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Semistability for Time-Varying Discontinuous Dynamical Systems with Application to Agreement Problems in Switching Networks

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Abstract— **Semistability is the property whereby the solutions of a dynamical system converge to Lyapunov stable equilibrium points determined by the system initial conditions. In this paper, we extend the theory of semistability to discontinuous time-varying dynamical systems. In particular, Lyapunov-based tests for semistability, weak semistability, as well as uniform semistability for nonautonomous differential inclusions are established. Using these results we develop a framework for designing semistable protocols in switching dynamical networks with time-dependent and state-dependent communication topologies.**

I. Introduction

Due to recent technological advances in sensing, actuation, communication, and computation, a considerable research effort has been devoted to the control of networks and control over networks. Network systems involve distributed decision-making for coordination of dynamic agents involving information flow enabling enhanced operational effectiveness via cooperative control in autonomous systems. These dynamical network systems cover a very broad spectrum of applications including cooperative control of unmanned air vehicles (UAV's) [1], autonomous underwater vehicles (AUV's) [2], distributed sensor networks [3], air and ground transportation systems [4], swarms of air and space vehicle formations [5], and congestion control in communication networks [6], to cite but a few examples.

In many applications involving multiagent systems, groups of agents are required to agree on certain quantities of interest. In particular, it is important to develop information consensus protocols for networks of dynamic agents wherein a unique feature of the closed-loop dynamics under any control algorithm that achieves consensus is the existence of a continuum of equilibria representing a state of equipartitioning or *consensus*. Under such dynamics, the limiting consensus state achieved is not determined completely by the dynamics, but depends on the initial system state as well.

Since every neighborhood of a nonisolated equilibrium contains another equilibrium, a non-isolated equilibrium cannot be asymptotically stable. Hence, asymptotic stability is not the appropriate notion of stability for systems having a continuum of equilibria. For such systems possessing a continuum of equilibria, *semistability* [7], [8] is the relevant notion of stability. Semistability is the property whereby every trajectory that starts in a neighborhood of a Lyapunov stable equilibrium converges to a (possibly different) Lyapunov stable equilibrium. It is important to note that semistability is not equivalent to set stability of the equilibrium set [9]. From a practical viewpoint, it is not sufficient to only guarantee that a network converges to a state of consensus since steady state convergence is not sufficient to guarantee that small perturbations from the limiting state will lead to only small transient excursions from a state of consensus. It is also necessary to guarantee that the equilibrium states representing consensus are Lyapunov stable, and consequently, semistable.

Since communication links among multiagent systems are often unreliable and time-varying due to multipath effects and exogenous disturbances, the information exchange topologies in network systems are often dynamic. In particular, link failures or creations in network multiagent systems result in switchings of the communication topology. This is the case, for example, if information between agents is exchanged by means of line-of-sight sensors that experience periodic communication dropouts due to agent motion. Variation in network topology introduces control input discontinuities, which in turn give rise to discontinuous dynamical systems. In addition, the communication topology may be time-varying. In this case, the vector field defining the dynamical system is a discontinuous function of the state and time, and hence, system stability can be analyzed using discontinuous Lyapunov theory involving concepts such as weak and strong stability notions, differential inclusions, and generalized gradients of locally Lipschitz functions and proximal subdifferentials of lower semicontinuous functions [10].

To address agreement problems in switching networks with time-dependent and state-dependent topologies, in this paper we extend the theory of semistability to discontinuous time-varying dynamical systems. In particular, we develop necessary and sufficient conditions to guarantee weak and strong invariance of Fillipov solutions under the assumption that the discontinuous time-varying vector field is uniformly bounded. Moreover, we present Lyapunov-based tests for (strong) semistability, weak semistability, as well as uniform semistability for nonautonomous differential inclusions. It is important to note that our results are different from the results in the literature [11], [12] since the Lipschitz conditions in [11], [12] are not valid for autonomous differential inclusions considered in [13]. However, the autonomous differential inclusions considered in [13] are indeed a special case of the nonautonomous differential inclusion discussed in this paper.

