Proceedings of the
47th IEEE Conference on Decision and Control
Cancun, Mexico, Dec. 9-11, 2008

A Parameter-Dependent Lyapunov Function Based Approach to H_∞-Control of LPV Discrete-Time Systems with Delays

Shaosheng Zhou
School of Automation
Hangzhou Dianzi University
Hangzhou, Zhejiang, 310018
P. R. China

Wei Xing Zheng
School of Computing and Mathematics
University of Western Sydney
Penrith South DC NSW 1797
Australia

***Abstract*— In this paper we study the problem of H_∞ control of linear parameter-varying (LPV) discrete-time systems with delays. In an LPV system, the state-space matrices are a function of time-varying parameters which are assumed to be real-time measurable. We utilize a parameter-dependent Lyapunov function to establish a delay-dependent H_∞ performance condition for the LPV system with unknown but bounded delays. On the basis of the H_∞ performance condition established, we develop a linear matrix inequality (LMI) based H_∞ control strategy. We show that solving the related LMI optimization problem paves the way for designing a H_∞ controller for the LPV discrete-time system with delays. We also use a numerical example to demonstrate the application of the presented H_∞ controller design method.**

I. Introduction

There has been a growing research interest in the problems of analysis and control design for systems subject to time delays in the state and/or control input during the recent years [7], [10], [17], [22]. The existing results on the issues are often classified into two types according to their dependence on the size of the delays, namely, delay-independent results and delay-dependent results. Delay-dependent conditions on stability and stabilization of delayed control systems are usually less conservative than delay independent ones, especially when the size of the delay is small. Hence, in recent years delay-dependent stability and stabilization with some performances for time-delay systems has become an active area of research, with many interesting results having been obtained by various analysis and synthesis methods [3], [4], [11], [12], [14], [15], [16], [18], [19], [23].

It is well known that many physical processes such as power systems, aircraft systems, chemical processes and so on belong to the type of parameter-varying systems [8], [9]. For this reason, the analysis and control design of this class of systems have been extensively studied in recent years. Numerous results have been given for this class of systems [2], [13], [20], [21], [24]. The control syntheses for linear parameter-varying (LPV) systems have been studied in [5], [6], [21]. Recently, the controller design problem was investigated in [1] for time-varying discrete systems by using parameter-independent Lyapunov functions. However, to the best of the authors' knowledge, no work on delay-dependent H_∞ controller design for parameter-varying discrete delayed systems via parameter-dependent Lyapunov functions has been reported.

This paper is concerned with the H_∞ control problem for a class of parameter-varying discrete systems with time delays under some commonly adopted assumptions. One assumption is that the state-space matrices of the systems are dependent on a vector of time-varying real parameters, while the other assumption is that these parameters are real-time measurable so that they can be fed to the controller. With the introduction of a parameter-dependent Lyapunov function,

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the delay-dependent method and auxiliary variable technique are employed to establish new H_∞ performance conditions expressed by matrix inequalities. It is shown that by adopting a gain-scheduled controller design strategy, solving a set of linear matrix inequalities (LMIs) corresponding to delay-dependent H_∞ performance conditions provides H_∞ controllers for parameter-varying delayed systems.

Notation. Throughout this paper, a real symmetric matrix $P > 0 (\geq 0)$ denotes P being a positive definite (or positive semidefinite) matrix, and $A > (\geq) B$ means $A - B > (\geq) 0$. I is used to denote an identity matrix with proper dimension. Matrices, if their dimensions are not explicitly stated, are assumed to have compatible dimensions for algebraic operations. The space of square summable vector sequences is denoted by $l_2[0, \infty)$. A sequence

$$v = \{v_k\} \in l_2[0, \infty)$$

if

$$\|v\|_2 = \sqrt{\sum_{k=0}^{\infty} v_k^\tau v_k} < \infty.$$

II. Problem Formulation

Consider the parameter-varying discrete-time systems with time delays:

$$\begin{bmatrix} x_{k+1} \\ z_k \end{bmatrix} = \begin{bmatrix} A(h(k)) & A_d(h(k)) & A_\omega(h(k)) \\ B(h(k)) & B_d(h(k)) & B_\omega(h(k)) \end{bmatrix}$$

$$\begin{matrix} A_u(h(k)) & A_{ud}(h(k)) \\ B_u(h(k) & B_{ud}(h(k)) \end{matrix} \Bigg] \begin{bmatrix} x_k \\ x_{k-d} \\ \omega_k \\ u_k \\ u_{k-d} \end{bmatrix} \quad (1)$$

where $x_k \in \mathbb{R}^n$ is the state, $u_k \in \mathbb{R}^m$ is the control input, $z_k \in \mathbb{R}^q$ is the controlled output variable, $\omega_k \in \mathbb{R}^p$ is the noise signal which is assumed to be an arbitrary signal such that $\{\omega_k\} \in l_2[0, \infty)$. The nonnegative integer d is the unknown time delay of the system and satisfies

$$1 \leq d \leq \bar{d}$$

where $\bar{d}$ is a known positive integer. The time-varying parameter vector

$$h(k) = (h_1(k), \ldots, h_s(k))$$

is assumed to be measured online. $h(k)$ is allowed to vary in the unit simplex

$$\Xi := \left\{ h(k) \mid \ h_i(k) \geq 0, \ i = 1, \ldots, s, \ \sum_{i=1}^{s} h_i(k) = 1 \right\} \quad (2)$$

