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Conditions for a Massless Plasma Analysis to Predict Stabilization of the Tokamak Plasma Vertical Instability

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Abstract— **This paper describes the problem of feedback control for stabilization of the plasma vertical instability in a tokamak. Such controllers are typically designed based on a model that assumes the plasma mass m is identically zero. However, the assumption of m = 0 can lead to a controller C that appears to be stabilizing according to the massless analysis but in fact will increase the instability of the physical system.**

In this work, we consider the most commonly used type of controller, a proportional-derivative controller. Suppose C is a PD controller which stabilizes the vertical instability with plasma mass assumed to be zero. We give easy-to-check necessary and sufficient conditions for C to also stabilize the physical system, in which the plasma actually has a small mass.

I. Introduction

Tokamaks are torus-shaped devices designed to confine a plasma composed of ionized hydrogen isotopes while the plasma is heated to initiate fusion reactions. This paper describes the problem of feedback control for stabilization of the vertical instability in tokamaks. Such controllers are typically designed based on a massless model of the plasma. Our goal is to produce a few additional constraints on the design to ensure that the controller performs adequately in the physical system, in which plasmas actually have a small mass. The solution of this problem is of more than academic interest, since the assumption of zero mass represents a point of bifurcation in the plasma response model [1]. Without careful handling, this bifurcation can lead to a completely erroneous analysis of the system and massively destabilizing control design.

A. Background on the Model

The dynamics of the plant comprising a tokamak confining an assumed axisymmetric plasma is constructed from the basic electromagnetic equation [1]

$$M\dot{I} + R\delta I + \Psi_z \dot{z_C} + \Psi_r \dot{r_C} = U\delta V \qquad (1)$$

where M and R are the mutual inductance and resistance of the toroidal conductors whose currents define the states of (1), and Ψ_z, Ψ_r represent the partial derivatives of flux values at those conductors with respect to vertical (z_C) and radial (r_C) motion of the plasma current centroid ("center of mass" of the distributed plasma current). Toroidal currents in (respectively, voltages on) conductors are represented by the vector I (resp., V) while $\delta I = I - I_{\text{eq}}$ (resp., $\delta V = V - V_{\text{eq}}$) represents a perturbation of the currents (voltages) from their values defining a nominal plasma equilibrium. The vector I includes both currents in active control coils and in toroidal conducting vessel elements. We make the standard approximation that plasma current is conserved on the time scale of the vertical instability. In the following, we use the notation $\delta I = [\delta I_c \;\; \delta I_v]^T$ to represent a partitioning of the current vector into the n_c active control coils and the n_v passive (vacuum vessel) currents, $U = [\mathbf{I}_{n_c} \;\; \mathbf{0}_{n_c \times n_v}]^T$, where $\mathbf{I}_{n_c}$ and $\mathbf{0}_{n_c \times n_v}$ are identity and zero matrices respectively. The vertical motion of the centroid for a plasma having mass m can be represented by the inertial momentum equation

$$m\ddot{z_C} = f_z \delta z_C + f_I \delta I \qquad (2)$$

where $\delta z_C = z_C - z_{C,\text{eq}}$ represents perturbed values relative to the plasma current centroid vertical coordinate at the nominal plasma equilibrium, $f_z = \partial F_z / \partial z_C$, $f_I = \partial F_z / \partial I$, and F_z is the total vertical force on the plasma. We note that $\Psi_z = f_I^T$ [1].

The equations (1) and (2) can be combined to form the overall plant model. From equation (1) we obtain

$$M_{\#}\dot{I} + R\delta I + \Psi_z \dot{z_C} = U\delta V \qquad (3)$$

where $M_{\#} = M + \Psi_r(\partial r_C / \partial I)$ and $\partial r_C / \partial I$ is derived from a linearization of the plasma response around the chosen nominal plasma equilibrium. Defining the variables $v_z = \dot{z_C} = d(\delta z_C)/dt$, $x_z = [v_z^T \;\; \delta z_C^T]^T$, we can write (2) as

$$\begin{pmatrix} 0 & 1 \\ m & 0 \end{pmatrix} \dot{x_z} + \begin{pmatrix} -1 & 0 \\ 0 & -f_z \end{pmatrix} x_z + \begin{pmatrix} 0 \\ -f_I \end{pmatrix} \delta I = 0\,.$$

Combining with (3), we obtain the matrix equation

$$\tilde{M}\dot{x} + \tilde{R}x = \tilde{U}\delta V\,, \qquad (4)$$

where

$$x = \begin{pmatrix} v_z \\ \delta z_C \\ \delta I \end{pmatrix},\; \tilde{M} = \begin{pmatrix} 0 & 1 & 0 \\ m & 0 & 0 \\ 0 & \Psi_z & M_{\#} \end{pmatrix}, \qquad (5)$$

$$\tilde{R} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -f_z & -f_I \\ 0 & 0 & R \end{pmatrix},\; \tilde{U} = \begin{pmatrix} 0 \\ 0 \\ U \end{pmatrix}.$$

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If the equilibrium plasma boundary is sufficiently vertically elongated, so that $f_z > 0$, it can be shown [1] that the system (4) possesses a single positive real eigenvalue. The eigenvector corresponding to the unstable root corresponds to a nearly rigid vertical motion of the plasma current distribution, hence the name. Stabilization of the vertical instability requires a feedback control loop that produces radial magnetic field across the plasma in response to changes in some measure of the plasma vertical position, typically the plasma current centroid position z_C [2]. The vertical control portion of a tokamak shape and stability feedback system often takes the (PD) form

$$\delta V = -G_p(z_C - z_{C,\text{ref}}) - G_d\, dz_C/dt\,, \tag{6}$$

where δV is the additional (nonequilibrium) voltage applied to the PF coils, $z_C - z_{C,\text{ref}}$ is the displacement of z_C from some reference position $z_{C,\text{ref}}$, and dz_C/dt is the vertical velocity of the plasma. The gains G_p and G_d are vectors which map the scalar errors to the set of active control coils. Under many conditions, this feedback can completely stabilize the vertical instability [2].