II. Mathematical Preliminaries

The notation used in this paper is fairly standard. Specifically, $\mathbb{R}$ denotes the set of real numbers, $\mathbb{R}^n$ denotes the set of $n \times 1$ real column vectors, and $(\cdot)^{\mathrm{T}}$ denotes transpose. For $A \in \mathbb{R}^{n\times m}$ we write $\operatorname{rank} A$ to denote the rank of A, and $\partial\mathcal{S}$ and $\overline{\mathcal{S}}$ to denote the boundary and the closure of the subset $\mathcal{S} \subset \mathbb{R}^n$, respectively. Furthermore, we write $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$ for the Euclidean vector norm and inner product, respectively, on $\mathbb{R}^n$, $\mathcal{B}_\varepsilon(\alpha)$, $\alpha \in \mathbb{R}^n$, $\varepsilon > 0$, for the open ball *centered* at α with *radius* ε, $\operatorname{dist}(p, \mathcal{M})$ for the distance from

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a point p to the set $\mathcal{M}$, that is, $\mathrm{dist}(p,\mathcal{M}) \triangleq \inf_{x\in\mathcal{M}} \|p-x\|$, $x(t) \to \mathcal{M}$ as $t \to \infty$ to denote that $x(t)$ approaches the set $\mathcal{M}$, that is, for each $\varepsilon > 0$ there exists $T > 0$ such that $\mathrm{dist}(x(t),\mathcal{M}) < \varepsilon$ for all $t > T$, and $x(t) \rightrightarrows \mathcal{M}$ as $t \to \infty$ to denote $x(t)$ approaches the set $\mathcal{M}$ uniformly in the initial time $t_0 \in \mathbb{R}$.

In this paper, we consider time-varying differential equations given by

$$\dot{x}(t) = f(t, x(t)), \quad x(t_0) = x_0, \quad t \geq t_0, \tag{1}$$

where $t \in \mathbb{R}$, $x(t) \in \mathbb{R}^q$, and $f : \mathbb{R} \times \mathbb{R}^q \to \mathbb{R}^q$ is Lebesgue measurable and locally essentially bounded [14], [15], that is, bounded on a bounded neighborhood of every point, excluding sets of measure zero. An absolutely continuous function $x : [t_0, \tau] \to \mathbb{R}^q$ is said to be a *Filippov solution* [14], [15] of (1) on the interval $[t_0, \tau]$ with initial condition $x(t_0) = x_0$, if $x(t)$ satisfies

$$\dot{x}(t) \in \mathcal{K}[f](t, x(t)), \quad \text{a. a.} \quad t \in [t_0, \tau], \tag{2}$$

where the Filippov set-valued map $\mathcal{K}[f] : [0,\infty) \times \mathbb{R}^q \to \mathcal{B}(\mathbb{R}^q)$ is defined by

$$\mathcal{K}[f](t,x) \triangleq \bigcap_{\delta>0} \bigcap_{\mu(\mathcal{S})=0} \overline{\mathrm{co}}\,\{f(t, \mathcal{B}_\delta(x)\backslash\mathcal{S})\}, \quad (t,x) \in [t_0,\infty) \times \mathbb{R}^q, \tag{3}$$

where $\mathcal{B}(\mathbb{R}^q)$ denotes the collection of all subsets of $\mathbb{R}^q$, $\mu(\cdot)$ denotes the Lebesgue measure in $\mathbb{R}^q$, and "$\overline{\mathrm{co}}$" denotes the convex closure. Dynamical systems of the form given by (2) are called *differential inclusions* in the literature [16] and for each state $x \in \mathbb{R}^q$, they specify a *set* of possible evolutions rather than a single one. Note that it follows from [10] that there exists a set $\mathcal{N}_f \subset \mathbb{R}^q$ of measure zero such that, for every set $\mathcal{W} \subseteq \mathbb{R}^q$ of zero measure,

$$\mathcal{K}[f](t,x) = \overline{\mathrm{co}}\left\{\lim_{i\to\infty} f(t,x_i) : x_i \to x,\ x_i \notin \mathcal{N}_f \cup \mathcal{W}\right\}. \tag{4}$$

Since the Filippov set-valued map given by (3) is upper semicontinuous with nonempty, convex, and compact values, and is also locally bounded, it follows that Filippov solutions to (1) exist [15].

An equilibrium point of (1) is a point $x_{\mathrm{e}} \in \mathbb{R}^q$ such that $0 \in \mathcal{K}[f](t_0, x_{\mathrm{e}})$ for all $t_0 \geq 0$. We denote the set of equilibrium points of (1) by $\mathcal{E}$. The upper semicontinuity of the set-valued map $\mathcal{K}[f]$ implies that $\mathcal{E}$ is closed.

