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On passivity based control of stochastic port-Hamiltonian systems

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Abstract— **This paper introduces stochastic port-Hamiltonian systems and clarifies some of their properties. Stochastic port-Hamiltonian systems are extension of port-Hamiltonian systems which are used to express various deterministic passive systems. Some properties such as passivity of port-Hamiltonian systems do not generally hold for the stochastic port-Hamiltonian systems. Firstly, we show a necessary and sufficient condition to preserve the stochastic Hamiltonian structure of the original system under time-invariant coordinate transformations. Secondly, we derive a condition to maintain stochastic passivity of the system. Finally, we introduce stochastic generalized canonical transformations and propose a stabilization method based on stochastic passivity.**

I. INTRODUCTION

Physical systems are practically important and they have good properties for the control design such as passivity and energy conservation law. Many control approaches for these systems based on their intrinsic features are proposed. As one of the representation of the physical systems, port-Hamiltonian systems are introduced [1], [2]. It includes not only the conventional Hamiltonian systems [3] but also passive electro-mechanical systems, mechanical systems with nonholonomic constraints [4] and so on.

Meanwhile, theories and techniques for the deterministic dynamical systems described by ordinary differential equations are applied to the stochastic ones described by stochastic differential equations [5]. In the literatures [6], [7], [8], the notions of conserved quantities and symmetry are formulated for the stochastic systems described by stochastic differential equations written in the sense of Itô and Stratonovich. Lyapunov function approach to the stochastic stability of stochastic systems are introduced in [9], [10]. In this theory, nonnegative supermartingales are used as stochastic Lyapunov functions and asymptotic convergence of sample trajectories, or flows of the stochastic differential equations is proven by the martingale convergence theorem. The notion of the stochastic passivity for the stochastic systems is introduced in [11]. One can utilize the well-known results of the stabilization method by the output feedback for the passive systems [12], [13], [14] to achieve the asymptotic stability for the stochastic nonlinear systems in probability.

The aim of this paper is to introduce stochastic port-Hamiltonian systems which are extension of port-Hamiltonian systems, and clarify some of their properties. Some properties such as invariance under a class of transformations and passivity of port-Hamiltonian systems do not generally hold for the stochastic port-Hamiltonian systems. Firstly, we show a necessary and sufficient condition to preserve the stochastic Hamiltonian structure of the original system under time-invariant coordinate transformations. Secondly, we derive a condition to maintain stochastic passivity of the system. Thirdly, we introduce stochastic generalized canonical transformations which are extension of generalized canonical transformations proposed in [15]. Stochastic generalized canonical transformations are pairs of coordinate and feedback transformations under which the stochastic port-Hamiltonian structure is preserved. Finally, we propose a stabilization method based on stochastic passivity such that we transform a stochastic port-Hamiltonian system to a passive one and then stabilize the system by the output feedback based on stochastic passivity.

II. STOCHASTIC PORT-HAMILTONIAN SYSTEMS

This section proposes stochastic port-Hamiltonian systems which are extension of port-Hamiltonian systems [3], [1], [2] and clarifies some of their properties.

Port-Hamiltonian systems are described by the following form (1) and they are utilized to express various deterministic passive systems

$$\begin{cases} \dot{x} = (J(x) - R(x)) \dfrac{\partial H(x)}{\partial x}^{\mathrm{T}} + g(x)u \\ y = g(x)^{\mathrm{T}} \dfrac{\partial H(x)}{\partial x}^{\mathrm{T}} \end{cases} . \tag{1}$$

We extend these systems into the stochastic dynamical systems which are described by the following stochastic differential equation written in the sense of Itô,

$$\begin{cases} \mathrm{d}x = (J(x) - R(x)) \dfrac{\partial H(x)}{\partial x}^{\mathrm{T}} \mathrm{d}t + g(x)u \ \mathrm{d}t + h(x)\mathrm{d}w \\ y \ = g(x)^{\mathrm{T}} \dfrac{\partial H(x)}{\partial x}^{\mathrm{T}} \end{cases} . \tag{2}$$

Here $x(t) \in \mathbb{R}^n$, $u(t), y(t) \in \mathbb{R}^m$ describe the state, the input and the output, respectively. The structure matrix $J(x) \in \mathbb{R}^{n\times n}$ and the dissipation matrix $R(x) \in \mathbb{R}^{n\times n}$ are skew-symmetric and symmetric positive semi-definite, respectively. Hamiltonian $H(x) \in \mathbb{R}$ is smooth function, which describes the total energy of the system. $w(t) \in \mathbb{R}^r$ is a standard Wiener process defined on a probability space $(\Omega, \mathcal{F}, \mathcal{P})$, where Ω is a sample space, $\mathcal{F}$ is the sigma algebra of the observable random events and $\mathcal{P}$ is a probability measure on Ω. $g : \mathbb{R}^n \to \mathbb{R}^{n\times m}$ and $h : \mathbb{R}^n \to \mathbb{R}^{n\times r}$ are

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smooth functions. We define the systems (2) as **stochastic port-Hamiltonian systems**. In this paper, according to [16] we introduce the notion of stability in probability for the systems (2) as follows.