In what follows, we will drop the argument k for some k-dependent variables and matrices for illustration convenience. The state-space data in (1) are assumed to be affine in h, that is,

$$\begin{bmatrix} A(h) & A_d(h) & A_\omega(h) & A_u(h) & A_{ud}(h) \\ B(h) & B_d(h) & B_\omega(h) & B_u(h) & B_{ud}(h) \end{bmatrix}$$

$$= \sum_{i=1}^{s} h_i \begin{bmatrix} A_i & A_{id} & A_{i\omega} & A_{iu} & A_{iud} \\ B_i & B_{id} & B_{i\omega} & B_{iu} & B_{iud} \end{bmatrix} \quad (3)$$

where all the sub-block matrices on the right hand side of (3) are known constant matrices.

We are interested in designing a gain-scheduled controller

$$u_k = \mathcal{K}(h) x_k := \sum_{i=1}^{s} h_i K_i x_k \quad (4)$$

In the sequel, for notional simplicity, we will also drop the arguments of some h-dependent matrices in the case that no notional confusion is caused. Then, the closed-loop system Σ^c from system of (1) and (4) can be described by

$$\begin{bmatrix} x_{k+1} \\ z_k \end{bmatrix} = \begin{bmatrix} \mathcal{A} & \mathcal{A}_d & A_\omega \\ \mathcal{B} & \mathcal{B}_d & B_\omega \end{bmatrix} \begin{bmatrix} x_k \\ x_{k-d} \\ \omega_k \end{bmatrix} \quad (5)$$

where

$$\begin{aligned} \mathcal{A} &= A + A_u \mathcal{K}, \quad & \mathcal{A}_d &= A_d + A_{ud} \mathcal{K}, \\ \mathcal{B} &= B + B_u \mathcal{K}, \quad & \mathcal{B}_d &= B_d + B_{ud} \mathcal{K} \end{aligned} \quad (6)$$

The objective of this paper is to design a controller in the form of (4) such that the following specifications are met for the closed-loop system Σ^c of (5)–(6):

(S1): The closed-loop system Σ^c of (5)–(6) is globally asymptotically stable for any $h \in \Xi$ when $\omega_k \equiv 0$.

(S2): The l_2-gain between the external disturbance ω_k and the controlled output z_k is less than γ, that is, for any nonzero $\omega \in l_2[0,\infty)$ and zero initial condition $x_0 = 0$,

$$\|z\|_2 < \gamma \|\omega\|_2 \tag{7}$$

In the following, we will refer systems satisfying (S1) and (S2) to be as stable and with H_∞ norm bound γ.

III. Stability Analysis

This section discusses a new characterization involving parameter-dependent Lyapunov function for the closed-loop system Σ^c to be stable and with H_∞ norm bound γ.

Theorem 1: The closed-loop system Σ^c is stable and with H_∞ norm bound γ, if there exist symmetric matrices H, Q and Z, matrix V and h-dependent matrix $\mathcal{P}(h)$ satisfying

$$\alpha_1 I \le \mathcal{P}(h) \le \alpha_2 I \tag{8}$$

for any $h \in \Xi$ and some scalars $\{\alpha_i > 0\}_{i=1}^{2}$ such that for any h, $h^+ \in \Xi$,

$$\begin{bmatrix} \bar{d}H + V + V^T + Q - \mathcal{P}^{-1}(h) & -V & 0 & \mathcal{A}^T & \bar{d}(\mathcal{A}-I)^T Z & \mathcal{B}^T \\ * & -Q & 0 & \mathcal{A}_d^T & \bar{d}\mathcal{A}_d^T Z & \mathcal{B}_d^T \\ * & * & -\gamma^2 I & A_\omega^T & \bar{d}A_\omega^T Z & B_\omega^T \\ * & * & * & -\mathcal{P}(h^+) & 0 & 0 \\ * & * & * & * & -\bar{d}Z & 0 \\ * & * & * & * & * & -I \end{bmatrix} < 0 \tag{9}$$

$$\begin{bmatrix} H & V \\ V^T & Z \end{bmatrix} \ge 0 \tag{10}$$

where

$$h^+ := (h_1(k+1), h_2(k+1), \ldots, h_s(k+1)) \in \Xi$$

and $*$'s denote the corresponding transposed block matrices due to symmetry.