B. Plasmas With and Without Mass

For convenience of design, the mass of the plasma is neglected in virtually every design of stabilizing controllers for the vertical instability [1]. In neglecting plasma mass, the inertial momentum equation (2) is used with $m = 0$ to derive the algebraic relation $\partial z_C/\partial I = -f_I/f_z$, which reduces by two the state dimension of (4). However, the assumption of $m = 0$ can lead to erroneous conclusions and, in particular, to controllers that appear to be stabilizing according to the massless analysis but will not actually stabilize the physical system. Such controllers can in fact cause the closed loop system to be far more unstable than the original open-loop system [1]. For the simple PD controller (6), additional physical insight is typically used in the control design process to ensure a correctly stabilizing controller. Absent this insight or when designing a more sophisticated (e.g. LQG) controller, the question remains as to how to guarantee that a stabilizing controller developed with the assumption $m = 0$ will also be stabilizing for the actual physical plant, for which $m > 0$.

In [1], stability properties of the open-loop system (1) were characterized and several conditions necessary for a PD controller of the form (6) to feedback-stabilize the physical (with-mass) system were derived. In the present work, we address the practical problem of characterizing when a controller that has been designed using the standard but strictly incorrect zero plasma mass assumption is able to actually stabilize the physical system with mass $m > 0$. We provide necessary and sufficient conditions for a PD controller (6) to guarantee stability with mass $m > 0$ if it stabilizes the system with $m = 0$.

C. Mathematical Problem Statement

We can derive an expression for the characteristic polynomial for the plasma with mass model (4) with PD feedback (6) which is key to analyzing stability [3]. In our situation, this polynomial can be defined as the determinant

$$\det\begin{pmatrix} -1 & \lambda & 0 \\ \lambda m & -f_z & -f_I \\ 0 & \lambda\Psi_z + \lambda g_d + g_p & \lambda M_\# + R \end{pmatrix}, \tag{7}$$

where the vectors $g_p = [G_p^T \quad 0]^T$ and $g_d = [G_d^T \quad 0]^T$ contain zeros in entries corresponding to the passive (vessel) conductors. The closed loop system is asymptotically stable if and only if this determinant is nonzero for all values of λ in the closed RHP, which in turn holds if and only if $\det G(\lambda) \neq 0$ where

$$G(\lambda) = \begin{pmatrix} \lambda^2 m - f_z & -f_I \\ \lambda\Psi_z + \lambda g_d + g_p & \lambda M_\# + R \end{pmatrix}.$$

Equivalently, the closed loop is stable if and only if the matrix $G(\lambda)$ is invertible for all λ in the closed RHP.

At this point we note that both $M_\#$ and R are positive definite; hence $M_\#$ is invertible and $[\lambda M_\# + R]$ is invertible for all λ in the closed RHP [3].

We would like to answer the following:

Problem 1.1: Suppose one has chosen a controller (g_d, g_p) so that the system is asymptotically stable for mass $m = 0$. What are necessary and sufficient conditions so that the system will be asymptotically stable for all masses in some interval containing zero? Furthermore, we would like a method for computing the maximum allowable mass.

We have explained how the asymptotic stability of the physical system is equivalent to the invertibility of $G(\lambda)$ for all λ in the closed RHP. Next we will express this condition in terms of another function of λ.

Definition 1.2: We define a function

$$S_m(\lambda) := \lambda^2 m - f_z + f_I\,[\lambda M_\# + R]^{-1}\,[\lambda\Psi_z + \lambda g_d + g_p]\,.$$

The reason for defining this function is the following:

Remark 1.3: For any complex number λ where $[\lambda M_\# + R]$ is invertible, we have

$$G(\lambda) \text{ is invertible } \iff S_m(\lambda) \neq 0\,.$$

Proof. This is immediate since $S_m(\lambda)$ is just the Schur complement of the matrix $G(\lambda)$. See, for example, page 21 of [4]. ■

In addition, this function has the following useful properties:

Lemma 1.4: For all $m \geq 0$,

1) $S_m(\lambda)$ is a rational function of λ,
2) $S_m(\lambda) = S_0(\lambda) + \lambda^2 m$,
3) $S_0(\lambda)$ is analytic and real valued at ∞. In particular,

$$S_0(\infty) = -f_z + f_I M_\#^{-1}[\Psi_z + g_d]\,.$$

Proof. From the definition of $S_m(\lambda)$, the first and second claim are obviously true. It remains to prove the last claim. We have

$$S_0(\lambda) = -f_z + f_I[\lambda M_\# + R]^{-1}[\lambda\Psi_z + \lambda g_d + g_p]\,.$$

Observe that for any nonzero λ where $[\lambda M_\# + R]$ is invertible we have

$$\begin{aligned}&[\lambda M_\# + R]^{-1}[\lambda\Psi_z + \lambda g_d + g_p]\\ &\quad = \lambda^{-1}[M_\# + \lambda^{-1}R]^{-1}\lambda[\Psi_z + g_d + \lambda^{-1}g_p]\\ &\quad = [M_\# + \lambda^{-1}R]^{-1}[\Psi_z + g_d + \lambda^{-1}g_p]\,.\end{aligned}$$