Definition 1: The equilibrium solution $x \equiv 0$ of the systems (2) is stable in probability if and only if for any $\epsilon > 0$ and $\delta > 0$, there exists $r = r(\epsilon, \delta) > 0$ such that if the initial condition $x(t^0)$ satisfies $\|x(t^0)\| < r(\epsilon, \delta)$, then

$$\mathcal{P}\left\{\sup_{t \le t_0} \|x(t)\| > \epsilon\right\} < \delta.$$

Definition 2: The equilibrium solution $x \equiv 0$ of the systems (2) is asymptotically stable in probability if and only if it is stable in probability and for any $\epsilon > 0$,

$$\lim_{T \to \infty} \mathcal{P}\left\{\sup_{t \le T} \|x(t)\| > \epsilon\right\} = 0.$$

The following lemma is obtained in [15] for the port-Hamiltonian systems.

Lemma 1: [15] The port-Hamiltonian system (1) *is transformed into another one by any time-invariant coordinate transformation.*

This lemma implies that the port-Hamiltonian structure is preserved under any time-invariant coordinate transformation. However, this lemma does not always hold in the case of the stochastic port-Hamiltonian system. One can prove the following theorem which characterizes the class of time-invariant coordinates which preserve the stochastic port-Hamiltonian structure.

Theorem 1: The stochastic port-Hamiltonian system (2) *is transformed into another stochastic port-Hamiltonian system by a time-invariant coordinate transformation* $\bar{x} = \Phi(x)$ *if and only if there exists a skew-symmetric matrix* $K(x)$ *and a symmetric matrix* $S(x)$ *such that* $R(x) + S(x)$ *is positive semi-definite and they satisfy*

$$\frac{1}{2}\mathrm{tr}\left\{\frac{\partial}{\partial x}\left(\frac{\partial \Phi^i}{\partial x}\right)^{\mathrm{T}} hh^{\mathrm{T}}\right\} = \frac{\partial \Phi^i}{\partial x}(K - S)\frac{\partial H}{\partial x}^{\mathrm{T}} \quad (i = 1, 2, \cdots, n) \quad , \tag{3}$$

where $\mathrm{tr}\{\cdot\}$ *represents the trace of the argument and* $(\cdot)^i$ *represents the* i*-th row of the argument.*

Proof: Firstly, the necessity of Eq. (3) is shown. By utilizing the Itô formula [17], [16], the dynamics of the system in the new coordinate $\bar{x}$ is calculated as

$$\begin{aligned}
\mathrm{d}\bar{x}^i &= \frac{\partial \Phi^i}{\partial x}\mathrm{d}x + \frac{1}{2}\mathrm{tr}\left\{\frac{\partial}{\partial x}\left(\frac{\partial \Phi^i}{\partial x}\right)^{\mathrm{T}} hh^{\mathrm{T}}\right\}\mathrm{d}t \\
&= \left[\frac{\partial \Phi^i}{\partial x}(J - R)\frac{\partial H}{\partial x}^{\mathrm{T}} + \frac{1}{2}\mathrm{tr}\left\{\frac{\partial}{\partial x}\left(\frac{\partial \Phi^i}{\partial x}\right)^{\mathrm{T}} hh^{\mathrm{T}}\right\}\right]\mathrm{d}t \\
&\quad + \frac{\partial \Phi^i}{\partial x}gu\mathrm{d}t + \frac{\partial \Phi^i}{\partial x}h\mathrm{d}w
\end{aligned} \tag{4}$$

Suppose that a stochastic port-Hamiltonian system (2) is transformed into another one by a time-invariant coordinate $\bar{x} = \Phi(x)$. Then, the following equation holds for all u and w

R.H.S. of Eq. (4)

$$\begin{aligned}
&\equiv \left[(\bar{J}(\bar{x}) - \bar{R}(\bar{x}))\frac{\partial H(\Phi^{-1}(\bar{x}))^{\mathrm{T}}}{\partial \bar{x}}\right]^i \mathrm{d}t + \left[\bar{g}(\bar{x})u\right]^i \mathrm{d}t \\
&\quad + \left[\bar{h}(\bar{x})\mathrm{d}w\right]^i.
\end{aligned} \tag{5}$$

This implies

$$\frac{\partial \Phi}{\partial x}g \equiv \bar{g}, \qquad \frac{\partial \Phi}{\partial x}h \equiv \bar{h}. \tag{6}$$

The symbol $[\cdot]^{-1}$ represents the inverse matrix of the argument and if no confusion arises, $\cdot^{-1}$ represents the inverse function of the argument function in what follows. The first term of the right hand side of Eq. (5) is calculated as follows:

$$\begin{aligned}
&\left[(\bar{J}(\bar{x}) - \bar{R}(\bar{x}))\frac{\partial H(\Phi^{-1}(\bar{x}))^{\mathrm{T}}}{\partial \bar{x}}\right]^i \\
&= \left[\frac{\partial \Phi}{\partial x}\left[\frac{\partial \Phi}{\partial x}\right]^{-1}(\bar{J} - \bar{R})\left[\frac{\partial \Phi}{\partial x}\right]^{-\mathrm{T}}\frac{\partial \Phi}{\partial x}^{\mathrm{T}}\frac{\partial H(\Phi^{-1}(\bar{x}))^{\mathrm{T}}}{\partial \bar{x}}\right]^i \\
&= \frac{\partial \Phi^i}{\partial x}\left[\frac{\partial \Phi}{\partial x}\right]^{-1}(\bar{J} - \bar{R})\left[\frac{\partial \Phi}{\partial x}\right]^{-\mathrm{T}}\frac{\partial H(x)^{\mathrm{T}}}{\partial x}.
\end{aligned} \tag{7}$$

