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On the reachability of single-input positive switched systems

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***Abstract*— In the paper reachability property for positive switched systems which commute among n single-input n-dimensional systems is investigated. Starting from a (hard to check) necessary and sufficient condition for reachability [6], and by introducing some new algebraic tools, sufficient reachability conditions which are easy to test are derived.**

I. Introduction

Modeling of physical phenomena comes as the result of a weighted balance among different, and often conflicting, needs. In particular, accuracy typically brings the drawback of computational complexity. For this reason, general complex models are often replaced by simpler and possibly linear models, each of them suitable for describing the system evolution under specific working conditions. This simple fact stimulated, in the last fifteen years, a long stream of research concerned with the analysis and design of "switched linear systems", by this meaning systems whose describing equations change, according to some switching law, within a (possibly infinite) family of (linear) subsystems. Reachability and controllability properties for these systems have been investigated in [4], [11], [12].

On the other hand, positive linear systems naturally arise in various fields such as bioengineering (compartmental models), economic modelling, and stochastic processes (Markov chains), where the state variables represent quantities that have no meaning unless nonnegative [3].

In this perspective, switched positive systems are mathematical models which keep into account two different needs: the need for a system model which is obtained as a family of simple subsystems, each of them accurate enough to capture the system laws under specific operating conditions, and the nonnegativity constraint on the system variables. This is the case when trying to describe certain physiological and pharmacokinetic processes, as, for instance, the insulin-sugar metabolism. Of course, the need for this class of systems in specific research contexts has stimulated an interest in theoretical issues related to them, and, in particular, structural properties of continuous-time positive switched systems have been recently investigated in [5], [6], [7]. In detail, necessary conditions for reachability have been investigated in [5] (monomial reachability) and in [6] (pattern reachability), while necessary and sufficient conditions for the reachability of continuous-time positive switched systems of dimension n, which commute among n single-input subsystems, have been investigated in [7]. These conditions, even though valuable from a theoretical point of view, appear quite difficult to check. This difficulty has stimulated research interests in the detailed analysis of the dominant modes of the exponential of a Metzler matrix [8]. By relaying on these results, we derived some sufficient conditions for reachability which are easy enough to check [9]. This contribution aims at further exploring this interesting and fruitful direction, by providing new conditions.

Before proceeding, we introduce some notation. For every $k \in \mathbb{N}$, we set $\langle k \rangle := \{1, 2, \ldots, k\}$. The (i, j)th entry of a matrix A is $[A]_{i,j}$. If A is block partitioned, $\mathrm{block}_{(i,j)}[A]$ denotes its (i, j)th block. In the special case of a vector $\mathbf{v}$, we let $[\mathbf{v}]_i$ denote its ith entry and $\mathrm{block}_i[\mathbf{v}]$ its ith block. $\mathbb{R}_+$ is the semiring of nonnegative real numbers. A matrix A with entries in $\mathbb{R}_+$ is a *nonnegative matrix* ($A \geq 0$); if $A \geq 0$ and $A \neq 0$, A is a *positive matrix* ($A > 0$), while if all its entries are positive it is a *strictly positive matrix* ($A \gg 0$). The same notation is adopted for nonnegative, positive and strictly positive vectors. A *Metzler matrix* is a real square matrix, whose off-diagonal entries are nonnegative. Every Metzler matrix has a real eigenvalue $\lambda_{\max}(A)$ satisfying $\lambda_{\max}(A) > \mathrm{Re}(\lambda)$ for every other $\lambda \in \sigma(A)$.

Given any matrix $A \in \mathbb{R}^{q\times r}$, by the *nonzero pattern* of A we mean the set of index pairs corresponding to its nonzero entries, namely $\overline{\mathrm{ZP}}(A) := \{(i, j) : [A]_{i,j} \neq 0\}$. Conversely, the *zero pattern* $\mathrm{ZP}(A)$ is the set of indices corresponding to the zero entries of A. The adaptation of these concepts to the vector case is straightforward. We let $\mathbf{e}_i$ denote the ith vector of the canonical basis in $\mathbb{R}^n$. A vector $\mathbf{v} \in \mathbb{R}^n_+$ is an *ith monomial vector* if $\overline{\mathrm{ZP}}(\mathbf{v}) = \overline{\mathrm{ZP}}(\mathbf{e}_i) = \{i\}$. For any set $\mathcal{S} \subseteq \langle n \rangle$, we set $\mathbf{e}_{\mathcal{S}} := \sum_{i\in\mathcal{S}} \mathbf{e}_i$ and we let $P_{\mathcal{S}}$ be the $n\times|\mathcal{S}|$ selection matrix that singles out the columns of the identity matrix corresponding to the indices in $\mathcal{S}$. Consequently, for any vector $\mathbf{v} \in \mathbb{R}^n_+$, with $\overline{\mathrm{ZP}}(\mathbf{v}) = \mathcal{S}$, $\mathbf{v}_{\mathcal{S}} := P^T_{\mathcal{S}}\mathbf{v}$ is the restriction of $\mathbf{v}$ to its positive components.

To every $n \times n$ Metzler matrix A we associate [2], [10] a *directed graph* $\mathcal{G}(A)$ with vertices indexed by $1, 2, \ldots, n$. There is an arc (j, i) from j to i if and only if $[A]_{ij} \neq 0$. We say that vertex i is *accessible* from j if there exists a path (i.e., a sequence of adjacent arcs $(j, i_1), (i_1, i_2), \ldots, (i_{k-1}, i)$) in $\mathcal{G}(A)$ from j to i (equivalently, $\exists\ k \in \mathbb{N}$ such that $[A^k]_{ij} \neq 0$). Two distinct vertices are said to *communicate* if each of them is accessible from the other. By definition, each vertex communicates with itself. The concept of communicating vertices allows to partition the set of vertices $\langle n \rangle$ into *communicating classes*, say $\mathcal{C}_1, \ldots, \mathcal{C}_\ell$. To any class $\mathcal{C}_i$ we associate two index sets:

$$\begin{aligned}\mathcal{A}(\mathcal{C}_i) &:= \{j : \text{ the class } \mathcal{C}_j \text{ has access to the class } \mathcal{C}_i\}\\ \mathcal{D}(\mathcal{C}_i) &:= \{j : \text{ the class } \mathcal{C}_j \text{ is accessible from the class } \mathcal{C}_i\}.\end{aligned}$$

Each class $\mathcal{C}_i$ is assumed to access to itself. If i is a vertex in $\mathcal{G}(A)$, we denote by $\mathcal{C}(i)$ the class i belongs to.

The *reduced graph* $\mathcal{R}(A)$ [10] associated with A (with

$\mathcal{G}(A)$) is the (acyclic) graph having the classes $\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_\ell$ as vertices. There is an arc (j,i) in $\mathcal{R}(A)$ if and only if $i \in \mathcal{D}(\mathcal{C}_j)$. Any (acyclic) path $(i_1,i_2),(i_2,i_3),\ldots,(i_{k-1},i_k)$ in $\mathcal{R}(A)$ identifies a *chain of classes* $(\mathcal{C}_{i_1}, \mathcal{C}_{i_2}, \ldots, \mathcal{C}_{i_k})$, having $\mathcal{C}_{i_1}$ as *initial class* and $\mathcal{C}_{i_k}$ as *final class*.