As $\lambda \to \infty$ we have

$$[\Psi_z + g_d + \lambda^{-1}g_p] \;\to\; [\Psi_z + g_d]$$

and since $M_\#$ is invertible,

$$[M_\# + \lambda^{-1}R]^{-1} \;\to\; M_\#^{-1}\,.$$

The result follows. ∎

All of this (Remark 1.3 and Lemma 1.4) combines to prove that we would solve Problem 1.1 if we could solve the following more general problem:

Problem 1.5: Suppose that for all $m \geq 0$,

1) $s_m(\lambda)$ is a rational function of λ,
2) $s_m(\lambda) = s_0(\lambda) + \lambda^2 m$,
3) $s_0(\lambda)$ is analytic and real valued at ∞.

Suppose $s_0(\lambda)$ has no zeros in the closed RHP. What are necessary and sufficient conditions so that there exists $m_* > 0$ with the property that $s_m(\lambda)$ has no λ zeroes in the closed RHP for all $m \in [0, m_*)$? Furthermore, we would like a method for computing the maximum choice for m_*.

Since it is merely a question concerning rational functions, Problem 1.5 constitutes a considerable abstraction of Problem 1.1. For this reason, we write $s_m(\lambda)$ to denote a general rational function with the desired properties (as in the statement of Problem 1.5), and write $S_m(\lambda)$ to mean the specific function (which has these properties) given in Definition 1.2.

In this paper we will solve Problem 1.5, thereby solving Problem 1.1.

II. Main Results

In this section we describe our main results. In §II-A we give the solution to Problem 1.5, and in §II-B we translate this into the language of the tokamak and plasma system, thus giving the solution to Problem 1.1. In §II-C we outline a practical method for determining the maximum plasma mass for which the zero-mass feedback controller is stabilizing.

A. Solution in Terms of Rational Functions

The solution to Problem 1.5 is given by the following theorem, whose proof we postpone until §III.

Theorem 2.1: Suppose that for $m \geq 0$,

1) $s_m(\lambda)$ is a rational function of λ,
2) $s_m(\lambda) = s_0(\lambda) + \lambda^2 m$,
3) $s_0(\lambda)$ is analytic and real valued at ∞.

Define a constant c by writing

$$s_0(\lambda) = s_0(\infty) + c\lambda^{-1} + b(\lambda)\,,\ \text{with } \lambda b(\lambda) \to 0 \text{ as } \lambda \to \infty.$$

Suppose that $s_0(\infty) \neq 0$ and $c \neq 0$. Further, suppose that $s_0(\lambda)$ has no zeros in the closed RHP. Then there exists $m_* > 0$ such that $s_m(\lambda)$ has no zeros in the closed RHP for all $m \in [0, m_*)$ if and only if $s_0(\infty) > 0$ and $c < 0$.

It remains to give a closed form expression for the largest possible choice of m_*. The following definition will help us write down such an expression.

Definition 2.2: Let $s_0(\lambda)$ be a rational function. We call ω_0 a *critical value* if the graph of $s_0(i\omega)$ crosses the positive real axis at $\omega = \omega_0$. Clearly there are finitely many critical values.

The following proposition, whose proof we also postpone until §III, gives the expression we seek.

Proposition 2.3: In Theorem 2.1, the largest choice for m_* is

$$m_* = \min_{k=1,\ldots,\ell} \left\{ \frac{s_0(i\omega_k)}{\omega_k^2} \right\}, \qquad (8)$$

where $\omega_1, \ldots, \omega_\ell$ are the nonzero critical values. If there are no nonzero critical values, then the quantity on the righthand side of (8) is defined to be ∞.

B. Vertical Stability of the Tokamak Plasma

Here are necessary and sufficient conditions for a massless plasma analysis to predict the vertical stability of a plasma with small mass (thus solving Problem 1.1).

Theorem 2.4: Consider the tokamak and plasma system discussed in §I. Suppose the closed loop system is stable for mass zero. Define

$$\begin{aligned}\xi &= -f_z + f_I M_\#^{-1}[\Psi_z + g_d]\\ \eta &= f_I M_\#^{-1} g_p - f_I M_\#^{-1} R M_\#^{-1}[\Psi_z + g_d]\,.\end{aligned}$$

Suppose that $\xi \neq 0$ and $\eta \neq 0$. Then there exists $m_* > 0$ such that the system is stable for all $m \in [0, m_*)$ if and only if $\xi > 0$ and $\eta < 0$.

Based on a small example set (see §II-C) and on physical intuition, we speculate that the maximum mass m_* will always be so large as to impose no practical constraint beyond satisfaction of the basic necessary and sufficient conditions $\xi > 0$ and $\eta < 0$ just described.

We remark that the necessity of these conditions has already been shown in [1]. To prove this result we require a lemma which explicitly computes the constant c as described in Theorem 2.1.