By using Eqs. (4), (5) and (7), we have

$$\begin{aligned}
\frac{1}{2}\mathrm{tr}\left\{\frac{\partial}{\partial x}\left(\frac{\partial \Phi^i}{\partial x}\right)^{\mathrm{T}} hh^{\mathrm{T}}\right\} &= \frac{\partial \Phi^i}{\partial x}\left[\Big(\left[\frac{\partial \Phi}{\partial x}\right]^{-1}\bar{J}\left[\frac{\partial \Phi}{\partial x}\right]^{-\mathrm{T}} - J\Big)\right. \\
&\quad \left. -\Big(\left[\frac{\partial \Phi}{\partial x}\right]^{-1}\bar{R}\left[\frac{\partial \Phi}{\partial x}\right]^{-\mathrm{T}} - R\Big)\right]\frac{\partial H(x)^{\mathrm{T}}}{\partial x}.
\end{aligned} \tag{8}$$

We define the matrices $K(x)$ and $S(x)$ from Eq. (8) as

$$\begin{aligned}
K(x) &:= \left[\frac{\partial \Phi}{\partial x}\right]^{-1}\bar{J}(\Phi(x))\left[\frac{\partial \Phi}{\partial x}\right]^{-\mathrm{T}} - J(x), \\
S(x) &:= \left[\frac{\partial \Phi}{\partial x}\right]^{-1}\bar{R}(\Phi(x))\left[\frac{\partial \Phi}{\partial x}\right]^{-\mathrm{T}} - R(x).
\end{aligned}$$

Then, $K(x)$ is skew-symmetric since $J(x)$ and $\bar{J}(\Phi^{-1}(x))$ are so and, for $R(x)$ is symmetric and $\bar{R}(\Phi^{-1}(x))$ is symmetric positive semi-definite, $S(x)$ is symmetric and $R(x)+S(x)$ is symmetric positive semi-definite. This proves the necessity of Eq. (3).

Secondly, the sufficiency of Eq. (3) is shown. The output y can be calculated using Eq. (6) in the new coordinate $\bar{x}$ as

$$\begin{aligned}
y &= g^{\mathrm{T}}\left[\frac{\partial \Phi}{\partial x}\right]^{\mathrm{T}}\left[\frac{\partial \Phi}{\partial x}\right]^{-\mathrm{T}}\frac{\partial H}{\partial x}^{\mathrm{T}} \\
&= \left(\frac{\partial \Phi}{\partial x}g\right)^{\mathrm{T}}\left(\frac{\partial H}{\partial x}\frac{\partial \Phi^{-1}(\bar{x})}{\partial \bar{x}}\right)^{\mathrm{T}} = \bar{g}^{\mathrm{T}}\frac{\partial H}{\partial \bar{x}}^{\mathrm{T}}.
\end{aligned} \tag{9}$$

Therefore, the output y has the same form in $\bar{x}$ as in x. Now suppose that Eq. (3) holds. Then, by utilizing Eqs. (3) and

(4), the dynamics of the system can be calculated in the new coordinate $\bar{x}$ as

$$\begin{aligned}
\mathrm{d}\bar{x}^i &= \left[\frac{\partial \Phi^i}{\partial x}(J-R)\frac{\partial H}{\partial x}^{\mathrm{T}} + \frac{\partial \Phi^i}{\partial x}(K-S)\frac{\partial H}{\partial x}^{\mathrm{T}}\right]\mathrm{d}t \\
&+ \frac{\partial \Phi^i}{\partial x} g u \mathrm{d}t + \frac{\partial \Phi^i}{\partial x} h \mathrm{d}w \\
&= \Bigg[\Bigg(\underbrace{\frac{\partial \Phi}{\partial x}(J+K)\frac{\partial \Phi}{\partial x}^{\mathrm{T}}}_{=:\bar{J}} - \underbrace{\frac{\partial \Phi}{\partial x}(R+S)\frac{\partial \Phi}{\partial x}^{\mathrm{T}}}_{=:\bar{R}}\Bigg) \\
&\times \frac{\partial H(\Phi^{-1}(\bar{x}))}{\partial \bar{x}}^{\mathrm{T}}\Bigg]^i \mathrm{d}t + \underbrace{\frac{\partial \Phi^i}{\partial x} g}_{=:\bar{g}^i} u \mathrm{d}t + \underbrace{\frac{\partial \Phi^i}{\partial x} h}_{=:\bar{h}^i} \mathrm{d}w \quad (10)
\end{aligned}$$

$$\begin{aligned}
&=: \left[(\bar{J}(\bar{x}) - \bar{R}(\bar{x}))\frac{\partial H(\Phi^{-1}(\bar{x}))^{\mathrm{T}}}{\partial \bar{x}}\right]^i \mathrm{d}t + \left[\bar{g}(\bar{x})u\right]^i \mathrm{d}t \\
&+ \left[\bar{h}(\bar{x})\mathrm{d}w\right]^i. \qquad (11)
\end{aligned}$$