Let $\mathcal{S}$ be a given closed subset of $\mathbb{R}^q$. Then the pair $(\mathcal{S}, \mathcal{K}[f])$ is called *weakly positively invariant* (resp., *strongly invariant*) if for all initial conditions (t_0, x_0) with $x_0 \in \mathcal{S}$, $\mathcal{S}$ contains a Filippov solution (resp., all Filippov solutions) $x(\cdot)$ of (1) on $[t_0, \infty)$ satisfying $x(t_0) = x_0$. The *Clarke generalized gradient* of $V : \mathbb{R}^q \to \mathbb{R}$ at x is the set

$$\partial V(x) \triangleq \mathrm{co}\left\{\lim_{i\to\infty} \nabla V(x_i) : x_i \to x, x_i \notin \mathcal{N} \cup \mathcal{S}\right\}, \tag{5}$$

where ∇ denotes the nabla operator, $\mathcal{N}$ is the set of measure zero points where ∇V does not exist, and $\mathcal{S}$ is an arbitrary set of measure zero in $\mathbb{R}^q$.

An equilibrium point $x_{\mathrm{e}} \in \mathcal{E}$ of (1) is *Lyapunov stable* if for every $t_0 \in \mathbb{R}$ and every $\varepsilon > 0$, there exists $\delta = \delta(t_0, \varepsilon) > 0$ such that for every $\|x_0 - x_{\mathrm{e}}\| \leq \delta$, the Fillippov solutions $x(t)$, $t \geq t_0$, with the initial condition $x(t_0) = x_0$ satisfy $\|x(t) - x_{\mathrm{e}}\| < \varepsilon$ for all $t \geq t_0$. An equilibrium point $x_{\mathrm{e}} \in \mathcal{E}$ of (1) is *uniformly Lyapunov stable* if for every $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that for every $\|x_0 - x_{\mathrm{e}}\| \leq \delta$, the Fillippov solutions $x(t)$, $t \geq t_0$, with the initial condition $x(t_0) = x_0$ satisfy $\|x(t) - x_{\mathrm{e}}\| < \varepsilon$ for all $t \geq t_0$ and for all $t_0 \in \mathbb{R}$. The following definitions are needed.

Definition 2.1: i) An equilibrium point $x_{\mathrm{e}} \in \mathcal{E}$ of (1) is *weakly semistable* (resp., *semistable*) if for every $t_0 \in \mathbb{R}$, x_{e} is Lyapunov stable and there exists $\delta = \delta(t_0) > 0$ such that, for every $\|x_0 - x_{\mathrm{e}}\| \leq \delta$, a Fillippov solution (resp., every Filippov solution) $x(t)$, $t \geq t_0$, with the initial condition $x(t_0) = x_0$ satisfies $\lim_{t\to\infty} x(t) = z$ and $z \in \mathcal{E}$ is a Lyapunov stable equilibrium point. The system (1) is *weakly semistable* (resp., *semistable*) if all the equilibrium points of (1) are weakly semistable (resp., semistable).

ii) An equilibrium point $x_{\mathrm{e}} \in \mathcal{E}$ of (1) is *uniformly weakly semistable* (resp., *uniformly semistable*) if x_{e} is uniformly Lyapunov stable and there exists $\delta > 0$ such that, for every $x_0 \in \mathbb{R}^q$ satisfying $\|x_0 - x_{\mathrm{e}}\| \leq \delta$, there exists a uniformly Lyapunov stable equilibrium point $z_{x_0} \in \mathcal{E}$ such that a Filippov solution (resp., every Filippov solution) $x(t)$, $t \geq t_0$, with the initial condition $x(t_0) = x_0$ satisfies $\lim_{t\to\infty} x(t) = z_{x_0}$ uniformly in $t_0 \in \mathbb{R}$, that is, for every $\varepsilon > 0$, there exists $T = T(\varepsilon) > 0$ such that $\|x(t) - z_{x_0}\| < \varepsilon$ for every $t \geq t_0 + T(\varepsilon)$. The system (1) is *uniformly weakly semistable* (resp., *uniformly semistable*) if all the equilibrium points of (1) are uniformly weakly semistable (resp., uniformly semistable).