Lemma 2.5: The function $S_0(\lambda)$ given in Definition 1.2 has the following expansion in $1/\lambda$.

$$S_0(\lambda) = -f_z + f_I M_{\#}^{-1}[\Psi_z + g_d] + \frac{f_I M_{\#}^{-1} g_p - f_I M_{\#}^{-1} R M_{\#}^{-1}[\Psi_z + g_d]}{\lambda} + \ldots$$

Proof. We set $M = M_{\#}$ and $v = \Psi_z + g_d$ for ease of notation. Now we have

$$S_0(\lambda) = -f_z + f_I \left[\lambda M + R\right]^{-1} \left[\lambda v + g_p\right]$$

Observe that, for $|\lambda|$ sufficiently large,

$$\begin{aligned} [\lambda M + R]^{-1} &= \left[\lambda M(I + (\lambda M)^{-1} R)\right]^{-1} \\ &= \left[I + (\lambda M)^{-1} R\right]^{-1} (\lambda M)^{-1} \\ &= \left[I - (\lambda M)^{-1} R + \ldots\right] (\lambda M)^{-1} \\ &= \frac{M^{-1}}{\lambda} - \frac{M^{-1} R M^{-1}}{\lambda^2} + \ldots, \end{aligned}$$

and hence

$$\begin{aligned} &[\lambda M + R]^{-1} [\lambda v + g_p] \\ &= \left[\frac{M^{-1}}{\lambda} - \frac{M^{-1} R M^{-1}}{\lambda^2} + \ldots\right] [\lambda v + g_p] \\ &= M^{-1} v + \frac{M^{-1} g_p - M^{-1} R M^{-1} v}{\lambda} \\ &\quad + \ldots, \end{aligned}$$

and the result follows. ■

We remark in passing that the above also gives an alternate proof to part 3 of Lemma 1.4.

Proof of Theorem 2.4. We apply Theorem 2.1 (which will be proved in §III) to the function $S_m(\lambda)$ given in Definition 1.2 and take into account the explicit computations of $S_0(\infty)$ and c given in Lemmas 1.4 and 2.5. ■

C. Outline of the Method with an Example

Here we outline a practical method for computing the maximum mass stabilizable using gains chosen based on the massless plasma approximation.

1) Choose a controller (g_d, g_p) so that the system is asymptotically stable for mass zero .
2) Check that the conditions given in Theorem 2.4 hold.
3) Plot the graph $\{S_0(i\omega) : -\infty < \omega < \infty\}$ and estimate the critical values $\omega_1, \ldots, \omega_\ell$ (see definitions 1.2 and 2.2).
4) Compute the maximum allowable mass using Proposition 2.3.

As an illustration, we describe an application of this method to a model of the KSTAR Tokamak [5], [6]. A cross-section of the KSTAR Tokamak is shown in Figure 1, with a plasma cross-section shown in the interior. The active control coils 1 through 14 outside of the vacuum vessel are used to establish the plasma equilibrium. The internal coils 15 and

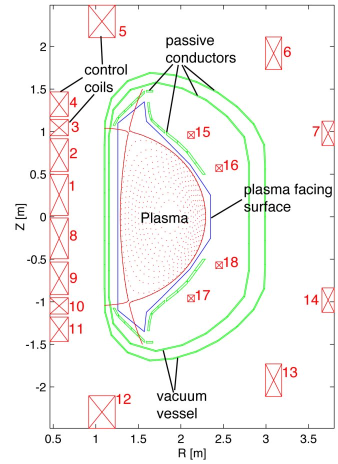

Fig. 1. Cross-section of the KSTAR Tokamak.

17 are dedicated to vertical position (stability) control and coils 16 and 18 used for radial position control.

To achieve vertical stability of the plasma we set the following gains:

```
gd(15)  = 0;
gd(17)  = 0;
gp(15)  = 800;
gp(17)  = -800;
```

Step 1: Our software verifies that these gain values stabilize the closed loop system at mass zero.

Step 2: As required by Theorem 2.4 we compute the values of ξ and η:

$$\begin{aligned} \xi &= 6.2 \cdot 10^6 \\ \eta &= -208.0 \cdot 10^6 \end{aligned}$$

Since $\xi > 0$ and $\eta < 0$, we know that the closed loop system is stable for sufficiently small mass. (Therefore it makes sense to compute the maximum allowable mass.)

Step 3: We obtain the following plot for $S_0(i\omega)$:

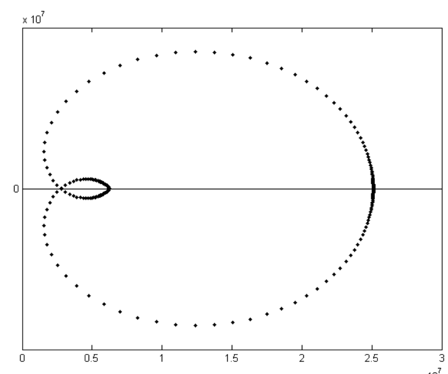

Fig. 2. Plot of $\{S_0(i\omega) : -\infty < \omega < \infty\}$

The graph in Figure 2 crosses the real axis at 4 values of ω. The $(\omega, S_0(i\omega))$ pairs are:

$$(-11.0,\ 2.8\cdot 10^6)$$
$$(0,\ 2.5\cdot 10^7)$$
$$(+11.0,\ 2.8\cdot 10^6)$$
$$(\infty,\ 6.2\cdot 10^6)$$

The crossings at $\omega_1 := -11.0$ and $\omega_2 := +11.0$ yield nonzero critical values, since $w_k \in \mathbb{R}\setminus\{0\}$ and $S_0(i\omega_k) > 0$ for $k = 1, 2$.

Step 4: For $k = 1, 2$ we compute $S_0(i\omega_k)/\omega_k^2$. We obtain:

$$2.3\cdot 10^4,$$
$$2.3\cdot 10^4$$

(It was expected that these two values would be equal since $\omega_1 = -\omega_2$. In general, the nonzero critical values will come in pairs and the above computation only needs to be performed once for each pair.) The maximum allowable mass m_* is the minimum of the values we just computed (see Proposition 2.3). In this simple example there is only one value. We conclude that $m_* = 2.3\cdot 10^4$ kg, a quantity far greater than the masses occurring in the tokamak which are on the order of a milligram.