Then, $\bar{J}(\bar{x})$ is skew-symmetric since $J(\Phi^{-1}(\bar{x}))$ and $K(\Phi^{-1}(\bar{x}))$ are so, and $\bar{R}(\bar{x})$ is symmetric positive semi-definite because of the assumption that $R(\Phi^{-1}(\bar{x})) + S(\Phi^{-1}(\bar{x}))$ is so. This proves the sufficiency of Eq. (3). ■

Remark 1: Consider the port-Hamiltonian system (1) and we apply Theorem 1 to the system. In this case, $h(x) \equiv 0$ holds. Then, Eq. (3) is rewritten by

$$\frac{\partial \Phi^i}{\partial x}(K-S)\frac{\partial H}{\partial x}^{\mathrm{T}} = 0, \quad (i = 1, 2, \cdots, n). \qquad (12)$$

The condition (12) holds for any H and Φ if we select the matrices $K(x) \equiv 0$ and $S(x) \equiv 0$. This implies that any time-invariant coordinate transformation preserves the Hamiltonian structure. It shows that Theorem 1 implies the existing result for the port-Hamiltonian system as a special case.

We now turn to another fundamental property of the stochastic port-Hamiltonian system which is an extension of the passivity of the port-Hamiltonian system. The passivity of the port-Hamiltonian system is stated by the following lemma.

Lemma 2: [2] Consider the port-Hamiltonian system (1)*. Suppose that Hamiltonian $H(x)$ is positive semidefinite. Then, the system is passive with the storage function $H(x)$. Furthermore, if $R(x) \equiv 0$, then, the system is lossless with the storage function $H(x)$.*

In the literature [11], the notion of **stochastic passivity** which corresponds to the passivity for the deterministic system is introduced for the stochastic system as follows.

Definition 3: [11] Consider the nonlinear stochastic differential system written in the sense of Itô

$$\begin{cases} \mathrm{d}x = f(x,u)\mathrm{d}t + h(x)\mathrm{d}w \\ y \ = s(x,u) \end{cases}, \qquad (13)$$

where $x(t) \in \mathbb{R}^n$, $u(t), y(t) \in \mathbb{R}^m$ describe the state, the input and the output, respectively. $w(t) \in \mathbb{R}^r$ is a standard Wiener process defined on a probability space $(\Omega, \mathcal{F}, \mathcal{P})$. $f: \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n, h: \mathbb{R}^n \to \mathbb{R}^{n\times r}$ and $s: \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$ are smooth functions.

The stochastic system (13) is said to be **stochastic passive** if there exists s positive semidefinite function $V: \mathbb{R}^n \to \mathbb{R}$, called the storage function satisfying

$$\mathcal{L}V(x) \leq s(x,u)^{\mathrm{T}} u. \qquad (14)$$

Here $\mathcal{L}(\cdot)$ represents the infinitesimal generator [16] of the stochastic process of the system (13) defined as

$$\mathcal{L}(\cdot) := \frac{\partial(\cdot)}{\partial x} f + \frac{1}{2}\mathrm{tr}\left\{\frac{\partial}{\partial x}\left(\frac{\partial(\cdot)}{\partial x}\right)^{\mathrm{T}} h h^{\mathrm{T}}\right\}. \qquad (15)$$

Unlike in the case of the port-Hamiltonian system, the stochastic port-Hamiltonian system is not always stochastic passive even if Hamiltonian $H(x)$ is positive semi-definite. The following lemma characterizes the stochastic passivity of the stochastic port-Hamiltonian system.

Lemma 3: Consider the stochastic port-Hamiltonian system (2)*. Suppose that Hamiltonian $H(x)$ is positive semi-definite. Then, the system is stochastic passive if and only if*

$$\frac{1}{2}\mathrm{tr}\left\{\frac{\partial}{\partial x}\left(\frac{\partial H(x)}{\partial x}\right)^{\mathrm{T}} h(x)h(x)^{\mathrm{T}}\right\} \leq \frac{\partial H}{\partial x} R \frac{\partial H}{\partial x}^{\mathrm{T}} \qquad (16)$$

holds.

Proof: Firstly, the necessity of Eq. (16) is shown. By utilizing Eq. (15), $\mathcal{L}H(x)$ is calculated as

$$\begin{aligned}
\mathcal{L}H(x) &= \frac{\partial H}{\partial x}\left((J-R)\frac{\partial H}{\partial x}^{\mathrm{T}} + gu\right) \\
&+ \frac{1}{2}\mathrm{tr}\left\{\frac{\partial}{\partial x}\left(\frac{\partial H}{\partial x}\right)^{\mathrm{T}} h h^{\mathrm{T}}\right\} \\
&= -\frac{\partial H}{\partial x} R \frac{\partial H}{\partial x}^{\mathrm{T}} + y^{\mathrm{T}} u + \frac{1}{2}\mathrm{tr}\left\{\frac{\partial}{\partial x}\left(\frac{\partial H}{\partial x}\right)^{\mathrm{T}} h h^{\mathrm{T}}\right\} \\
&. \qquad (17)
\end{aligned}$$

Since J is skew-symmetric, the following equation holds

$$\frac{\partial H}{\partial x} J \frac{\partial H}{\partial x}^{\mathrm{T}} = 0.$$

Suppose that the system (2) is stochastic passive. Then, from Eq. (14), we obtain

$$\mathcal{L}H(x) \leq y^{\mathrm{T}} u. \qquad (18)$$

By utilizing Eqs. (17) and (18), we can prove the necessity of Eq. (16).