An $n \times n$ Metzler matrix A is *reducible* if there exists a permutation matrix P such that

$$P^T AP = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix},$$

where A_{11} and A_{22} are square (nonvacuous) matrices, otherwise it is *irreducible*. It follows that 1×1 matrices are always irreducible. In general, given a square Metzler matrix A, a permutation matrix P can be found such that

$$P^T AP = \begin{bmatrix} A_{11} & A_{12} & \ldots & A_{1\ell} \\ & A_{22} & \ldots & A_{2\ell} \\ & & \ddots & \vdots \\ & & & A_{\ell\ell} \end{bmatrix}, \tag{1}$$

where each A_{ii} is irreducible. (1) is usually known as *Frobenius normal form* of A [2]. Clearly, the irreducible matrices $A_{11}, A_{22}, \ldots, A_{\ell\ell}$ correspond to the communicating classes $\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_\ell$ of $\mathcal{G}(P^T AP)$ (coinciding with those of $\mathcal{G}(A)$, after a suitable relabelling). Consequently, we will refer to the dominant eigenvalue of the block A_{kk} as to the dominant eigenvalue of the class $\mathcal{C}_k$.

When dealing with the graph of a matrix in Frobenius normal form (1), for every $i \in \langle \ell \rangle$, $\mathcal{A}(\mathcal{C}_i) \subseteq \{i, i+1, \ldots, \ell\}$, while $\mathcal{D}(\mathcal{C}_i) \subseteq \{1,2,\ldots,i\} = \langle i \rangle$, so that $\mathcal{A}(\mathcal{C}_i) \cap \mathcal{D}(\mathcal{C}_i) = \{i\}$. A class $\mathcal{C}_i$ is *final* if $\mathcal{D}(\mathcal{C}_i) = \{i\}$, and *distinguished* [10] if $\lambda_{\max}(A_{ii}) > \lambda_{\max}(A_{jj})$ for every $j \in \mathcal{D}(\mathcal{C}_i), j \neq i$.

Basic definitions and results about cones may be found, e.g., in [1]. We recall here only a couple of useful results. A cone $\mathcal{K}$ is said to be *polyhedral* if it can be expressed as the set of nonnegative linear combinations of a finite set of *generating vectors*. This amounts to saying that $k \in \mathbb{N}$ and $C \in \mathbb{R}^{n \times k}$ can be found, such that $\mathcal{K}$ coincides with the set of nonnegative combinations of the columns of C (for short, $\mathcal{K} := \text{Cone}(C)$). A polyhedral cone $\mathcal{K}$ in $\mathbb{R}^n$ is *simplicial* if it admits n linearly independent generating vectors, i.e. $\mathcal{K} = \text{Cone}(C)$ for some nonsingular matrix C. When so, a vector $\mathbf{v}$ belongs to the boundary of the simplicial cone $\mathcal{K}$ if and only if $\mathbf{v} = C\mathbf{u}$ for some $\mathbf{u} > 0$, with $\overline{\text{ZP}}(\mathbf{u}) \neq \langle n \rangle$.

II. Reachability property

A *single-input continuous-time positive switched system* is described by the following equation

$$\dot{\mathbf{x}}(t) = A_{\sigma(t)}\mathbf{x}(t) + b_{\sigma(t)}u(t), \qquad t \in \mathbb{R}_+, \tag{2}$$

where $\mathbf{x}(t)$ and $u(t)$ denote the n-dimensional state variable and the scalar input, respectively, at the time instant t, and σ is a switching sequence, taking values in a finite set $\mathcal{P}$. In this paper, we steadily address the case $\mathcal{P} = \langle n \rangle = \{1, 2, \ldots, n\}$.

We assume that the switching sequence is piece-wise constant, and hence in every time interval $[0,t]$ there is a finite number of discontinuities, which corresponds to a finite number of switching instants $0 = t_0 < t_1 < \ldots < t_k < t$. Also, we assume that, at the switching time t_ℓ, σ is right continuous. For each $i \in \mathcal{P}$, the pair (A_i, b_i) represents a continuous-time positive system, which means that A_i is an $n \times n$ Metzler matrix and $b_i \in \mathbb{R}^n_+$.

As a first step, we recall the definition of reachability for positive switched systems.

Definition 1: [6], [7] A state $\mathbf{x}_f \in \mathbb{R}^n_+$ is said to be *reachable* if there exist some time instant $t > 0$, a switching sequence $\sigma : [0,t[\to \mathcal{P}$ and an input $u : [0,t[\to \mathbb{R}_+$ that lead the state trajectory from $\mathbf{x}(0) = 0$ to $\mathbf{x}(t) = \mathbf{x}_f$. The positive switched system (2) is said to be *reachable* if every state $\mathbf{x}_f \in \mathbb{R}^n_+$ is reachable.

As shown in [5], if system (2) is reachable, there exists a relabeling of the subsystems $(A_i, b_i), i \in \langle n \rangle$, such that

$$A_i\mathbf{e}_i = \alpha_i\mathbf{e}_i \qquad \text{and} \qquad b_i = \beta_i\mathbf{e}_i, \tag{3}$$

for suitable $\alpha_i \geq 0$ and $\beta_i > 0$. This will be a steady assumption in the following. For this specific class of systems every reachable state can be reached by resorting to a piece-wise constant nonnegative input signal. To prove this, we recall that the state at the time t, starting from the zero initial condition, under the action of the soliciting input $u(\tau), \tau \in [0,t[$, and of the switching sequence $\sigma : [0,t[$, with switching instants $0 = t_0 < t_1 < \ldots < t_k < t$ and switching values $i_0, i_1, \ldots, i_k$, can be expressed as follows:

$$\begin{aligned} \mathbf{x}(t) &= e^{A_{i_k}(t-t_k)} \ldots e^{A_{i_1}(t_2-t_1)} \int_{t_0}^{t_1} e^{A_{i_0}(t_1-\tau)} b_{i_0} u(\tau) \mathrm{d}\tau + \\ &+ e^{A_{i_k}(t-t_k)} \ldots e^{A_{i_2}(t_3-t_2)} \int_{t_1}^{t_2} e^{A_{i_1}(t_2-\tau)} b_{i_1} u(\tau) \mathrm{d}\tau + \\ &+ \ldots + \int_{t_k}^{t} e^{A_{i_k}(t-\tau)} b_{i_k} u(\tau) \mathrm{d}\tau. \end{aligned} \tag{4}$$