Remark 1: Theorem 1 provides a delay-dependent H_∞ performance condition for the closed-loop system Σ^c. The main idea behind the derivation of Theorem *1* is the introduction of the combined parameter-dependent and delay-dependent Lyapunov function V_k. The Lyapunov function V_k consists of three parts. The third part is of delay-dependent form by which a delay dependent result can be derived.

IV. Controller Design

In this section, we present a sufficient condition for the existence of H_∞ controller in the form of (4) based on Theorem 1. Note that the conditions in (8), (9) and (10), as such, cannot be directly employed for controller design. One way to facilitate Theorem 1 for the construction of a controller is to convert (8), (9) and (10) into a finite set of linear matrix inequality constraints. To this end, one must further restrict the choice of the parameter-dependent Lyapunov functions. The following theorem gives one possible way to do so.

Theorem 2: The closed-loop system Σ^c of *(5)–(6)* is stable and with H_∞ norm bound γ, if there exist symmetric matrices $\{P_i\}_{i=1}^{s}$, $H_1 > 0$, Q_1, Z_1, matrices $\{\Psi_j\}_{i=1}^{s}$, V_1 and Ω such that for all $i, j, g \in \{1, \ldots, s\}$,

$$\begin{bmatrix} w_i & * & * & * & * & * & * \\ -V_1^T & Q_1 - (\Omega + \Omega^T) & * & * & * & * & * \\ 0 & 0 & -\gamma^2 I & * & * & * & * \\ A_i\Omega + A_{iu}\Psi_j & A_{id}\Omega + A_{iud}\Psi_j & A_{j\omega} & -P_g & * & * & * \\ \bar{d}A_i\Omega + \bar{d}A_{iu}\Psi_j - \bar{d}\Omega & \bar{d}A_{id}\Omega + \bar{d}A_{iud}\Psi_j & \bar{d}A_{j\omega} & 0 & -\bar{d}Z_1 & * & * \\ B_i\Omega + B_{iu}\Psi_j & B_{id}\Omega + B_{iud}\Psi_j & B_{j\omega} & 0 & 0 & -I & * \\ \Omega & 0 & 0 & 0 & 0 & 0 & -Q_1 \end{bmatrix} < 0 \tag{11}$$

$$\begin{bmatrix} H_1 & V_1 \\ V_1^T & \Omega + \Omega^T - Z_1 \end{bmatrix} \ge 0 \tag{12}$$

where

$$w_i = \bar{d}H_1 + V_1 + V_1^T + P_i - (\Omega + \Omega^T) \tag{13}$$

When linear matrix inequalities (11) are feasible, the gain of a desired state feedback controller in (4) is given by

$$K_j = \Psi_j \Omega^{-1},\ j \in \{1, \ldots, s\} \quad (14)$$

Remark 2: Theorem 2 provides a sufficient condition for the solvability of H_∞ control problem for the parameter-varying delayed system. The desired state feedback controller can be obtained by solving the linear matrix inequalities (11) and (12).

V. Numerical Results

Consider the system Σ in (1), (2) and (3) with the following data:

$$A_1 = \begin{bmatrix} 1.1 & 0.8 & 0 \\ 0.6 & 0 & 0.32 \\ 0 & -0.63 & 0.3 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} 0.9 & 1.2 & 0 \\ 0.6 & 0 & 0.29 \\ 0 & -0.57 & 0.3 \end{bmatrix}$$

$$A_{1u} = \begin{bmatrix} 1.2 & -1.5 \\ 0.15 & 0.2 \\ 0.1 & -1.7 \end{bmatrix}$$

$$A_{2u} = \begin{bmatrix} 2.4 & 1.2 \\ 1.2 & -0.135 \\ 3.6 & -2.1 \end{bmatrix}$$

$$A_{1d} = \begin{bmatrix} 0.06 & 0.04 & 0.02 \\ 0.08 & 0 & 0.04 \\ 0 & 0.03 & -0.042 \end{bmatrix}$$

$$A_{1\omega} = \begin{bmatrix} 0.01 \\ 0.015 \\ 0.005 \end{bmatrix}$$

$$A_{2d} = \begin{bmatrix} -0.05 & 0 & -0.02 \\ 0 & 0.04 & 0 \\ -0.012 & 0.05 & 0.013 \end{bmatrix}$$

$$A_{2\omega} = \begin{bmatrix} -0.012 \\ -0.016 \\ 0 \end{bmatrix}$$

$$A_{1ud} = \begin{bmatrix} 0.14 & 0.035 \\ 0.14 & 0.07 \\ 0.21 & -0.28 \end{bmatrix}$$

$$A_{2ud} = \begin{bmatrix} 0.09 & 0.03 \\ -0.06 & -0.03 \\ 0.012 & 0.15 \end{bmatrix}$$