III. Proofs

In this section we provide the proofs of the results stated in §II-A. Together, §III-A, §III-B, and §III-C make up the proof of Theorem 2.1, which is the meat of the situation and underlies all our results. Then in §III-D, we give the proof of Proposition 2.3. Throughout this section we will use basic results from complex function theory. One reference, among many possible references, is [7].

A. Counting the zeros of $s_m(\lambda)$

We want to understand the behavior of the zeros of $s_m(\lambda)$ as we vary the parameter m. We start with a lemma.

Lemma 3.1: Suppose that for $m \geq 0$,

1) $s_m(\lambda)$ is a rational function of λ,
2) $s_m(\lambda) = s_0(\lambda) + \lambda^2 m$,
3) $s_0(\lambda)$ is analytic and real valued at ∞.

Suppose $s_0(\infty) \neq 0$. Then we have the following:

1) If $s_0(\lambda)$ has k zeros counting multiplicity, then $s_m(\lambda)$ has $k+2$ zeros counting multiplicity for all $m > 0$. Moreover, all of these zeros move continuously in m.
2) We may write the zeros of $s_0(\lambda)$ as $\{\alpha_0^1, \ldots, \alpha_0^k\}$ and for fixed $m > 0$ the zeros of $s_m(\lambda)$ as $\{\alpha_m^1, \ldots, \alpha_m^k, \beta_m^1, \beta_m^2\}$ so that $m \mapsto \alpha_m^i$ gives a continuous function $[0,\infty) \to \mathbb{C}$ for $i = 1, \ldots, k$, and $m \mapsto \beta_m^i$ gives a continuous function $(0,\infty) \to \mathbb{C}$ for $i = 1, 2$. We call β_m^1 and β_m^2 the *excess zeros*.
3) As $m \to 0$ we have $\beta_m^i \to \infty$ for $i = 1, 2$.

Proof. Since s_0 is rational with $s_0(\infty) \neq 0$, we can write $s_0 = p/q$ for coprime polynomials of equal degree; hence

$$s_m(\lambda) = \frac{p(\lambda) + \lambda^2 m q(\lambda)}{q(\lambda)}\,. \tag{9}$$

When $m > 0$, the numerator of (9) is a polynomial of degree $k+2$; furthermore, (9) is in lowest terms since a common zero of $p(\lambda) + \lambda^2 m q(\lambda)$ and $q(\lambda)$ would be a zero of $p(\lambda)$. It follows that $s_m(\lambda)$ has $k+2$ zeros counting multiplicity.

Now we show that the zeros move continuously in m. Fix $m \in [0,\infty)$. Let $\gamma_m^1, \ldots, \gamma_m^r$ be an enumeration of the zeros of $s_m(\lambda)$; repeats are allowed. Let $\varepsilon > 0$. Without loss of generality, suppose that $\varepsilon > 0$ is small enough so that all the closed balls $\overline{B}_\varepsilon(\gamma_m^i)$ of radius ε centered at the roots of $s_m(\lambda)$ are disjoint (except that possibly some of them are identical) and that furthermore, that $s_m(\lambda)$ has no poles in any of these closed ε-balls. Take $\delta > 0$ such that $|\hat{m} - m| < \delta$ implies that $s_{\hat{m}}(\lambda)$ has no poles in any of the closed ε-balls and no zeros on their boundaries. (When we write $|\hat{m} - m| < \delta$, it is implicit that $\hat{m} \geq 0$.) When $|m - \hat{m}| < \delta$, the number of zeros $s_{\hat{m}}(\lambda)$ in $B_\varepsilon(\gamma_m^i)$ is given by

$$N(\gamma_m^i; \hat{m}) \ :=\ \frac{1}{2\pi i}\int_{|\xi - \gamma_m^i| = \varepsilon} \frac{s'_{\hat{m}}(\xi)}{s_{\hat{m}}(\xi)}\, d\xi\,.$$

Since $s'_{\hat{m}}(\xi)/s_{\hat{m}}(\xi)$ is continuous in $\hat{m}$, we immediately see that $N(\gamma_m^i; \hat{m})$ is a continuous function of $\hat{m}$ defined on the set $\{\hat{m} : |\hat{m} - m| < \delta\}$. Since $N(\gamma_m^i; \hat{m})$ takes integer values, we immediately see that $N(\gamma_m^i; \hat{m}) = N(\gamma_m^i; m)$ whenever $|\hat{m} - m| < \delta$. When $\hat{m}$ and m are within distance δ of each other, we have paired each γ_m^i with a zero of $s_{\hat{m}}(\lambda)$ which is within ε distance of γ_m^i. We conclude that the zeros of $s_m(\lambda)$ move continuously in m. Part (2) of the lemma follows immediately from part (1).