Proposition 1: Consider a continuous-time positive switched system (2), which switches among n single-input subsystems $(A_i, b_i), i \in \langle n \rangle$, satisfying (3) for suitable $\alpha_i \geq 0$ and $\beta_i > 0$. Given a time instant $t > 0$, a positive vector $\mathbf{v} \in \mathbb{R}^n_+$, $k \in \mathbb{Z}_+$, $k+1$ time instants $0 = t_0 < t_1 < \ldots < t_k < t$ and $k+1$ indices $i_0, i_1, \ldots, i_k \in \langle n \rangle$, the following facts are equivalent ones:
i) there exists a nonnegative input $u(\cdot)$ such that (4) holds for $\mathbf{x}(t) = \mathbf{v}$;
ii) there exists a piece-wise constant input $u(\cdot)$, taking some suitable constant value $u_i \geq 0$ in every time interval $[t_i, t_{i+1})$, such that

$$\begin{aligned} \mathbf{v} \;\; &= \;\; e^{A_{i_k}(t-t_k)} \ldots e^{A_{i_1}(t_2-t_1)} \int_{t_0}^{t_1} e^{A_{i_0}(t_1-\tau)} b_{i_0} d\tau \cdot u_0 \\ &+ \;\; \ldots + \int_{t_k}^{t} e^{A_{i_k}(t-\tau)} b_{i_k} d\tau \cdot u_k; \end{aligned} \tag{5}$$

iii) $\mathbf{v} \in \text{Cone}[e^{A_{i_k}(t-t_k)} b_{i_k} | e^{A_{i_k}(t-t_k)} e^{A_{i_{k-1}}(t_k-t_{k-1})} b_{i_{k-1}} | \ldots | e^{A_{i_k}(t-t_k)} \ldots e^{A_{i_1}(t_2-t_1)} e^{A_{i_0}(t_1-t_0)} b_{i_0}]$.

Proof: We preliminarily notice that, by the assumption on the pairs $(A_i, b_i), i \in \langle n \rangle$, $e^{A_i t} b_i = e^{\alpha_i t}\beta_i\mathbf{e}_i$, $\forall\, t \in \mathbb{R}_+$.
i) $\Leftrightarrow$ ii) Under the proposition's assumptions,

$$\int_{t_i}^{t_{i+1}} e^{A_i(t_{i+1}-\tau)} b_i u(\tau) d\tau = \int_{t_i}^{t_{i+1}} e^{\alpha_i(t_{i+1}-\tau)} \beta_i \mathbf{e}_i u(\tau) d\tau$$

$$= \left[\int_{t_i}^{t_{i+1}} e^{\alpha_i(t_{i+1}-\tau)}u(\tau)d\tau\right]\cdot\beta_i\mathbf{e}_i,$$

where the term inside the square brackets is nonnegative. But then, $u_i \geq 0$ can always be found such that

$$\int_{t_i}^{t_{i+1}} e^{\alpha_i(t_{i+1}-\tau)}u(\tau)d\tau = \int_{t_i}^{t_{i+1}} e^{\alpha_i(t_{i+1}-\tau)}d\tau\cdot u_i.$$

This proves that i) $\Rightarrow$ ii), the converse being obvious.

ii) $\Leftrightarrow$ iii) It suffices to notice that

$$\int_{t_i}^{t_{i+1}} e^{A_i(t_{i+1}-\tau)}b_i d\tau = \left[\int_{t_i}^{t_{i+1}} e^{\alpha_i(t_{i+1}-\tau)}d\tau\right]\cdot\beta_i\mathbf{e}_i =$$

$$= e^{\alpha_i(t_{i+1}-t_i)}c_i\beta_i\mathbf{e}_i = e^{A_i(t_{i+1}-t_i)}b_i c_i,$$

for some suitable $c_i > 0$. ■

Remark 1: Proposition 1 shows that, when dealing with an n-dimensional single-input system, commuting among n subsystems and satisfying (3), a vector $\mathbf{v} > 0$ is reachable if and only if there exist $k \in \mathbb{Z}_+$, switching values $i_0, i_1, \dots, i_k \in \langle n\rangle$ and positive time intervals $\tau_0, \tau_1, \dots, \tau_k$, such that $\mathbf{v}$ belongs to

$$\mathrm{Cone}[e^{A_{i_k}\tau_k}b_{i_k}|e^{A_{i_k}\tau_k}e^{A_{i_{k-1}}\tau_{k-1}}b_{i_{k-1}}|\dots|e^{A_{i_k}\tau_k}\dots e^{A_{i_0}\tau_0}b_{i_0}]$$
$$= \mathrm{Cone}[e_{i_k}|e^{A_{i_k}\tau_k}e_{i_{k-1}}|\dots|e^{A_{i_k}\tau_k}\dots e^{A_{i_1}\tau_1}e_{i_0}]$$

When so (see Lemma A.2 in [7]), once we set $\mathcal{S} := \overline{\mathrm{ZP}}(\mathbf{v})$, we have $\overline{\mathrm{ZP}}(e^{A_{i_k}}\mathbf{e}_{\mathcal{S}}) = \mathcal{S}$ and $\mathbf{v} = e^{A_{i_k}\tau_k}\mathcal{B}_k$, for some vector $\mathcal{B}_k > 0$ with $\overline{\mathrm{ZP}}(\mathcal{B}_k) \subseteq \mathcal{S}$. So, if we introduce

$$\mathcal{I}_{\mathcal{S}} := \{i \in \langle n\rangle : \overline{\mathrm{ZP}}(e^{A_i}\mathbf{e}_{\mathcal{S}}) = \mathcal{S}\}, \qquad (6)$$

a necessary condition for a vector $\mathbf{v}$, with $\overline{\mathrm{ZP}}(\mathbf{v}) = \mathcal{S}$, to be reachable is that $I_{\mathcal{S}} \neq \emptyset$.

We now recall the main result about the reachability of this class of positive switched systems.

Proposition 2: [6] Given an n-dimensional continuous-time positive switched system (2), commuting among n single-input subsystems $(A_i, b_i), i \in \langle n\rangle$, satisfying (3), the following facts are equivalent:

i) the switched system (2) is reachable;

ii) for every set $\mathcal{S} \subset \langle n\rangle$, with $1 < |\overline{\mathrm{ZP}}(\mathcal{S})| < n$, $\mathcal{I}_{\mathcal{S}} \neq \emptyset$ and either

iia) $\exists\, j(\mathcal{S}) \in \mathcal{I}_{\mathcal{S}}$ such that $\overline{\mathrm{ZP}}(b_{j(\mathcal{S})}) \subset \mathcal{S}$,

or

iib) for every $\mathbf{v} \in \mathbb{R}^n_+$, with $\overline{\mathrm{ZP}}(\mathbf{v}) = \mathcal{S}$, there exist $m \in \mathbb{N}$, $(\tau_1, \dots, \tau_m) \in \mathbb{R}^m_+$ and $i_1, \dots, i_m \in \mathcal{I}_{\mathcal{S}}$, such that $\mathbf{v}$ can be obtained as the nonnegative combination of no more than $|\mathcal{S}|-1$ columns of $e^{A_{i_1}\tau_1}\dots e^{A_{i_m}\tau_m}P_{\mathcal{S}}$, with $P_{\mathcal{S}}$ the $n \times |\mathcal{S}|$ selection matrix corresponding to $\mathcal{S}$.