Definition 2.2 ([11]): Let $\mathcal{S}$ be a closed subset of $\mathbb{R}^q$. Give $u \notin \mathcal{S}$, let $x \in \partial\mathcal{S}$ be such that $\|x - u\| = \inf_{s\in\mathcal{S}} \|s - u\|$. Then x is called a *projection* of u onto $\mathcal{S}$. The set of all such projections is denoted by $\mathrm{proj}(u, \mathcal{S})$. The vector $u - x$ (and all its nonnegative multiples) defines a *proximal normal direction* to $\mathcal{S}$ at x. The set of all vectors constructed in this way (for fixed x, by varying u) is called the *proximal normal cone* to $\mathcal{S}$ at x, and is denoted by $\mathcal{N}_{\mathcal{S}}^{\mathrm{P}}(x)$. Thus, $\mathcal{N}_{\mathcal{S}}^{\mathrm{P}}(x) = \{\alpha(u - x) : x \in \mathrm{proj}(u, \mathcal{S}), \alpha \geq 0\}$.

Definition 2.3: The *contingent set* of a Fillipov solution $x(\cdot) : [t_0, t_1] \to \mathbb{R}^q$ of (1) satisfying $x(t_0) = x_0$ is the set of all limit points of the seuqences of the form $\{\frac{x_i(t_i)-x_0}{t_i-t_0}\}_{i=1}^{\infty}$ where $\{t_i\}_{i=1}^{\infty}$ is a sequence in $[t_0, t]$ converging to t_0.

III. Lyapunov-Based Semistability Analysis for Time-Varying Discontinuous Dynamical Systems

In this section, we develop Lyapunov-based semistability theory for time-varying discontinuous dynamical systems of the form given by (1). The following lemma is needed for the main results of the paper.

Lemma 3.1: Let $\mathcal{S}$ be a closed subset of $\mathbb{R}^q$ and consider $(t, x) \in [t_0, t_0 + a] \times \overline{\mathcal{B}}_b(x_0)$ for (2). Assume that for every $(t, z) \in [t_0, t_0 + d] \times \overline{\mathcal{B}}_b(x_0)$ there exists $w \in \mathrm{proj}(z, \mathcal{S})$ such that $\langle v, z - w\rangle \leq 0$ for every $v \in \mathcal{K}[f](t, z)$, where $d = \min\{a, \frac{b}{m}\}$ and $m = \sup\{\|v\| : v \in \mathcal{K}[f](t,x), (t,x) \in [t_0, t_0 + a] \times \overline{\mathcal{B}}_b(x_0)\}$. Then $\mathrm{dist}(x(t), \mathcal{S}) \leq \mathrm{dist}(x(t_0), \mathcal{S})$ for every $t \in [t_0, t_0 + d]$ and every Filippov solution $x(\cdot)$ of (2) on $[t_0, t_0 + d]$ satisfying $x(t_0) = x_0$.

Next, we present necessary and sufficient conditions for characterizing weak invariance. It is important to note that our results are different from the results in [11], [12] since the Lipschitz conditions in [11], [12] do not hold for the nonautonomous differential inclusion discussed in this paper; see Examples 3.1 and 3.2 below. A similar observation holds for Proposition 3.4.

Proposition 3.1: Let $\mathcal{S}$ be a closed subset of $\mathbb{R}^q$. Assume that there exists $M > 0$ such that for every $(t, x) \in \mathbb{R}^{q+1}$ and every $v \in \mathcal{K}[f](t, x)$, $\|v\| \leq M$. Then $(\mathcal{S}, \mathcal{K}[f])$ is weakly positively invariant if and only if, for every $w \in \partial\mathcal{S}$ and every $\zeta \in \mathcal{N}^{\rm P}_{\mathcal{S}}(w)$,

$$\min_{v\in\mathcal{K}[f](t,w)}\langle\zeta,v\rangle\leq 0,\quad t\in\mathbb{R}.\tag{6}$$

The following propositions are needed for the next result. For the first proposition recall that the *epigraph* of a function $f:\mathcal{X}\to\mathbb{R}$ is defined by the α-sublevel set ${\rm Ep}(f)\triangleq\{(x,\alpha)\in\mathcal{X}\times\mathbb{R}:f(x)\leq\alpha\}$ [17, p. 23].

Proposition 3.2: Assume that there exists $M>0$ such that for every $(t,x)\in\mathbb{R}^{q+1}$ and almost every $v\in\mathcal{K}[f](t,x)$, $\|v\|\leq M$. Furthermore, assume that there exist a continuously differentiable function $V(\cdot)$ and a continuous function $W(\cdot)$ such that the following statements hold:

i) $\alpha(\|x\|)\leq V(x)\leq\beta(\|x\|)$, $x\in\mathbb{R}^q$, where $\alpha(\cdot)$ and $\beta(\cdot)$ are class $\mathcal{K}_\infty$ functions.

ii) $\min_{v\in\mathcal{K}[f](t,x)}\langle\nabla V(x),v\rangle\leq -W(x)$ for all $x\in\mathbb{R}^q$ and $t\in\mathbb{R}$, where $W(x)\geq 0$ for all $x\in\mathbb{R}^q$.

