(t) > 0$ for all $[0, T_1]$ (the case $y_j(t) < 0$ can be proven analogously).

Since $y_j(t) > 0$ for all $[0, T_1]$ then $\operatorname{sign}[y_j(t - h(t))] = 1$ for all $t \in [h_0, T_1]$. Hence, $\dot{y}_j(t) \leq d_i R_{2i}^{\max} - p + f_i^{\max}$ and from $y_j(t) \leq R_{2i}^{\max}$ we obtain the inequality $y_j(t) \leq R_{2i}^{\max} - t(p_i - d_i R_{2i}^{\max} - f_i^{\max})$, which contradicts with our supposition.

III. Now, show that $|y_i(t)| < \varepsilon_{2i}$ for all $t > t_i^0$. Really, using the upper estimation of $y_i(t)$

$$\dot{y}_i(t) \leq d_i y_i(t) + p_i + f_i^{\max}, y_i(t_i^0) = 0$$

we have $y_i(t_i^0 + h_0) \leq E_D(p_i + f_i^{\max}) < \varepsilon_{2i}$ and for $t > t_i^0 + h_0$

$$\dot{y}_i(t) \leq d_i \varepsilon_{2i} - p_i + f_i^{\max} \leq d_i R_{2i}^{\max} - p_i + f_i^{\max} < 0$$

i.e. the function $y_i(t)$ is decreasing. So, $\|y(t)\| \leq \varepsilon_2$ for all $t > T_1$.

IV. From $\|x_1(t)\| \leq e^{-\mu(t-T_1)} H R_1^{\max} + \mu^{-1} H |A_{12}| \varepsilon_2$we have

$$\|x_1(t)\| \leq \frac{0.5}{\lambda_{\max}(H)} H \varepsilon_1 + \mu^{-1} H |A_{12}| \varepsilon_2 = \varepsilon_1$$

for all $t > T_1 + T_2$.

V. The inequalities $\|x_1(t)\| \leq \varepsilon_1$ and $\|y(t)\| \leq \varepsilon_2$ for $t > T_1 + T_2$ imply $\|x(t)\| \leq \varepsilon$.

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