Condition iib) provided in Proposition 2 cannot be easily verified. Specifically, there is no obvious way of testing whether indices $i_1, \dots, i_m$ and positive time intervals $\tau_1, \dots, \tau_m$ can be found, such that a given positive vector $\mathbf{v}$, with $\overline{\mathrm{ZP}}(\mathbf{v}) = \mathcal{S}$, belongs to $\mathrm{Cone}(e^{A_{i_1}\tau_1}\dots e^{A_{i_m}\tau_m}P_{\mathcal{S}})$ and it can be obtained by combining less than $|\mathcal{S}|$ columns of $e^{A_{i_1}\tau_1}\dots e^{A_{i_m}\tau_m}P_{\mathcal{S}}$. This would require trying all index sequences, of increasing length, meanwhile varying the lengths of the switching intervals τ_i. Of course, as there is no result about what it may be convenient to do (increasing or decreasing the τ_i's) and when one should give up (is there a maximum number of indices m after which no successful result can be obtained, unless it has been obtained earlier?), we need to explore alternative means for solving this problem and find sufficient conditions for the problem solvability.

If we assume that the set $\mathcal{S} \subset \langle n\rangle$ and the indices $i_1, i_2, \dots, i_m \in \mathcal{I}_{\mathcal{S}}$ are given, the problem one has to address can be equivalently stated as follows.

PROBLEM STATEMENT: Given any positive vector $\mathbf{v} \in \mathbb{R}^n_+$, with $\overline{\mathrm{ZP}}(\mathbf{v}) = \mathcal{S}$, find conditions ensuring that $\mathbf{v}_{\mathcal{S}} = P_{\mathcal{S}}^T\mathbf{v}$ belongs to the boundary of the simplicial cone, $\mathrm{Cone}[P_{\mathcal{S}}^T e^{A_{i_1}\tau_1}\dots e^{A_{i_m}\tau_m}P_{\mathcal{S}}]$ for some $\tau_i \geq 0$.

This restatement allows to address our problem in an apparently restrictive, but in fact equivalent, formulation (just set $\mathcal{S} = \langle n\rangle$, $I_{\mathcal{S}} = \mathcal{P}$ and, consequently, $\mathbf{v}_{\mathcal{S}} = \mathbf{v}$).

NEW PROBLEM STATEMENT: we search for conditions ensuring that $\mathbf{v} \in \mathbb{R}^n_+, \mathbf{v} \gg 0$, can be obtained as

$$\mathbf{v} = e^{A_{i_1}\tau_1}\dots e^{A_{i_m}\tau_m}\mathbf{u},\ \exists\ \tau_i \geq 0, \mathbf{u} > 0 \text{ with } \mathrm{ZP}(\mathbf{u}) \neq \emptyset.$$

The goal of this contribution is to find sufficient conditions for the solvability of this new problem, for $m = 1$ and $m = 2$, which will lead, through Proposition 2, to sufficient conditions for system reachability.

III. ASYMPTOTIC EXPONENTIAL CONES

Definition 2: [7], [8] Given an $n \times n$ Metzler matrix A, we define its *asymptotic exponential cone*, $\mathrm{Cone}_\infty(e^{At})$, as the polyhedral cone generated by the vectors $\mathbf{v}_i^\infty$, which represent the asymptotic directions of the columns of e^{At}:

$$\mathbf{v}_i^\infty := \lim_{t\to+\infty}\frac{e^{At}\mathbf{e_i}}{\|e^{At}\mathbf{e_i}\|}, \qquad i = 1, 2, \dots, n.$$

$\mathrm{Cone}_\infty(e^{At})$ always exists, is a polyhedral convex cone in $\mathbb{R}^n_+$, and is never the empty set. $\mathrm{Cone}(e^{At})$ is simplicial for every $t \geq 0$, while $\mathrm{Cone}_\infty(e^{At})$ is typically not. A first result about asymptotic exponential cones was obtained in [8]:

Lemma 1: Given an $n\times n$ Metzler matrix A and a strictly positive vector $\mathbf{v} \in \mathbb{R}^n_+$, the following facts are equivalent:

i) there exists $\tau > 0$ such that $\mathbf{v}$ belongs to $\partial\mathrm{Cone}(e^{A\tau})$;

ii) $\mathbf{v} \notin \mathrm{Cone}_\infty(e^{At})$.

A powerful characterization of $\mathrm{Cone}_\infty(e^{At})$ was obtained in [8]. This characterization may be further refined. To this end we need a result describing the dominant mode of each single column of the exponential e^{At} of a Metzler matrix $A \in \mathbb{R}^{n\times n}$. If we assume that A is in Frobenius normal form (1), then [8], at every time instant $t > 0$

$$e^{At} =: \mathcal{A}(t) = \begin{bmatrix} \mathcal{A}_{11}(t) & \mathcal{A}_{12}(t) & \dots & \mathcal{A}_{1\ell}(t) \\ & \mathcal{A}_{22}(t) & \dots & \mathcal{A}_{2\ell}(t) \\ & & \ddots & \vdots \\ & & & \mathcal{A}_{\ell\ell}(t) \end{bmatrix},$$

where $\mathcal{A}_{ii}(t)$ is strictly positive for every i, while for $i \neq j$ the matrix $\mathcal{A}_{ij}(t)$ is either strictly positive or zero. Even more, all entries of each nonzero block $\mathcal{A}_{ij}(t)$ exhibit the same dominant mode (of exponential type) weighted by a positive coefficient. In detail,

Proposition 3: [8] Let A be an $n \times n$ Metzler matrix in Frobenius normal form (1). Then there exist (not necessarily distinct) positive eigenvectors of A, $\tilde{\mathbf{v}}_j \in \mathbb{R}^n_+$, of unit norm, and real modes $m_j(t) = \frac{t^{\bar{m}_j}}{\bar{m}_j!} e^{\lambda_j^* t}$, with $\lambda_j^* \in \mathbb{R}$ and $\bar{m}_j \in \mathbb{Z}_+$, and strictly positive row vectors $\mathbf{c}_i \in \mathbb{R}^{1\times n_i}_+$, $j \in \langle \ell \rangle$, such that

$$\mathcal{A}(t) = e^{At} = [\, \tilde{\mathbf{v}}_1 \ \ \dots \ \ \tilde{\mathbf{v}}_\ell \,] \begin{bmatrix} m_1(t) & & \\ & \ddots & \\ & & m_\ell(t) \end{bmatrix} \cdot \begin{bmatrix} \mathbf{c}_1 & & \\ & \ddots & \\ & & \mathbf{c}_\ell \end{bmatrix} + \mathcal{A}_{lc}(t),$$

and $\forall\ i \in \langle n \rangle$ if we let $\mathcal{C}_j$ be the class of vertex i, then

$$\lim_{t\to+\infty} \frac{\mathcal{A}_{lc}(t)\mathbf{e}_i}{m_j(t)} = 0.$$

Moreover, $\lambda_j^* = \max\{\lambda_{\max}(A_{kk}) : k \in \mathcal{D}(\mathcal{C}_j)\}$, and $\bar{m}_j + 1$ is the maximum number of classes $\mathcal{C}_k$ with $\lambda_{\max}(A_{kk}) = \lambda_j^*$ that lie in a single chain in $\mathcal{R}(A)$ having $\mathcal{C}_j$ as initial class. Also, $\tilde{\mathbf{v}}_j$ is a positive eigenvector of A corresponding to λ_j^* and it exhibits the following zero pattern properties:

- $k(j) := \max\{k : \mathrm{block}_k[\tilde{\mathbf{v}}_j] > 0\} \equiv \max\{k \in \mathcal{D}(\mathcal{C}_j) : \lambda_{\max}(A_{kk}) = \lambda_j^*$, and there is a chain from $\mathcal{C}_j$ to $\mathcal{C}_k$ including other $\bar{m}_j$ classes $\mathcal{C}_h$ with $\lambda_{\max}(A_{hh}) = \lambda_j^*\}$;
- for every $k \le k(j)$, $\mathrm{block}_k[\tilde{\mathbf{v}}_j] = 0$ if $k \notin \mathcal{D}(\mathcal{C}_j)$, and $\mathrm{block}_k[\tilde{\mathbf{v}}_j] \gg 0$ otherwise.