Then $(V^{-1}([0,c]),\mathcal{K}[f])$ is weakly positively invariant and, for every $x_0\in\mathbb{R}^q$, there exists a Filippov solution $x(\cdot)$ to (1) on $[t_0,\infty)$ with $x(t_0)=x_0$ such that $x(t)\to W^{-1}(0)$ as $t\to\infty$, where $c>0$.

Proof. Since $V(\cdot)$ is continuously differentiable, it follows from Proposition 2 of [16, p. 32] that $\{\nabla V(x)\}=\partial V(x)$, $x\in\mathbb{R}^q$. Thus, it follows from *ii)* that $\min_{v\in\mathcal{K}[f](t,x)}\langle p,v\rangle\leq 0$, $p\in\partial V(x)$, $x\in\mathbb{R}^q$. Consider the epigraph of $V(\cdot)$ defined by ${\rm Ep}(V)\triangleq\{(x,z)\in\mathbb{R}^q\times\mathbb{R}:V(x)\leq z\}$. Note that ${\rm Ep}(V)$ is closed. Let $(\zeta,\lambda)\in\mathbb{R}^q\times\mathbb{R}$ belong to $\mathcal{N}^{\rm P}_{{\rm Ep}(V)}(x,z)$ for some $(x,z)\in{\rm Ep}(V)$. We show that for $(\zeta,\lambda)\in\mathcal{N}^{\rm P}_{{\rm Ep}(V)}(x,z)$, there exists $v\in\mathcal{K}[f](t,x)$ such that $\langle\zeta,v\rangle\leq 0$.

First, we show that $\lambda\leq 0$. Let y be in the domain of V and $(y^*,0)\in\mathcal{N}^{\rm P}_{{\rm Ep}(V)}(y,V(y))$ with $y^*\neq 0$. Without loss of generality, assume that $\|y^*\|=1$. Then there exists $(x,V(y))\notin{\rm Ep}(V)$ such that $\|(x,V(y))-(y,V(y))\|=\inf_{(s,V(s))\in{\rm Ep}(V)}\|(x,V(s))-(s,V(s))\|$ and $(x-y)/\|x-y\|=y^*$, where $(y,V(y))\in{\rm Ep}(V)$. By Proposition 2.1 of [18] we can assume, without loss of generality, that $(y^*,0)\in\partial{\rm dist}((x,V(y)),{\rm Ep}(V))$. Note that for every $(\hat{x},V(\hat{y}))$, it follows from the definition of an epigraph that ${\rm dist}((\hat{x},V(\hat{y})),{\rm Ep}(V))\leq{\rm dist}((\hat{x},V(\hat{y})-t),{\rm Ep}(V))$ for every $t>0$. Suppose that there exists $(\hat{x},V(\hat{y}))$ arbitrarily close to $(x,V(y))$ and $t>0$ arbitrarily small so that ${\rm dist}((\hat{x},V(\hat{y})),{\rm Ep}(V))<{\rm dist}((\hat{x},V(\hat{y})-t),{\rm Ep}(V))$. Then it follows from Theorem 1.4 of [18] that there exists $(\zeta,\lambda)\in\partial{\rm dist}((\bar{x},V(\bar{y})),{\rm Ep}(V))$, where $(\bar{x},V(\bar{y}))$ is arbitrarily close to $(x,V(y))$ such that $\langle(\zeta,\lambda),(\hat{x},V(\hat{y})-t)-(\hat{x},V(\hat{y}))\rangle>0$, which implies that $\lambda<0$. For the case where ${\rm dist}((\hat{x},V(\hat{y})),{\rm Ep}(V))={\rm dist}((\hat{x},V(\hat{y})-t),{\rm Ep}(V))$, $t>0$, it follows that $\langle(\zeta,\lambda),(\hat{x},V(\hat{y})-t)-(\hat{x},V(\hat{y}))\rangle=0$, which implies that $\lambda=0$. Hence, $\lambda\leq 0$.