Secondly, the sufficiency of Eq. (16) is shown. Substituting Eq. (16) for Eq. (17), we obtain Eq. (18). According to Eq. (14), this proves the sufficiency of Eq. (16). ■

Due to Lemma 3, we obtain the following corollary.

Corollary 1: Consider the stochastic port-Hamiltonian system (2). *Suppose that Hamiltonian $H(x)$ is positive semi-definite and $R(x) \equiv 0$. Then, the system is stochastic lossless* [1] *if and only if*

$$\mathrm{tr}\left\{\frac{\partial}{\partial x}\left(\frac{\partial H(x)}{\partial x}\right)^{\mathrm{T}} h(x)h(x)^{\mathrm{T}}\right\} = 0$$

holds. Suppose the system (2) *is stochastic lossless and moreover $u \equiv 0$. Then, Hamiltonian $H(x)$ is a conserved quantity of the stochastic system.*

Remark 2: Consider the port-Hamiltonian system (1) and we apply Lemma 3 to the system. In this case, $h(x) \equiv 0$ holds. Then, the condition (16) always holds. This implies that Lemma 3 reduces to Lemma 2 for the deterministic port-Hamiltonian system as a special case.

III. Recovery of stochastic passivity via stochastic generalized canonical transformations

The generalized canonical transformations are proposed in [15] to transform both the input and the output of the port-Hamiltonian system (1) into those of another port-Hamiltonian system which has another Hamiltonian. This section extends such transformations to stochastic versions for the stochastic port-Hamiltonian systems. We clarify the conditions of the transformations by which the transformed system preserves the stochastic Hamiltonian structure and, furthermore, obtains stochastic passivity.

Firstly, we define the **stochastic generalized canonical transformations**. Then, we show the conditions which these transformations should satisfy.

Definition 4: A set of transformations

$$\begin{aligned} \bar{x} &= \Phi(x) \\ \bar{H} &= H(x) + U(x) \\ \bar{y} &= y + \alpha(x) \\ \bar{u} &= u + \beta(x) \end{aligned} \qquad (19)$$

that changes the coordinate x to $\bar{x}$, Hamiltonian H to $\bar{H}$, the output y to $\bar{y}$ and the input u to $\bar{u}$ is said to be a stochastic generalized canonical transformation for the stochastic port-Hamiltonian system if it transforms the system described by (2) into another one. Here $U : \mathbb{R}^n \to \mathbb{R}$, $\alpha : \mathbb{R}^n \to \mathbb{R}^m$ and $\beta : \mathbb{R}^n \to \mathbb{R}^m$ are appropriate functions, respectively.

Theorem 2: Consider the stochastic port-Hamiltonian system (2). *For a given scalar function $U(x)$ and a given vector function $\beta(x)$, the set of transformations defined by* (19) *yields a stochastic generalized canonical transformation if and only if there exists a skew-symmetric matrix $P(x)$, a symmetric matrix $Q(x)$ such that $R(x) + Q(x)$ is positive semi-definite and a function $\Phi(x)$ and they satisfy*

$$\frac{1}{2}\mathrm{tr}\left\{\frac{\partial}{\partial x}\left(\frac{\partial \Phi^i}{\partial x}\right)^{\mathrm{T}} hh^{\mathrm{T}}\right\} = \frac{\partial \Phi^i}{\partial x}\left[(J-R)\frac{\partial U}{\partial x}^{\mathrm{T}} + g\beta + (P-Q)\frac{\partial (H+U)}{\partial x}^{\mathrm{T}}\right] \quad (i = 1, 2, \cdots, n). \qquad (20)$$

Further the change of the output $\alpha(x)$ defined in (19) *is given by*

$$\alpha(x) = g(x)^{\mathrm{T}}\frac{\partial U(x)}{\partial x}^{\mathrm{T}}. \qquad (21)$$

Proof: Firstly, the necessity of the theorem is shown. In the same manner as Eq. (5), the dynamics of the system in the new coordinate $\bar{x}$ is calculated as

$$\mathrm{d}\bar{x}^i = \left[\frac{\partial \Phi^i}{\partial x}(J-R)\frac{\partial H}{\partial x}^{\mathrm{T}} + \frac{1}{2}\mathrm{tr}\left\{\frac{\partial}{\partial x}\left(\frac{\partial \Phi^i}{\partial x}\right)^{\mathrm{T}} hh^{\mathrm{T}}\right\}\right]\mathrm{d}t + \frac{\partial \Phi^i}{\partial x}gu\mathrm{d}t + \frac{\partial \Phi^i}{\partial x}h\mathrm{d}w \qquad (22)$$

Suppose a function $\Phi(x)$ which yields a stochastic generalized canonical transformation exists and a stochastic port-Hamiltonian system (2) is transformed into another one using a stochastic generalized canonical transformation with U, β and Φ such that (20) holds. Then, the following equation holds for all u and w

R.H.S. of Eq. (22)

$$\begin{aligned} &\equiv \left[(\bar{J}-\bar{R})\frac{\partial \bar{H}(\Phi^{-1}(\bar{x}))^{\mathrm{T}}}{\partial \bar{x}}\right]^i \mathrm{d}t + \left[\bar{g}\bar{u}\right]^i\mathrm{d}t + \left[\bar{h}\mathrm{d}w\right]^i \\ &= \frac{\partial \Phi^i}{\partial x}\left[\frac{\partial \Phi}{\partial x}\right]^{-1}(\bar{J}-\bar{R})\left[\frac{\partial \Phi}{\partial x}\right]^{-\mathrm{T}}\frac{\partial (H(x)+U(x))^{\mathrm{T}}}{\partial x}\mathrm{d}t \\ &\quad + \left[\bar{g}(u+\beta)\right]^i\mathrm{d}t + \left[\bar{h}\mathrm{d}w\right]^i. \end{aligned} \qquad (23)$$