Remark 2: The first part of the previous result has been proved in [8] (see Proposition 6.1). The second part, concerning the zero pattern properties of the vectors $\tilde{\mathbf{v}}_j$'s can be derived from Theorem 5.4 and Proposition 4.4 in [8]. Notice that $\{\tilde{\mathbf{v}}_1, \tilde{\mathbf{v}}_2, \dots, \tilde{\mathbf{v}}_\ell\} = \{\mathbf{v}_1^\infty, \mathbf{v}_2^\infty, \dots, \mathbf{v}_n^\infty\}$: in the former case we pick up a single vector for each class, while in the latter we consider each single column independently. Consequently, $\mathrm{Cone}_\infty(e^{At}) = \mathrm{Cone}(\tilde{\mathbf{v}}_1, \tilde{\mathbf{v}}_2, \dots, \tilde{\mathbf{v}}_\ell)$.

Lemma 2: Given an $n \times n$ Metzler matrix A in Frobenius normal form (1), let $V := \{\tilde{\mathbf{v}}_1, \tilde{\mathbf{v}}_2, \dots, \tilde{\mathbf{v}}_\ell\}$ be the set of the asymptotic directions of the columns of e^{At}, ordinately corresponding to the classes $\mathcal{C}_1, \dots, \mathcal{C}_\ell$, as they are defined in Proposition 3. Then the set V' of all vectors in V which correspond to a distinguished class is linearly independent and $\mathrm{Cone}(V') = \mathrm{Cone}(V) = \mathrm{Cone}_\infty(e^{At})$.

Proof: We know from Proposition 3 that each $\tilde{\mathbf{v}}_j$ is a positive eigenvector of A, corresponding to some eigenvalue λ_j^* and to some class $\mathcal{C}_j$. Since eigenvectors corresponding to distinct eigenvalues are linearly independent, V is linearly dependent if and only if there exists some eigenvalue λ such that the set of (not necessarily distinct) eigenvectors in V which correspond to λ, say $V_\lambda := \{\tilde{\mathbf{v}}_{i_1}, \dots, \tilde{\mathbf{v}}_{i_s}\}$, $s \le \ell$, is linearly dependent. Express V_λ as the union of two disjoint subsets $V_\lambda := V_{\rm dist} \cup V_{\rm rem}$, with $V_{\rm dist}$ containing those eigenvectors of V_λ which correspond to some distinguished class $\mathcal{C}_i$, and $V_{\rm rem}$ including those eigenvectors of V_λ which do not correspond to a distinguished class.

Since the asymptotic direction of a column corresponding to a distinguished class $\mathcal{C}_j$ always exhibits a strictly positive jth block, and from any distinguished class corresponding to λ it is not possible to access another distinguished class corresponding to the same eigenvalue, it follows that if $\tilde{\mathbf{v}}_j \in V_{\rm dist}$, then $\mathrm{block}_j[\tilde{\mathbf{v}}_h] = 0$ for every other $\tilde{\mathbf{v}}_h \in V_{\rm dist}$ (see [8] for further details). This ensures that the vectors in $V_{\rm dist}$ are all linearly independent. So, if V_λ is a set of linearly dependent vectors, then it must be $V_{\rm rem} \ne \emptyset$.

Choose $\tilde{\mathbf{v}}_j \in V_{\rm rem}$, and let $\mathcal{C}_j$ be the communicating class corresponding to $\tilde{\mathbf{v}}_j$. Clearly, $\mathcal{C}_j$ must access at least one distinguished class whose dominant eigenvalue is λ. If $\bar{m}_j + 1$ is the maximum number of distinguished classes with dominant eigenvalue λ which can be encountered along a chain of classes starting from $\mathcal{C}_j$ and there exist k such chains, then by Proposition 3, $\tilde{\mathbf{v}}_j$ is necessarily a linear combination of the k linearly independent eigenvectors (belonging to $V_{\rm dist}$) which correspond to λ and to those k distinguished classes.

Now, since each vector in $V_{\rm dist}$, as previously observed, has one strictly positive block which is zero in all the other vectors of $V_{\rm dist}$, in order for $\tilde{\mathbf{v}}_j$ to be positive such a linear combination must have only positive coefficients. By applying this reasoning to all vectors in $V_{\rm rem}$, we can claim that the cone generated by the vectors in V_λ is equal to the cone generated by the vectors in $V_{\rm dist}$ alone. ■

By the previous lemma, $\mathrm{Cone}_\infty(e^{At})$ is the polyhedral cone generated by the full column rank positive matrix whose columns are the coordinate vectors of the elements of V'. In the following, we will refer to such a matrix as to

$$V_\infty = [\, \hat{\mathbf{v}}_1^\infty \ \ \hat{\mathbf{v}}_2^\infty \ \ \dots \ \ \hat{\mathbf{v}}_r^\infty \,]. \tag{7}$$

We recall that each $\hat{\mathbf{v}}_i^\infty$ is a positive eigenvector of unit norm corresponding to a distinguished class. Notice, also, that every final class $\mathcal{C}_i$, by accessing no other class, is surely distinguished, and the corresponding dominant eigenvector (of unit norm) has only zero blocks, except for the ith which is strictly positive. We now address the special case when all generating vectors of $\mathrm{Cone}_\infty(e^{At})$ are (positive) eigenvectors of A corresponding to the same eigenvalue.

Lemma 3: Let $A \in \mathbb{R}^{n\times n}$ be a Metzler matrix in Frobenius normal form (1), and let $V_\infty \in \mathbb{R}^{n\times r}$ be the positive full column rank matrix, described as in (7), such that $\mathrm{Cone}_\infty(e^{At}) = \mathrm{Cone}(V_\infty)$. The following facts are equivalent ones:

i) all vectors in $\mathrm{Cone}(V_\infty)$ are eigenvectors of A;
ii) all generating vectors of $\mathrm{Cone}(V_\infty)$ are (positive) eigenvectors of A corresponding to the same eigenvalue λ;
iii) all distinguished classes of A have the same dominant eigenvalue $\lambda = \lambda_{\max}(A)$;
iv) all distinguished classes of A are final and they exhibit the same dominant eigenvalue λ.