If $\lambda<0$, then $(\zeta/(-\lambda),-1)\in\mathcal{N}^{\rm P}_{{\rm Ep}(V)}(x,z)$, which implies that $-\zeta/\lambda\in\partial V(x)$. Now, it follows from *ii)* that there exists $v\in\mathcal{K}[f](t,x)$ such that $\langle(-\zeta/\lambda),v\rangle\leq 0$, and hence, $\langle\zeta,v\rangle\leq 0$. Alternatively, if $\lambda=0$, then $(\zeta,0)\in\mathcal{N}^{\rm P}_{{\rm Ep}(V)}(x,V(x))$. Now, it follows from Theorem 2.4 of [18] that there exist sequences $\{(\zeta_i,-\varepsilon_i)\}_{i=1}^\infty$, with $\varepsilon_i>0$, and $\{x_i\}_{i=1}^\infty$ such that $\lim_{i\to\infty}(\zeta_i,-\varepsilon_i)=(\zeta,0)$, $(\zeta_i,-\varepsilon_i)\in\mathcal{N}^{\rm P}_{{\rm Ep}(V)}(x_i,V(x_i))$, and $\lim_{i\to\infty}x_i=x$. Using the above result for the case where $\lambda<0$, it follows that there exists $v_i\in\mathcal{K}[f](t,x_i)$ such that $\langle\zeta_i,v_i\rangle\leq 0$. By assumption, the sequence $\{v_i\}_{i=1}^\infty$ is uniformly bounded. Hence, there exists a subsequence $\{n_i\}_{i=1}^\infty$ such that $\{v_{n_i}\}_{i=1}^\infty$ converges to the limit v. Furthermore, $v\in\mathcal{K}[f](t,x)$ since $\mathcal{K}[f]$ is upper semicontinuous. Thus, $\langle\zeta,v\rangle\leq 0$.

Since for $(\zeta,\lambda)\in\mathcal{N}^{\rm P}_{{\rm Ep}(V)}(x,z)$, there exists $v\in\mathcal{K}[f](t,x)$ such that $\langle\zeta,v\rangle\leq 0$, it follows from Proposition 3.1 that the pair $({\rm Ep}(V),\mathcal{K}[f]\times\{0\})$ is weakly invariant, and hence, for every $x_0\in\mathbb{R}^q$, there exists a Filippov solution $x(\cdot)$ to (1) on $[t_0,\infty)$ with $x(t_0)=x_0$ such that $V(x(t))\leq V(x_0)$ for all $t\geq t_0$, which implies that $(V^{-1}([0,c]),\mathcal{K}[f])$ is weakly invariant.

To show the second assertion, define a function $U:\mathbb{R}^q\times\mathbb{R}\to\mathbb{R}$ by $U(x,y)\triangleq V(x)+y$ and a set-valued map $\mathcal{F}(t,x,y)\triangleq\mathcal{K}[f](t,x)\times\{y:y=W(x)\}$. We claim that for every $\alpha\in\mathbb{R}^q$, there exists a Filippov solution (x,y) to the differential inclusion $\dot{z}\in\mathcal{F}(t,z)$ almost everywhere on $[t_0,\infty)$ with $x(t_0)=\alpha$ and $y(t_0)=0$ such that $U(x(t),y(t))\leq U(\alpha,0)$ for all $t\geq t_0$. Let $(\zeta,\eta)\in\partial U(x,y)$. Then $\zeta\in\partial V(x)$ and $\eta=1$. Since $\langle v,\zeta\rangle\leq -W(x)$ for some $v\in\mathcal{K}[f](t,x)$, it follows that $\langle v,\zeta\rangle+W(x)\leq 0$, or, equivalently, $\langle(v,W(x)),(\zeta,1)\rangle\leq 0$. Using similar arguments as above, it can be shown that the pair $({\rm Ep}(U),\mathcal{F}\times\{0\})$ is weakly invariant, which implies that for every $\alpha\in\mathbb{R}^q$, there exists a Filippov solution (x,y) to $\dot{z}\in\mathcal{F}(t,z)$ almost everywhere on $[t_0,\infty)$ with $x(t_0)=\alpha$ and $y(t_0)=0$ such that $U(x(t),y(t))\leq U(\alpha,0)$ for all $t\geq t_0$. Note that $U(x(t),y(t))\leq U(\alpha,0)$ for $t\geq t_0$ implies that $V(x(t))+\int_{t_0}^t W(x(\tau)){\rm d}\tau\leq V(\alpha)$, where $x(\cdot)$ is a Filippov solution to (1). Hence, $V(x(t))$ and $\int_{t_0}^t W(x(\tau)){\rm d}\tau$ are bounded. Furthermore, note that $\dot{x}(t)$ is uniformly bounded for almost all $t\geq t_0$. Now, using similar arguments as in the proof of Theorem 8.4 of [19], it can be shown that $x(t)\to W^{-1}(0)$ as $t\to\infty$. □