This implies

$$\frac{\partial \Phi}{\partial x}g \equiv \bar{g}, \qquad \frac{\partial \Phi}{\partial x}h \equiv \bar{h}. \qquad (24)$$

Using Eqs. (22), (23) and (24), we have

$$\frac{1}{2}\mathrm{tr}\left\{\frac{\partial}{\partial x}\left(\frac{\partial \Phi^i}{\partial x}\right)^{\mathrm{T}} hh^{\mathrm{T}}\right\} = \frac{\partial \Phi^i}{\partial x}\left[\left[\frac{\partial \Phi}{\partial x}\right]^{-1}(\bar{J}-\bar{R})\left[\frac{\partial \Phi}{\partial x}\right]^{-\mathrm{T}}\frac{\partial (H+U)}{\partial x}^{\mathrm{T}} - (J-R)\frac{\partial H}{\partial x}^{\mathrm{T}} + g\beta\right]. \qquad (25)$$

Here we define the matrices $P(x)$ and $Q(x)$ as

$$\begin{aligned} P(x) &:= \left[\frac{\partial \Phi}{\partial x}\right]^{-1}\bar{J}(\Phi(x))\left[\frac{\partial \Phi}{\partial x}\right]^{-\mathrm{T}} - J(x), \\ Q(x) &:= \left[\frac{\partial \Phi}{\partial x}\right]^{-1}\bar{R}(\Phi(x))\left[\frac{\partial \Phi}{\partial x}\right]^{-\mathrm{T}} - R(x). \end{aligned} \qquad (26)$$

[1] In this paper, we define the notion of stochastic lossless by Eq. (14) replacing "$\leq$" by "$=$" as in the similar manner of the deterministic case.

Then, $Q(x)$ is skew-symmetric since $J(x)$ and $\bar{J}(\Phi(x))$ are so and, for $R(x)$ is symmetric and $\bar{R}(\Phi(x))$ is symmetric positive semi-definite, $Q(x)$ is symmetric and $R(x)+Q(x)$ is symmetric positive semi-definite. By substituting Eq. (26) for Eq. (25), Equation (20) is obtained immediately.

The change of the output $\alpha(x)$ which yields a stochastic generalized canonical transformation (19) can be calculated as

$$\begin{aligned}\alpha &= \bar{g}^{\mathrm{T}}\frac{\partial \bar{H}(\Phi^{-1}(\bar{x}))}{\partial \bar{x}}^{\mathrm{T}} - g^{\mathrm{T}}\frac{\partial H(x)}{\partial x}^{\mathrm{T}} \\ &= g^{\mathrm{T}}\frac{\partial \Phi}{\partial x}^{\mathrm{T}}\left[\frac{\partial \Phi}{\partial x}\right]^{-\mathrm{T}}\frac{\partial (H+U)}{\partial x}^{\mathrm{T}} - g^{\mathrm{T}}\frac{\partial H(x)}{\partial x}^{\mathrm{T}} = g^{\mathrm{T}}\frac{\partial U}{\partial x}^{\mathrm{T}}.\end{aligned}$$

This proves the necessity of the theorem.

Secondly, the sufficiency of the theorem is shown. Now suppose the assumption of the theorem holds. Then, by substituting Eq. (20) for (22), the dynamics of the system can be calculated in the new coordinate $\bar{x}$ as

$$\begin{aligned}\mathrm{d}\bar{x}^i &= \Bigg[\frac{\partial \Phi^i}{\partial x}(J-R)\frac{\partial H}{\partial x}^{\mathrm{T}} + \frac{\partial \Phi^i}{\partial x}\Bigg[(J-R)\frac{\partial U}{\partial x}^{\mathrm{T}} + g\beta \\ &\quad + (P-Q)\frac{\partial (H+U)}{\partial x}^{\mathrm{T}}\Bigg]\Bigg]\mathrm{d}t + \frac{\partial \Phi^i}{\partial x}gu\mathrm{d}t + \frac{\partial \Phi^i}{\partial x}h\mathrm{d}w \\ &= \Bigg[\frac{\partial \Phi}{\partial x}\Big((J+P)-(R+Q)\Big)\frac{\partial \Phi}{\partial x}^{\mathrm{T}} \\ &\quad \times \frac{\partial (H(\Phi^{-1}(\bar{x}))+U(\Phi^{-1}(\bar{x})))}{\partial \bar{x}}^{\mathrm{T}}\Bigg]^i \mathrm{d}t \\ &\quad + \frac{\partial \Phi^i}{\partial x}g(u+\beta)\mathrm{d}t + \frac{\partial \Phi^i}{\partial x}h\mathrm{d}w. \qquad (27)\end{aligned}$$