Proof: i) $\Leftrightarrow$ ii) If i) holds, then all generating vectors of $\mathrm{Cone}(V_\infty)$, namely all columns of V_∞, are (positive) eigenvectors of A corresponding to the same eigenvalue. If not, by summing up two positive eigenvectors in V_∞ corresponding to two distinct eigenvalues we would get a vector in $\mathrm{Cone}(V_\infty)$ which is not an eigenvector of A. The converse is obvious.

ii) $\Rightarrow$ iii) Since we have already shown that the columns of V_∞ are the positive eigenvectors of unit norm corresponding to the distinguished classes of A, all distinguished classes of A have the same dominant eigenvalue. On the other hand, since there is at least one distinguished class whose dominant eigenvalue is $\lambda_{\max}(A)$, the result follows.

iii) $\Rightarrow$ iv) Suppose that there exists a distinguished class $\mathcal{C}_i$ which is not final and let $\lambda_{\max}(A_{ii})$ be the corresponding dominant eigenvalue. Clearly, $\mathcal{C}_i$ has access to some final class $\mathcal{C}_k$ with dominant eigenvalue $\lambda_{\max}(A_{kk})$. By the assumption that $\mathcal{C}_i$ is distinguished, it follows that $\lambda_{\max}(A_{ii}) > \lambda_{\max}(A_{kk})$, but by the assumption that all distinguished classes have the same dominant eigenvalue (and all final classes are distinguished) it follows that $\lambda_{\max}(A_{ii}) = \lambda_{\max}(A_{kk})$, a contradiction. Consequently, all distinguished classes must be final.

iv) $\Rightarrow$ ii) As the generating vectors of $\mathrm{Cone}(V_\infty)$ are (positive) eigenvectors of A corresponding to the distinguished classes of A, the result follows. ∎

In the remaining part of this section we aim to provide conditions for the solvability of the *new problem*, stated at the end of section II, assuming that $\mathbf{v} \gg 0$ has been assigned and m is either 1 or 2. From Lemma 1, we deduce that there exists an index $i \in \langle n \rangle$ such that $\mathbf{v} \in \partial\mathrm{Cone}(e^{A_i \tau})$ for some $\tau > 0$ if and only if $\mathbf{v} \notin \cap_{i=1}^n \mathrm{Cone}_\infty(e^{A_i t})$. So, we consider now strictly positive vectors $\mathbf{v}$ belonging to $\cap_{i=1}^n \mathrm{Cone}_\infty(e^{A_i t})$ and we search for conditions ensuring that $\mathbf{v}$ belongs to the boundary of $\mathrm{Cone}(e^{A_{i_1}\tau_1} e^{A_{i_2}\tau_2})$, for suitable indices $i_1, i_2 \in \langle n \rangle$ and positive time intervals $\tau_1, \tau_2 > 0$. Notice that we can always assume that $\mathbf{v}$ is not an eigenvector of A_{i_1}. Indeed, if this is the case, then we can also ensure that $e^{-A_{i_1}\tau_1}\mathbf{v} = e^{-\lambda t}\mathbf{v} = e^{A_{i_2}\tau_2}\mathbf{u}$, for some $\lambda \in \mathbb{R}$, and hence $\mathbf{v} \in \partial\mathrm{Cone}(e^{A_{i_2} t})$, thus ensuring that $\mathbf{v} \notin \mathrm{Cone}_\infty(e^{A_{i_2} t})$, a contradiction. As a consequence, we may always exclude as first matrix A_{i_1} any matrix which satisfies the equivalent conditions of Lemma 3, since in that case every vector in $\mathrm{Cone}_\infty(e^{A_{i_1} t})$ is necessarily an eigenvector of A_{i_1}. Clearly, if $\mathbf{v}$ is a common eigenvector of all matrices $A_i, i \in \langle n \rangle$, no choice is available for A_{i_1}.

In the following, for the sake of notational simplicity, we will replace the indices i_1 and i_2 with 1 and 2. For the aforementioned reasons, we will assume that the distinguished classes of A_1 correspond to at least two distinct eigenvalues and we will not consider strictly positive eigenvectors of A_1.

Proposition 4: Let $A_1, A_2 \in \mathbb{R}^{n\times n}$ be two Metzler matrices. If each positive eigenvector of A_1 does not belong to $\mathrm{Cone}_\infty(e^{A_2 t})$, then for every $\mathbf{v} \in \mathbb{R}^n_+, \mathbf{v} \gg 0$, there exists $\tau_1, \tau_2 \geq 0$ such that $\mathbf{v} \in \partial\mathrm{Cone}(e^{A_1\tau_1} e^{A_2\tau_2})$.

Proof: Let $V_{1\infty} = [\,\hat{\mathbf{v}}_1^\infty \ \ \hat{\mathbf{v}}_2^\infty \ \ \ldots \ \ \hat{\mathbf{v}}_r^\infty\,]$ be a full column rank matrix such that $\mathrm{Cone}_\infty(e^{A_1 t}) = \mathrm{Cone}(V_{1\infty})$, and each vector $\hat{\mathbf{v}}_i^\infty$ is a (positive) eigenvector of unit norm corresponding to some eigenvalue λ_i, with $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_r$. We already know from Lemma 1 that the result is true for every $\mathbf{v} \notin \mathrm{Cone}_\infty(e^{A_1 t}) \cap \mathrm{Cone}_\infty(e^{A_2 t})$, by setting either τ_1 or τ_2 equal to zero. Pick now $\mathbf{v} \in \mathrm{Cone}_\infty(e^{A_1 t}) \cap \mathrm{Cone}_\infty(e^{A_2 t})$. For two positive time instants τ_1, τ_2 to exist, having the desired properties, it must be $\mathbf{v} = e^{A_1\tau_1} e^{A_2\tau_2}\mathbf{u}$ for some $\mathbf{u} \in \partial\mathbb{R}^n_+$. But this amounts to saying that a time instant $\tau_1 > 0$ must exist such that the vector $\mathbf{w}(\tau_1) := e^{-A_1\tau_1}\mathbf{v}$ does not belong to $\mathrm{Cone}_\infty(e^{A_2 t})$. Of course, this is not the case when $\tau_1 = 0$, since $\mathbf{w}(0) = \mathbf{v} \in \mathrm{Cone}_\infty(e^{A_2 t})$. However, since $\mathbf{v} \in \mathrm{Cone}_\infty(e^{A_1 t})$, $\mathbf{v}$ can be expressed as the nonnegative combination of the columns of $V_{1\infty}$, namely as $\mathbf{v} = \sum_{i=1}^r c_i \hat{\mathbf{v}}_i^\infty, c_i \geq 0$. As a consequence, for $\tau_1 \to +\infty$, the vector $\mathbf{w}(\tau_1) = \sum_{i=1}^r c_i \hat{\mathbf{v}}_i^\infty e^{-\lambda_i \tau_1}$ will converge to some eigenvector $\mathbf{w}(+\infty)$ of A_1. Specifically, $\mathbf{w}(\tau_1)$ will align to the eigenvector $\sum_{i\in I} c_i \hat{\mathbf{v}}_i^\infty$, where $I := \{i \in \langle r \rangle : \lambda_i = \lambda_{i_{\min}}\}$ and $i_{\min} := \min\{i \in \langle r \rangle : c_i \neq 0\}$.