Proposition 3.3: Consider the time-varying discontinuous dynamical system (1). Assume that the Filippov solutions of (1) are bounded and every point in $\mathcal{E}$ is Lyapunov stable. Furthermore, assume that for a given $x_0\in\mathbb{R}^q$, there exists a Filippov solution to (1) satisfying $x(t)\to\mathcal{E}$ as $t\to\infty$. Then $x(t)\to z$ as $t\to\infty$, where $z\in\mathcal{E}$. Alternatively, assume that every point in $\mathcal{E}$ is uniformly Lyapunov stable and for given $x_0\in\mathbb{R}^q$, there exists a Filippov solution to (1) satisfying $x(t)\rightrightarrows\mathcal{E}$ as $t\to\infty$. Then $x(t)\rightrightarrows z$ as $t\to\infty$, where $z\in\mathcal{E}$.

Next, we present sufficient conditions for weak semistability and uniform weak semistability for (1).

Theorem 3.1: Assume that there exists $M>0$ such that for every $v\in\mathcal{K}[f](t,x)$, $\|v\|\leq M$. Furthermore, assume that there exist a continuously differentiable function $V(\cdot)$ and a continuous function $W(\cdot)$ such that *i)* and *ii)* of Proposition 3.2 hold, and $\mathcal{E}\subseteq W^{-1}(0)$. If every point in $W^{-1}(0)$ is a Lyapunov stable equilibrium of (1), then (1) is weakly semistable. Alternatively, if every point in $W^{-1}(0)$ is a uniformly Lyapunov stable equilibrium of (1), then (1) is uniformly weakly semistable.

Proof. It follows from Proposition 3.2 that there exists a Filippov solution $x(\cdot)$ to (1) such that $x(t)\to W^{-1}(0)$ as $t\to\infty$. Since every point in $W^{-1}(0)$ is a Lyapunov stable equilibrium of (1), it follows that $W^{-1}(0)\subseteq\mathcal{E}$. Furthermore, since, by assumption, $\mathcal{E}\subseteq W^{-1}(0)$, it follows that $W^{-1}(0)=\mathcal{E}$. Hence, $x(t)\to\mathcal{E}$ as $t\to\infty$ and every point in $\mathcal{E}$ is Lyapunov stable. Now, it follows from Proposition 3.3 that $x(t)\to z$ as $t\to\infty$, where $z\in\mathcal{E}$. By definition, (1) is weakly semistable. To show the second assertion, note that since $\dot{x}(t)$ is uniformly bounded, it follows using similar arguments as in the proof of Proposition 3.2 that $x(t)\rightrightarrows W^{-1}(0)$ as $t\to\infty$. Now, using similar arguments as above, it can be shown that (1) is uniformly weakly semistable. □

Remark 3.1: If all the conditions in Theorem 3.1 are satisfied and (1) has a unique Filippov solution, then it follows from Theorem 3.1 that (1) is semistable. Sufficient conditions for guaranteeing uniqueness of Filippov solutions can be found in [10], [15].

Example 3.1: Consider the time-varying discontinuous dynamical system given by

$$\begin{aligned}\dot{x}_1(t) &= \frac{1+2t^2}{1+t^2}\mathrm{sign}(x_2(t)-x_1(t)),\\ &\qquad x_1(t_0)=x_{10},\quad t\geq t_0, \qquad (7)\\ \dot{x}_2(t) &= \frac{1+2t^2}{1+t^2}\mathrm{sign}(x_1(t)-x_2(t)),\\ &\qquad x_2(t_0)=x_{20}, \qquad (8)\end{aligned}$$

where $x_1, x_2 \in \mathbb{R}$, $\mathrm{sign}(x) \triangleq x/|x|$ for $x \neq 0$, and $\mathrm{sign}(0) \triangleq 0$. Note that, for $x = [x_1, x_2]^{\mathrm{T}}$ and $t \geq t_0$,

$$\mathcal{K}[f](t,x) = \begin{cases} \{\frac{1+2t^2}{1+t^2}\}\times\{-\frac{1+2t^2}{1+t^2}\}, & x_2 > x_1,\\ \left[-\frac{1+2t^2}{1+t^2},\frac{1+2t^2}{1+t^2}\right]\times\left[-\frac{1+2t^2}{1+t^2},\frac{1+2t^2}{1+t^2}\right], & x_1 = x_2,\\ \{-\frac{1+2t^2}{1+t^2}\}\times\{\frac{1+2t^2}{1+t^2}\}, & x_1 > x_2.\end{cases}$$