It remains to show that $\beta_m^i \to \infty$ as $m \to 0$. Let β_m denote either of the β_m^i. If it is not the case that $\beta_m \to \infty$ as $m \to 0$, then there is a sequence of nonzero real numbers $m_n \to 0$ such that β_{m_n} converges to a complex number β_0, which must be a zero of $s_0(\lambda)$; that is, $\beta_0 = \alpha_0^j$ for some j. We seek a contradiction. Setting $m = 0$ and letting $\varepsilon > 0$ be sufficiently small as in the above argument (where now we have $r = k$ and $\gamma_m^i = \alpha_0^i$) we can choose $\delta > 0$ so that $N(\alpha_0^i; \hat{m}) = N(\alpha_0^i; 0)$ when $0 \leq \hat{m} < \delta$. By part (1) of the lemma, the total number of zeros of $s_0(\lambda)$ is equal to k. Hence for $0 \leq \hat{m} < \delta$, we have

$$\sum{}' N(\alpha_0^i; \hat{m}) = \sum{}' N(\alpha_0^i; 0) = k$$

where the sum is taken over a set of representatives for the collection of distinct ε-balls. However, if we choose N large enough so that

$$|m_N| < \delta\,, \quad |\beta_{m_N} - \alpha_0^j| < \varepsilon\,,$$

then we find that

$$\alpha_{m_N}^1, \ldots, \alpha_{m_N}^k, \beta_{m_N} \ \in\ \bigcup_{i=1}^{k} B_\varepsilon(\alpha_0^i)$$

are all zeros of $s_m(\lambda)$, and hence the value of $\sum' N(\alpha_0^i; \hat{m})$ is at least $k+1$, a contradiction. ■

B. The asymptotics of the excess zeros

If we assume that $\alpha_0^1, \dots, \alpha_0^k$ lie in the open LHP, then for small $m > 0$, we know that $\alpha_m^1, \dots, \alpha_m^k$ also lie in the open LHP. We want to know which half-plane the β_m^i lie in when $m > 0$ is small.

Proposition 3.2: Suppose that for $m \geq 0$,

1) $s_m(\lambda)$ is a rational function of λ,
2) $s_m(\lambda) = s_0(\lambda) + \lambda^2 m$,
3) $s_0(\lambda)$ is analytic and real valued at ∞.

Define a constant c by writing

$$s_0(\lambda) = s_0(\infty) + c\lambda^{-1} + b(\lambda)\,, \text{ with } \lambda b(\lambda) \to 0 \text{ as } \lambda \to \infty.$$

Let β_m denote either of the excess zeros (as in Lemma 3.1). If $s_0(\infty) > 0$, then as $m \to 0$ we have

$$\Re(\beta_m) \to \frac{c}{2s_0(\infty)}\,, \quad |\Im(\beta_m)| \to \infty\,.$$

Here, $\Re(z)$ and $\Im(z)$ denote the real and imaginary parts of a complex number z, respectively.

Proof. We write

$$s_m(\lambda) = s_0(\infty) + c\lambda^{-1} + b(\lambda) + \lambda^2 m$$

with $\lambda b(\lambda) \to 0$ as $\lambda \to \infty$. Let β_m be either of the excess zeros. We see that β_m satisfies

$$s_0(\infty) + c\beta_m^{-1} + b(\beta_m) + \beta_m^2 m = 0$$

and hence

$$\beta_m s_0(\infty) + c + \beta_m b(\beta_m) + \beta_m^3 m = 0\,.$$

As $m \to 0$, we have $\beta_m \to \infty$ by Lemma 3.1 and hence $\beta_m b(\beta_m) \to 0$. Therefore, as $m \to 0$, we have

$$\beta_m s_0(\infty) + c + \beta_m^3 m \to 0\,. \quad (10)$$

We write β_m in terms of its real and imaginary parts as $\beta_m = r_m + j_m$; for ease of notation we will write $r = r_m$ and $j = j_m$. Note that $\beta_m^3 = r^3 + 3rj^2 + 3r^2 j + j^3$ which has real part $r^3 + 3rj^2$ and imaginary part $3r^2 j + j^3$. Now we decompose (10) into real and imaginary parts to find:

$$c + r[s_0(\infty) + (r^2 + 3j^2)m] \quad \to \quad 0 \quad (11)$$

$$j[s_0(\infty) + (3r^2 + j^2)m] \quad \to \quad 0 \quad (12)$$

The proof of the proposition will follow from our analysis of (11) and (12). First we establish two claims.

Claim: For $m > 0$ small, j_m is bounded away from 0.

If this is not the case, then there exists $m_n \to 0$ such that $j_{m_n} \to 0$. For ease of notation, we will write $r = r_{m_n}$, $j = j_{m_n}$, and $m = m_n$. In this case we must have $|r| \to \infty$ since $\beta_m \to \infty$ by Lemma 3.1; hence (11) gives

$$s_0(\infty) + (r^2 + 3j^2)m \to 0\,.$$

Finally, since $3j^2 m \to 0$ this implies $s_0(\infty) + r^2 m \to 0$; that is, $r^2 m \to -s_0(\infty)$. This requires $s_0(\infty) \leq 0$ in order to be possible, which contradicts our hypothesis that $s_0(\infty) > 0$.

Claim: r_m is bounded

If this is not the case, then there is a sequence $m_n \to 0$ such that $|r_{m_n}| \to \infty$. We seek a contradiction. For ease of notation, we will write $r = r_{m_n}$, $j = j_{m_n}$ and $m = m_n$. From Equation (11) we have $s_0(\infty) + (r^2 + 3j^2)m \to 0$, which gives $(r^2 + 3j^2)m \to -s_0(\infty)$, and from (12) and the fact that $|j|$ is bounded away from 0, we have $(3r^2 + j^2)m \to -s_0(\infty)$. Combining these yields

$$(8j^2)m \to -2s_0(\infty)\,, \quad (8r^2)m \to -2s_0(\infty)\,.$$

Subtracting, we obtain

$$(|j|^2 + r^2)m \to 0\,,$$

so $r^2 m \to 0$ which implies $s_0(\infty) = 0$, contradicting $s_0(\infty) > 0$. Having successfully proven the claim that r_m is bounded, we dispense with our subsequences.