But then $\mathbf{w}(+\infty) \notin \mathrm{Cone}_\infty(e^{A_2 t})$, and since $\mathrm{Cone}_\infty(e^{A_2 t})$ is a closed set, it is possible to find some $0 < \bar{\tau}_1 < +\infty$ such that for every $\tau_1 > \bar{\tau}_1$, $\mathbf{w}(\tau_1) \notin \mathrm{Cone}_\infty(e^{A_2 t})$. So, by Lemma 1, for every such τ_1 it will be possible to find some $\tau_2 > 0$ such that $\mathbf{w}(\tau_1) \in \partial\mathrm{Cone}(e^{A_2\tau_2})$, and hence $\mathbf{v} \in \partial\mathrm{Cone}(e^{A_1\tau_1} e^{A_2\tau_2})$. ∎

The result of Proposition 4 can be generalized. Indeed, in order to ensure that when τ_1 is sufficiently large, we can always find $\tau_2 > 0$ such that $\mathbf{v} \in \partial\mathrm{Cone}(e^{A_1\tau_1} e^{A_2\tau_2})$, we are not specifically interested in constraining all positive eigenvectors of A_1 not to belong to $\mathrm{Cone}_\infty(e^{A_2 t})$, but only those eigenvectors of A_1 the vector $\mathbf{w}(\tau_1) := e^{-A_1\tau_1}\mathbf{v}$ asymptotically align with. In order to explore this issue, we preliminary need a technical lemma.

Lemma 4: Let $A_1 \in \mathbb{R}^{n\times n}$ be a Metzler matrix in Frobenius normal form (1), and let $V_\infty = [\hat{\mathbf{v}}_1^\infty \ \ldots \ \hat{\mathbf{v}}_r^\infty] \in \mathbb{R}^{n\times r}$ be the positive full column rank matrix, described as in (7), such that $\mathrm{Cone}_\infty(e^{A_1 t}) = \mathrm{Cone}(V_\infty)$. We assume that $\hat{\mathbf{v}}_i^\infty$, $i \in \langle r \rangle$, is the positive eigenvector corresponding to the dominant eigenvalue $\lambda_{\max}(A_{j_i j_i})$ of the distinguished class $\mathcal{C}_{j_i}$, $i \in \langle r \rangle$, and we do not introduce any specific ordering within the set of indices $\{j_1, \ldots, j_r\}$. Suppose that the $s \leq r$ distinct eigenvalues the previous eigenvectors correspond to are ordered as $\lambda_1 < \lambda_2 < \ldots < \lambda_s$, and define, for every $k \in \langle s \rangle$, the following sets:

- $I_k := \{i \in \langle r \rangle : \hat{\mathbf{v}}_i^\infty$ is an eigenvector corresponding to $\lambda_k\} = \{i \in \langle r \rangle : \lambda_{\max}(A_{j_i j_i}) = \lambda_k\}$;
- $\mathcal{D}_k := \bigcup_{i\in I_k} \mathcal{D}(\mathcal{C}_{j_i})$;
- $\mathcal{V} := \{k \in \langle s \rangle : \bigcup_{j\geq k} \mathcal{D}_j = \langle \ell \rangle\}$.

Then, for any $k \in \langle s \rangle$, there exists $\mathbf{c} \in \mathbb{R}^r_+$ such that

- $V_\infty \mathbf{c}$ is strictly positive;
- $k = \min\{i : \overline{\mathrm{ZP}}(\mathbf{c}) \cap I_i \neq \emptyset\}$.

if and only if $k \in \mathcal{V}$.

Proof: Notice, first, that since $\hat{\mathbf{v}}_i^\infty$, $i \in \langle r \rangle$, is the eigenvector corresponding to the dominant eigenvalue of the distinguished class $\mathcal{C}_{j_i}$, its nonzero pattern obeys the following rules (see Proposition 3):

$$\mathrm{block}_k[\hat{\mathbf{v}}_i^\infty] = \begin{cases} \gg 0, & \text{if } k \in \mathcal{D}(\mathcal{C}_{j_i}); \\ 0, & \text{otherwise.} \end{cases} \tag{8}$$

For any index $k \in \langle s \rangle$, the set $\mathcal{D}_k$ represents the set of indices of those classes that are reached by (at least) one distinguished class corresponding to λ_k. Clearly, as I_k is the set of indices i in $\langle r\rangle$ such that $\hat{\mathbf{v}}_i^\infty$ is an eigenvector corresponding to λ_k, then, by (8), $\overline{\mathrm{ZP}}(\sum_{i\in I_k}\hat{\mathbf{v}}_i^\infty) = \cup_{q\in\mathcal{D}_k}\mathcal{C}_q$. Finally, $\mathcal{V}$ represents the set of all indices $k\in\langle r\rangle$ for which $\overline{\mathrm{ZP}}(\sum_{i\in I_k\cup I_{k+1}\cup\ldots\cup I_r}\hat{\mathbf{v}}_i^\infty) = \langle n\rangle$, namely $\sum_{i\in I_k\cup I_{k+1}\cup\ldots\cup I_r}\hat{\mathbf{v}}_i^\infty$ is strictly positive.

As a consequence, if $k\notin\mathcal{V}$, then there is no way of finding some $\mathbf{c}\in\mathbb{R}_+^r$ such that $V_\infty\mathbf{c}\gg 0$ and $k:=\min\{i:\overline{\mathrm{ZP}}(\mathbf{c})\cap I_i\neq\emptyset\}$. Conversely, if $k\in\mathcal{V}$, there exists $\mathbf{c}\in\mathbb{R}_+^r$ such that $k=\min\{i:\overline{\mathrm{ZP}}(\mathbf{c})\cap I_i\neq\emptyset\}$ and $V_\infty\mathbf{c}\gg 0$ (e.g. $\mathbf{e}_\mathcal{S}$, with $\mathcal{S}=\cup_{q\in\mathcal{D}_k}\mathcal{C}_q$). ■

Proposition 5: Let $A_1, A_2\in\mathbb{R}_+^{n\times n}$ be two Metzler matrices, and adopt the same notation as in Lemma 4, with all the symbols $\hat{\mathbf{v}}_1^\infty,\ldots,\hat{\mathbf{v}}_r^\infty,\lambda_1,\ldots,\lambda_s, I_k$ and $\mathcal{V}$ referring to the matrix A_1, and assume $\lambda_1<\ldots<\lambda_s$. Set

$$\mathcal{K}_k := \mathrm{Cone}(\{\hat{\mathbf{v}}_i^\infty, i\in I_k\}).$$

If $\mathcal{K}_k\cap\mathrm{Cone}_\infty(e^{A_2t})=\{0\}$ for every $k\in\mathcal{V}$, then for every strictly positive vector $\mathbf{v}\in\mathrm{Cone}_\infty(e^{A_1t})\cap\mathrm{Cone}_\infty(e^{A_2t})$ there exists $\tau_1^*>0$ such that for every $\tau_1>\tau_1^*$ there exists $\tau_2>0$ such that $\mathbf{v}\in\partial\mathrm{Cone}(e^{A_1\tau_1}e^{A_2\tau_2})$.