$$B_1 = \begin{bmatrix} 0.5 & 1 & 0 \end{bmatrix}$$

$$B_2 = \begin{bmatrix} 0.1 & 0.5 & 1 \end{bmatrix}$$

$$B_{1u} = \begin{bmatrix} -0.5 & -1 \end{bmatrix}$$

$$B_{2u} = \begin{bmatrix} 2 & 0.05 \end{bmatrix}$$

$$B_{1\omega} = 0.01$$

$$B_{2\omega} = 0.6$$

$$B_{1d} = \begin{bmatrix} 0.5 & 0.1 & 0.07 \end{bmatrix}$$

$$B_{2d} = \begin{bmatrix} -0.5 & -0.1 & -0.07 \end{bmatrix}$$

$$B_{1ud} = \begin{bmatrix} 0.03 & 0.5 \end{bmatrix}$$

$$B_{2ud} = \begin{bmatrix} 0.5 & 0 \end{bmatrix}$$

$$\bar{d} = 3$$

The target is to design a gain-scheduled controller such that the closed-loop system is stable with a given H_∞ norm bound γ. The performance level is chosen as $\gamma = 3.9$. Using Matlab LMI Control Toolbox to solve the linear matrix inequalities (11), we have obtained the solutions as follows:

$$P_1 = \begin{bmatrix} 0.0050 & -0.0022 & -0.0119 \\ -0.0022 & 0.0013 & 0.0056 \\ -0.0119 & 0.0056 & 0.0289 \end{bmatrix}$$

$$P_2 = \begin{bmatrix} 0.0048 & -0.0022 & -0.0116 \\ -0.0022 & 0.0013 & 0.0056 \\ -0.0116 & 0.0056 & 0.0285 \end{bmatrix}$$

$$H_1 = 10^{-3} \times \begin{bmatrix} 0.0461 & -0.0139 & -0.1015 \\ -0.01390 & 0.0084 & 0.0328 \\ -0.1015 & 0.0328 & 0.2252 \end{bmatrix}$$

$$V_1 = 10^{-3} \times \begin{bmatrix} -0.1856 & 0.0526 & 0.4091 \\ 0.0524 & -0.0253 & -0.1221 \\ 0.4061 & -0.1205 & -0.8994 \end{bmatrix}$$

$$Q_1 = \begin{bmatrix} 0.0198 & -0.0109 & -0.0492 \\ -0.0109 & 0.0080 & 0.0287 \\ -0.0492 & 0.0287 & 0.1240 \end{bmatrix}$$

$$Z_1 = \begin{bmatrix} 0.0207 & -0.0115 & -0.0516 \\ -0.0115 & 0.0085 & 0.0305 \\ -0.0516 & 0.0305 & 0.1305 \end{bmatrix}$$

$$\Psi_1 = \begin{bmatrix} -0.0004 & -0.0001 & 0.0007 \\ 0.0023 & -0.0012 & -0.0055 \end{bmatrix}$$

$$\Psi_2 = \begin{bmatrix} -0.0004 & -0.0001 & 0.0007 \\ 0.0022 & -0.0012 & -0.0055 \end{bmatrix}$$

$$\Omega = \begin{bmatrix} 0.0107 & -0.0059 & -0.0266 \\ -0.0059 & 0.0043 & 0.0155 \\ -0.0267 & 0.0155 & 0.0671 \end{bmatrix}.$$

By Theorem 2, we have a gain-scheduled controller as

$$u(k) = \left\{ h_1(k) \begin{bmatrix} -0.0813 & -0.2929 & 0.0454 \\ 0.1581 & 0.1067 & -0.0442 \end{bmatrix} \right.$$
$$\left. + h_2(k) \begin{bmatrix} -0.0830 & -0.2921 & 0.0446 \\ 0.1571 & 0.1060 & -0.0443 \end{bmatrix} \right\} x(k)$$

For simulation, the disturbance signal is chosen as

$$\omega_k = \frac{-7\cos(k)}{1 + k^{0.51}}$$

which belongs to $l_2[0, \infty)$. We also choose

$$h_1 = \frac{1}{1 + \tan\left(\frac{k}{\pi}\right)}$$

and

$$h_2 = 1 - h_1.$$

VI. Conclusion

In this paper we have applied the parameter-dependent Lyapunov function approach to establish the new delay-dependent H_∞ performance conditions for a class of parameter-varying systems with time delays. We have used the delay-dependent conditions to develop delay-dependent H_∞ controllers for this class of systems. Finally, we have illustrated the applicability of the proposed approach through a numerical example.

Acknowledgements

This work was supported in part by the National Natural Science Foundation of P. R. China under Grants 60574080 and 60434020, and in part by a Research Grant from the Australian Research Council and a Research Grant from the University of Western Sydney, Australia.

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