In light of the fact that r_m is bounded, we now know that $|j_m| \to \infty$ since $|\beta_m| \to \infty$ by Lemma 3.1. It remains to show that the limit of r_m exists and find its limit. Equation (12) gives

$$s_0(\infty) + (3r^2 + j^2)m \to 0\,;$$

since r is bounded, this implies $s_0(\infty) + j^2 m \to 0$, and hence $j^2 m \to -s_0(\infty)$. Since r is bounded, Equation (11) yields

$$c + r[s_0(\infty) + 3j^2 m] \;\to\; 0\,.$$

Further, since $j^2 m \to -s_0(\infty)$, this implies

$$c + r[-2s_0(\infty)] \;\to\; 0\,,$$

and therefore

$$r \to \frac{c}{2s_0(\infty)}\,.$$

This completes the proof of the proposition. ■

Now we completely understand the situation when $s_0(\infty) > 0$.

Corollary 3.3: Under the hypotheses of Proposition 3.2 plus the assumption that $s_0(\lambda)$ has no zeros in the closed RHP, we have:

1) If $s_0(\infty) > 0$ and $c < 0$, then there exists $m_* > 0$ such that $s_m(\lambda)$ has no zeros in the closed RHP for all $m \in (0, m_*)$.
2) If $s_0(\infty) > 0$ and $c > 0$, then there exists $m_* > 0$ such that $s_m(\lambda)$ has at least one zero in the open RHP for all $m \in (0, m_*)$.

Proof. This follows immediately from the continuity of the zeros given by Lemma 3.1, and the asymptotics given in Proposition 3.2. ■

C. $s_0(\infty) < 0$ *begets instability*

From the analysis given up until this point, it is not clear what happens to the β_m^i as $m \to 0$ in the case where $s_0(\infty) < 0$; indeed, it is not even clear whether such a limit will exist. The following proposition gives a more delicate analysis of the case where $s_0(\infty) < 0$.

Proposition 3.4: Suppose that for $m \geq 0$,

1) $s_m(\lambda)$ is a rational function of λ,
2) $s_m(\lambda) = s_0(\lambda) + \lambda^2 m$,
3) $s_0(\lambda)$ is analytic and real valued at ∞.

Suppose $s_0(\infty) < 0$, and let $\delta > 0$ be given. Then there exists $m \in (0, \delta)$ such that $s_m(\lambda) = 0$ for some λ in the open RHP.

Proof. For any $m \geq 0$, we have

$$s_m(\lambda) = s_0(\lambda) + \lambda^2 m$$

and therefore $s_m(\lambda) = 0$ if and only if $s_0(\lambda) + \lambda^2 m = 0$. Making the substitution $\lambda = 1/z$ we find that this is true whenever

$$F_m(z) := m + z^2 s_0(1/z) = 0$$

for some $z \neq 0$. Since $s_0(1/z)$ is analytic at $z = 0$ we may write

$$s_0(1/z) = s_0(\infty) + z g(z)$$

where $g(z)$ is analytic at $z = 0$. By taking δ small enough, we may assume, without loss of generality, that $g(z)$ is analytic and this expression for $s_0(1/z)$ holds for all $z \in B_{2\delta}(0)$. This yields

$$m + z^2 s_0(1/z) = m + z^2 s_0(\infty) + z^3 g(z)$$

which we can write as

$$F_m(z) = f_m(z) + z^3 g(z)$$

when we define

$$f_m(z) := m + z^2 s_0(\infty)\,.$$

Choose $\varepsilon > 0$ such that

$$\varepsilon < \delta\,, \quad \frac{\varepsilon^2 |s_0(\infty)|}{2} < \delta\,,$$

and $|z| \leq \varepsilon$ implies

$$|z g(z)| < \frac{|s_0(\infty)|}{2}\,. \tag{13}$$

Fix

$$m := \frac{\varepsilon^2 |s_0(\infty)|}{2}$$

and observe that $m \in (0, \delta)$. Define

$$\Omega := \{z \in \mathbb{C} \mid \Re z > 0, |z| < \varepsilon\}$$

and

$$z_0 := \sqrt{\frac{m}{|s_0(\infty)|}}\,.$$

Since $s_0(\infty) < 0$, we have $f_m(z_0) = 0$. Furthermore,

$$z_0^2 = \frac{m}{|s_0(\infty)|} < \frac{2m}{|s_0(\infty)|} = \varepsilon^2\,,$$

and hence $z_0 \in (0, \varepsilon)$; that is, $z_0 \in \Omega$. Clearly, the only other zero of $f_m(z)$ is $-z_0 \notin \Omega$. Also, being a polynomial, $f_m(z)$ has no poles. Since $g(z)$ is analytic in $B_{2\delta}(0)$, we know that $F_m(z)$ has no poles in $B_{2\delta}(0) \supseteq \overline{\Omega}$.