$\bar{J}, \bar{R}, \bar{g}$ and $\bar{h}$ are given by

$$\begin{aligned}\bar{J}(\bar{x}) &= \frac{\partial \Phi(x)}{\partial x}(J(x)+P(x))\frac{\partial \Phi(x)}{\partial x}^{\mathrm{T}}\Bigg|_{x=\Phi^{-1}(\bar{x})} \\ \bar{R}(\bar{x}) &= \frac{\partial \Phi(x)}{\partial x}(R(x)+Q(x))\frac{\partial \Phi(x)}{\partial x}^{\mathrm{T}}\Bigg|_{x=\Phi^{-1}(\bar{x})} \\ \bar{g}(\bar{x}) &= \frac{\partial \Phi(x)}{\partial x}g(x)\Bigg|_{x=\Phi^{-1}(\bar{x})}, \\ \bar{h}(\bar{x}) &= \frac{\partial \Phi(x)}{\partial x}h(x)\Bigg|_{x=\Phi^{-1}(\bar{x})}. \qquad (28)\end{aligned}$$

Then, $\bar{J}(\bar{x})$ is skew-symmetric since $J(\Phi^{-1}(\bar{x}))$ and $P(\Phi^{-1}(\bar{x}))$ are so, and $\bar{R}(\bar{x})$ is symmetric positive semi-definite because of the assumption that $R(\Phi^{-1}(\bar{x})) + Q(\Phi^{-1}(\bar{x}))$ is so. By utilizing Eq. (21), the output in the new coordinate $\bar{y}$ is obtained as

$$\begin{aligned}\bar{y} &= g^{\mathrm{T}}\frac{\partial H(x)}{\partial x}^{\mathrm{T}} + g^{\mathrm{T}}\frac{\partial U(x)}{\partial x}^{\mathrm{T}} \\ &= g^{\mathrm{T}}\frac{\partial \Phi}{\partial x}^{\mathrm{T}}\left[\frac{\partial \Phi}{\partial x}\right]^{-\mathrm{T}}\frac{\partial (H+U)}{\partial x}^{\mathrm{T}} \\ &= \bar{g}^{\mathrm{T}}\frac{\partial \bar{H}(\Phi^{-1}(\bar{x}))}{\partial \bar{x}}^{\mathrm{T}}.\end{aligned}$$

This proves the sufficiency of the theorem. ■

Remark 3: Consider the port-Hamiltonian system (1) and we apply Theorem 2 to the system. In this case, $h(x) \equiv 0$ holds. Then, the condition (20) in Theorem 2 can be rewritten as

$$\frac{\partial \Phi}{\partial x}\left[(J-R)\frac{\partial U}{\partial x}^{\mathrm{T}} + g\beta + (P-Q)\frac{\partial (H+U)}{\partial x}^{\mathrm{T}}\right] = 0.$$

This coincides with the time-invariant version of Theorem 1 (i) in [18] which is a generalized version of a result from [15] incorporating the dissipative element R in (1). This implies that in considering the time-invariant case, Theorem 2 implies a result for the port-Hamiltonian system as a special case.

By utilizing Lemma 3 and Theorem 2, the following theorem states a condition under which a transformed stochastic port-Hamiltonian system by a stochastic generalized canonical transformation becomes stochastic passive with Hamiltonian $\bar{H}$ as a storage function and, furthermore, a stabilization method based on stochastic passivity.

Theorem 3: Consider the stochastic port-Hamiltonian system (2) and transform it by the stochastic generalized canonical transformation with appropriate functions $U(x)$ and $\beta(x)$ such that $H(x)+U(x) \geq 0$. Then, the transformed system becomes stochastic passive with new Hamiltonian $\bar{H} := H+U$ as a storage function if and only if

$$\begin{aligned}&\frac{1}{2}\mathrm{tr}\left\{\frac{\partial}{\partial x}\left(\frac{\partial (H+U)}{\partial x}\left[\frac{\partial \Phi}{\partial x}\right]^{-1}\right)^{\mathrm{T}} h(x)h(x)^{\mathrm{T}}\frac{\partial \Phi}{\partial x}^{\mathrm{T}}\right\} \\ &\qquad \leq \frac{\partial (H+U)}{\partial x}(R+Q)\frac{\partial (H+U)}{\partial x}^{\mathrm{T}} \qquad (29)\end{aligned}$$

holds. Furthermore, if $H+U$ is positive definite and a set defined as bellow holds $\Gamma \cap \Omega = \{0\}$. Then the unity feedback $\bar{u} = -\bar{y}$ renders the system asymptotically stable in probability. Here, the distribution Λ, the sets Γ and Ω are defined as

$$\begin{aligned}\Lambda &= \mathrm{span}\{\mathrm{ad}_{\bar{f}_0}^k \bar{g}_i | 0 \leq k \leq n-1, 1 \leq i \leq m\} \\ \Gamma &= \{\bar{x} \in \mathbb{R}^n | \mathcal{L}_0^k(H+U) = 0,\ k = 1, \cdots, r\} \\ \Omega &= \{\bar{x} \in \mathbb{R}^n | \mathcal{L}_0^k L_\lambda(H+U) = 0, \\ &\qquad \forall \lambda \in \Lambda,\ k = 0, \cdots, r-2\},\end{aligned}$$

where $\bar{f}_0 := (\bar{J}(\bar{x}) - \bar{R}(\bar{x}))\frac{\partial \bar{H}(\Phi^{-1}(\bar{x}))}{\partial \bar{x}}^{\mathrm{T}}$, $\bar{g}_i$ represents i-th row of $\bar{g}$ and an operator $\mathcal{L}_0$ denotes the operator (15) in which $\bar{f}$ is replaced by $\bar{f}_0$.