Proof: If $\mathbf{v}$ is a strictly positive vector in $\mathrm{Cone}_\infty(e^{A_1t})\cap\mathrm{Cone}_\infty(e^{A_2t})$, then, in particular, $\mathbf{v}\in\mathrm{Cone}_\infty(e^{A_1t})=\mathrm{Cone}(V_\infty)$, and hence $\mathbf{v}=V_\infty\mathbf{c}=\sum_{i=1}^r c_i\hat{\mathbf{v}}_i^\infty$, $\exists\ \mathbf{c}>0$. So, if $k:=\min\{i:\overline{\mathrm{ZP}}(\mathbf{c})\cap I_i\neq\emptyset\}$ then $k\in\mathcal{V}$. Also, once we set $\mathbf{w}(\tau_1):=e^{-A_1\tau_1}\mathbf{v}$, we easily notice that $\overline{\mathrm{ZP}}(\mathbf{w}(\tau_1))=\overline{\mathrm{ZP}}(\mathbf{v})$ and hence $\mathbf{w}(\tau_1)\gg 0$ for every $\tau_1\geq 0$. Moreover, if we express $\mathbf{v}$ as

$$\mathbf{v}=\sum_{i\in I_k}c_i\hat{\mathbf{v}}_i^\infty+\sum_{i\in I_{k+1}}c_i\hat{\mathbf{v}}_i^\infty+\ldots+\sum_{i\in I_s}c_i\hat{\mathbf{v}}_i^\infty,$$

then $\mathbf{w}(\tau_1)=\sum_{m=k}^s\left(\sum_{i\in I_m}c_i\hat{\mathbf{v}}_i^\infty\right)e^{-\lambda_m\tau_1}$. As τ_1 goes to $+\infty$, $\mathbf{w}(\tau_1)$ will align to the eigenvector $\mathbf{w}(+\infty):=\sum_{i\in I_k}c_i\ \hat{\mathbf{v}}_i^\infty$, corresponding to the eigenvalue $\lambda_k, k\in\mathcal{V}$, and belonging to $\mathcal{K}_k$. But then, by the assumption, $\mathbf{w}(+\infty)$ does not belong to $\mathrm{Cone}_\infty(e^{A_2t})$, and since $\mathrm{Cone}_\infty(e^{A_2t})$ is a closed set, it is possible to find some $0<\tau_1^*<+\infty$ such that for every $\tau_1>\tau_1^*$, $\mathbf{w}(\tau_1)\notin\mathrm{Cone}_\infty(e^{A_2t})$. So, by Lemma 1, for every such τ_1 there exists $\tau_2>0$ such that $\mathbf{w}(\tau_1)\in\partial\mathrm{Cone}(e^{A_2\tau_2})$, and hence $\mathbf{v}\in\partial\mathrm{Cone}(e^{A_1\tau_1}e^{A_2\tau_2})$. This proves i). ■

IV. Sufficient conditions for reachability

By resorting to the technical results of Section III, we derive sufficient conditions for reachability. Specifically, Proposition 4 immediately leads to the following condition.

Proposition 6: Consider an n-dimensional positive switched system (2), commuting among n single-input subsystems $(A_i, b_i), i\in\langle n\rangle$, satisfying (3) for suitable $\alpha_i\geq 0$ and $\beta_i>0$. If for every set $\mathcal{S}\subset\langle n\rangle$, with $1<|\mathcal{S}|<n$, we have $|\mathcal{I}_\mathcal{S}|\geq 2$ and there exist $i,j\in\mathcal{I}_\mathcal{S}, i\neq j$, such that each positive eigenvector of $P_\mathcal{S}^TA_iP_\mathcal{S}$ does not belong to the asymptotic exponential cone of $P_\mathcal{S}^TA_jP_\mathcal{S}$, then the system is reachable.

Proof: We prove the result by induction of the cardinality of $\mathcal{S}:=\overline{\mathrm{ZP}}(\mathbf{v})$, $\mathbf{v}$ being the vector to be reached. By assumption (3), all vectors $\mathbf{v}$ with $|\mathcal{S}|=|\overline{\mathrm{ZP}}(\mathbf{v})|=1$ are reachable. Suppose now that every positive vector $\mathbf{u}$, such that $|\overline{\mathrm{ZP}}(\mathbf{u})|<r$, is reachable. Let $\mathbf{v}$ be a positive vector whose nonzero pattern $\overline{\mathrm{ZP}}(\mathbf{v})=\mathcal{S}$ has cardinality $|\mathcal{S}|=r$. By the assumption and by Proposition 4, there exist $\tau_i,\tau_j>0$ such that $\mathbf{v}_\mathcal{S}=e^{P_\mathcal{S}^TA_iP_\mathcal{S}\tau_i}e^{P_\mathcal{S}^TA_jP_\mathcal{S}\tau_j}\mathbf{u}_\mathcal{S}$, with $\mathbf{u}_\mathcal{S}\in\mathbb{R}_+^{|\mathcal{S}|}$, $\mathrm{ZP}(\mathbf{u}_\mathcal{S})\neq\emptyset$. Consequently, there exist $\tau_i,\tau_j>0$ such that $\mathbf{v}=e^{A_i\tau_i}e^{A_j\tau_j}\mathbf{u}$, with $\mathbf{u}\in\mathbb{R}_+^n, \overline{\mathrm{ZP}}(\mathbf{u})\subsetneq\mathcal{S}$. By the inductive assumption, $\mathbf{u}$ is reachable and hence $\mathbf{v}$ is. ■

Similarly, from Proposition 5 one may deduce the following result, whose proof strictly reminds the previous one and hence is omitted.

Proposition 7: Consider an n-dimensional positive switched system (2), commuting among n single-input subsystems $(A_i, b_i), i\in\langle n\rangle$, satisfying (3) for suitable $\alpha_i\geq 0$ and $\beta_i>0$. Define for each set $\mathcal{S}\subset\langle n\rangle$ and each matrix $P_\mathcal{S}^TA_iP_\mathcal{S}$ the index sets $\mathcal{V}_{i,\mathcal{S}}$ and the cones $\mathcal{K}_k^{(i,\mathcal{S})}$, as in Lemma 4 and Proposition 5. If for every set $\mathcal{S}\subset\langle n\rangle$, with $1<|\mathcal{S}|<n$, we have $|\mathcal{I}_\mathcal{S}|\geq 2$ and there exist $i,j\in\mathcal{I}_\mathcal{S}, i\neq j$, such that $\mathcal{K}_k^{(i,\mathcal{S})}\cap\mathrm{Cone}_\infty(e^{P_\mathcal{S}^TA_jP_\mathcal{S}t})=\{0\}$ for every $k\in\mathcal{V}_{i,\mathcal{S}}$, then the system is reachable.

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