We will shortly apply Rouche's Theorem (see, for example, [7]) to conclude that $F_m(z)$ has exactly one zero in Ω. In order to satisfy the appropriate hypotheses, we must show that $|z^3 g(z)| < |f_m(z)|$ for all $z \in \partial\Omega$. In particular, this will also demonstrate that $F_m(z)$ has no zeros on $\partial\Omega$. Since $|z| \leq \varepsilon$ for all $z \in \overline{\Omega}$, taking into account (13), it suffices to show that

$$\frac{|f_m(z)|}{|z^2|} \geq \frac{|s_0(\infty)|}{2}$$

for all $z \in \partial\Omega$. First we consider the portion of $\partial\Omega$ that lies on the imaginary axis. If $z = i\omega$ with ω real and $|z| \leq \varepsilon$, then we have

$$\frac{|f_m(z)|}{|z^2|} = \frac{m}{\omega^2} + |s_0(\infty)| > \frac{|s_0(\infty)|}{2}\,.$$

(Note that the above relied again on the fact that $s_0(\infty) < 0$.) Now we treat the remaining portion of $\partial\Omega$. We have

$$\frac{|s_0(\infty)|}{2} = \frac{m}{\varepsilon^2}$$

and hence for $|z| = \varepsilon$,

$$\begin{aligned} \frac{|f_m(z)|}{|z^2|} &= \left|\frac{m}{z^2} + s_0(\infty)\right| \\ &\geq |s_0(\infty)| - \frac{m}{\varepsilon^2} \\ &= \frac{|s_0(\infty)|}{2}\,. \end{aligned}$$

Having satisfied the necessary hypotheses, we invoke Rouche's Theorem to conclude that $F_m(z)$ has a zero, say $z_0 \in \Omega$. It follows that $\gamma := 1/z_0$ is in the open RHP and $s_m(\gamma) = 0$. ■

Now the proof of Theorem 2.1 follows immediately from from Corollary 3.3 and Proposition 3.4.

D. The largest value for m_*

Having completed the proof of Theorem 2.1, we are now in a position to prove Proposition 2.3. First we need a preliminary lemma.

Lemma 3.5: Suppose that for $m \geq 0$,

1) $s_m(\lambda)$ is a rational function of λ,
2) $s_m(\lambda) = s_0(\lambda) + \lambda^2 m$.

If ω is a nonzero critical value (see Definition 2.2), then for

$$m = \frac{s_0(i\omega)}{\omega^2}\,,$$

the function $s_m(\lambda)$ has a zero on the imaginary axis.

Proof. By definition, ω being a nonzero critical value implies that $s_0(i\omega)$ is a positive real number and hence

$$m = \frac{s_0(i\omega)}{\omega^2}$$

is a positive real number with $s_0(i\omega) - \omega^2 m = 0$. Substituting $\lambda = i\omega$ into the equation $s_m(\lambda) = s_0(\lambda) + \lambda^2 m$ yields $s_m(i\omega) = s_0(i\omega) - \omega^2 m = 0$. That is, $s_m(\lambda)$ has a zero on the imaginary axis. ■

Proof of Proposition 2.3. Suppose that the hypotheses of Theorem 2.1 are satisfied, and further suppose that $s_0(\infty) > 0$ and $c < 0$. In this case, Theorem 2.1 says that there exists $m_* > 0$ so that $s_m(\lambda)$ has no zeros in the closed RHP for all $m \in [0, m_*)$. Choose the largest possible value for m_*, allowing the possibility that $m_* = \infty$. Now define

$$\hat{m} := \min_{k=\omega_1,\ldots,\omega_\ell} \left\{ \frac{s_0(i\omega_k)}{\omega_k^2} \right\},$$

where $\omega_1, \ldots, \omega_\ell$ are the critical values (see Definition 2.2), taking the convention that $\hat{m} = \infty$ if there are no critical values. We must show that $\hat{m} = m_*$. From Lemma 3.5 we already know $m_* \leq \hat{m}$. To show that $m_* \geq \hat{m}$, which would complete the proof, we must show that for all $m \in (0, \hat{m})$, the function $s_m(\lambda)$ has no zeros in the closed RHP.

By way of contradiction, suppose that there exists $m \in (0, \hat{m})$ such that $s_m(\lambda)$ has a zero in the closed RHP. By Corollary 3.3, $s_m(\lambda)$ has no zeros in the closed RHP when $m > 0$ is sufficiently small. Moreover, Lemma 3.1 tells us that when $m \in (0, \infty)$, all of the zeros of $s_m(\lambda)$ move continuously in m. Thus there is some intermediate value of m for which $s_m(\lambda)$ has a zero on the imaginary axis. That is, $s_m(i\omega) = 0$ for some $\omega \in \mathbb{R}$. We note that $\omega \neq 0$, since if ω was equal to zero, we would have $s_0(0) = s_m(0) = 0$, violating the hypothesis that $s_0(\lambda)$ has no zeros in the closed RHP. Since $s_m(i\omega) = s_0(i\omega) - \omega^2 m$, we have $s_0(i\omega) = \omega^2 m \in (0, \infty)$ and hence the graph of $s_0(i\omega)$ crosses the positive real axis at this particular value of ω. Thus ω is a critical value and we conclude that

$$m = \frac{s_0(i\omega)}{\omega^2} \geq \hat{m},$$

which contradicts the assumption that $m \in (0, \hat{m})$. ■

IV. Conclusions

In this paper, we have derived necessary and sufficient conditions (see Theorem 2.4) for a proportional-derivative controller to guarantee stabilization of the vertical instability of a tokamak plasma having some positive mass provided it stabilizes the instability with the plasma mass assumed to be zero. We have also devised a means of computing the maximum mass of a plasma that is stabilizable with this controller (see Proposition 2.3). In practice, these two results lead to an easy-to-follow procedure (see §II-C). Based on a small example set and on physical intuition, we speculate that this maximum mass will always be so large as to impose no practical constraint on plasma mass. This means that steps (3) and (4) in the method of §II-C may not be necessary in practice. However, step (2), which consists of verifying the inequalities $\xi > 0$ and $\eta < 0$ described in Theorem 2.4, must be performed in order to avoid reaching erroneous conclusions (see the first paragraph of §I).

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