Proof: Firstly, the former part of the theorem is shown. Due to Lemma 3, the necessary and sufficient condition is that the following equation holds in the new coordinate transformed by the stochastic generalized canonical transformation

$$\frac{1}{2}\mathrm{tr}\left\{\frac{\partial}{\partial \bar{x}}\left(\frac{\partial \bar{H}(\Phi^{-1}(\bar{x}))}{\partial \bar{x}}\right)^{\mathrm{T}} \bar{h}(\bar{x})\bar{h}(\bar{x})^{\mathrm{T}}\right\} \leq \frac{\partial \bar{H}}{\partial \bar{x}}\bar{R}\frac{\partial \bar{H}}{\partial \bar{x}}^{\mathrm{T}}. \tag{30}$$

Let us note that the following equation holds

$$\frac{\partial}{\partial \bar{x}}\left(\frac{\partial \bar{H}(\Phi^{-1}(\bar{x}))}{\partial \bar{x}}\right)^{\mathrm{T}} = \frac{\partial}{\partial x}\left(\frac{\partial \bar{H}(x)}{\partial x}\left[\frac{\partial \Phi(x)}{\partial x}\right]^{-1}\right)^{\mathrm{T}}\left[\frac{\partial \Phi(x)}{\partial x}\right]^{-1}. \tag{31}$$

By utilizing (28),(30) and (31), one can prove Eq. (29) holds immediately. The latter part of theorem is shown by directly applying Corollary 4.7 in [11]. ■

Example 1: Consider a typical mechanical system with random noises

$$\begin{cases} \begin{pmatrix} \mathrm{d}q \\ \mathrm{d}p \end{pmatrix} = \begin{pmatrix} 0 & I_m \\ -I_m & -D \end{pmatrix}\begin{pmatrix} \frac{\partial H(q,p)}{\partial q}^{\mathrm{T}} \\ \frac{\partial H(q,p)}{\partial p}^{\mathrm{T}} \end{pmatrix}\mathrm{d}t \\ \qquad + \begin{pmatrix} 0 \\ I_m \end{pmatrix} u\mathrm{d}t + \begin{pmatrix} 0 \\ h(q,p) \end{pmatrix}\mathrm{d}w \\ y = \frac{\partial H(q,p)}{\partial p}^{\mathrm{T}} = M(q)^{-1}p \end{cases} \tag{32}$$

with the Hamiltonian $H = \frac{1}{2}p^{\mathrm{T}}M(q)^{-1}p$, where $q, p \in \mathbb{R}^m$, a positive matrix $M(q)$ denotes the inertia matrix, a positive semidefinite matrix D denotes the friction coefficients and $w(t) \in \mathbb{R}^m$ is a standard Wiener process. I_m represents the $m \times m$ unit matrix. Here we suppose that $h = \mathrm{diag}\{k_h^1 p^1, \cdots, k_h^m p^m\}$ with real numbers $k_h^1, \cdots, k_h^m$. We assign any positive definite scalar function $U(q)$ so that $\bar{H}$ becomes positive definite. By using Theorem 2, we obtain a stochastic generalized canonical transformation as $\bar{q} = q$, $\bar{p} = p$, $\alpha = 0$, $\beta = \frac{\partial U(q)}{\partial q}^{\mathrm{T}} - Q_{22}M(q)^{-1}p$, where

$$P := 0, \quad Q := \begin{pmatrix} 0 & 0 \\ 0 & Q_{22} \end{pmatrix}, \quad Q_{22} = Q_{22}^{\mathrm{T}} \geq 0.$$

Then by using Theorem 3, the condition under which the transformed system becomes stochastic passive is

$$\frac{1}{2}\mathrm{tr}\left\{M(q)^{-1}hh^{\mathrm{T}}\right\} \leq p^{\mathrm{T}}M(q)^{-1}\left(D + Q_{22}\right)M(q)^{-1}p. \tag{33}$$

Since we have

$$\begin{aligned} \mathrm{tr}\left\{M(q)^{-1}hh^{\mathrm{T}}\right\} &= h^{\mathrm{T}}M(q)^{-1}h \leq \|M(q)^{-1}\|\|h\|^2 \\ &\leq \|M(q)^{-1}\| \max_{1\leq i\leq m}\{k_h^i\}^2\|p\|^2, \end{aligned}$$

a candidate $Q_{22} = \frac{\max_{1\leq i\leq m}\{k_h^i\}^2}{2\|M(q)^{-1}\|} I_m$ satisfies Eq. (33) and $u = -M(q)^{-1}p - \beta$ renders the system stochastic stable.

IV. CONCLUSION

This paper has introduced stochastic port-Hamiltonian systems and clarified some of their properties. Firstly, we have shown a necessary and sufficient condition to preserve the stochastic Hamiltonian structure of the original system under time-invariant coordinate transformations. Secondly, we have derived a condition to maintain stochastic passivity of the system. Thirdly, we have introduced stochastic generalized canonical transformations. We have also given a condition that the transformed system by this transformation becomes stochastic passive. Finally, a stabilization method based on passivity has been proposed.

In our recent result [19], we extend the results to the time-varying version and apply them to stochastic trajectory tracking control.

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