Clearly, $\|v\| \leq 2\sqrt{2}$ for almost all $v \in \mathcal{K}[f]$. Next, consider $V(x_1,x_2) = \frac{1}{2}(x_1-\alpha)^2 + \frac{1}{2}(x_2-\alpha)^2$, where $\alpha \in \mathbb{R}$. Then it follows from the time-dependent version of Theorem 1 of [20] that

$$\begin{aligned}&[x_1-\alpha, x_2-\alpha]^{\mathrm{T}}\mathcal{K}[f](t,x)\\ &= \mathcal{K}[[x_1-\alpha,x_2-\alpha]^{\mathrm{T}}f](t,x)\\ &= \mathcal{K}\left[-\frac{1+2t^2}{1+t^2}(x_1-x_2)\mathrm{sign}(x_1-x_2)\right](t,x)\\ &= -\frac{1+2t^2}{1+t^2}(x_1-x_2)\mathcal{K}[\mathrm{sign}(x_1-x_2)](x)\\ &= -\frac{1+2t^2}{1+t^2}(x_1-x_2)\mathrm{SGN}(x_1-x_2)\\ &= -\frac{1+2t^2}{1+t^2}|x_1-x_2|\\ &\leq -|x_1-x_2|, \quad t\in\mathbb{R}, \quad (x_1,x_2)\in\mathbb{R}^2, \qquad (9)\end{aligned}$$

where

$$\mathrm{SGN}(x) \triangleq \begin{cases} -1, & x<0,\\ [-1,1], & x=0,\\ 1, & x>0,\end{cases} \qquad (10)$$

which further implies that $\langle\nabla V(x_1,x_2), v\rangle \leq -|x_1-x_2|$ for every $v \in \mathcal{K}[f](t,x)$. Now, it follows from Theorem 1 of [15, p. 153] that $x_1 = x_2 = \alpha$ is Lyapunov stable. In fact, it can be shown that $x_1 = x_2 = \alpha$ is uniformly Lyapunov stable. Next, let $W(x_1,x_2) = |x_1-x_2|$ and note that $W^{-1}(0) = \{(x_1,x_2) \in \mathbb{R}^2 : x_1 = x_2\} = \mathcal{E}$. Now, it follows from Theorem 3.1 that (7) and (8) is weakly semistable. Moreover, it can be shown that (7) and (8) is uniformly weakly semistable. Figure 1 shows the solutions of (7) and (8) for $x_{10} = 4$, $x_{20} = -2$, and $t_0 = 0, 1, 2, 3$. △

The next proposition characterizes strong invariance of (1).

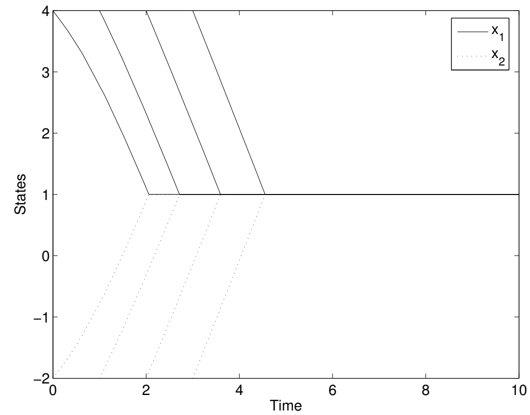

Fig. 1. State trajectories versus time for Example 3.1

Proposition 3.4: Consider the time-varying discontinuous system (1). Let $\mathcal{S}$ be a closed subset of $\mathbb{R}^q$ and assume that there exists $M > 0$ such that for every $(t,x) \in \mathbb{R}^{q+1}$,

$$\|f(t,x)\| \leq M, \qquad (11)$$

for almost all $t \in \mathbb{R}$. Then $(\mathcal{S}, \mathcal{K}[f])$ is strongly invariant if and only if, for every $\zeta \in \mathcal{N}_{\mathcal{S}}^{\mathrm{P}}(x)$ and every $x \in \mathcal{S}$,

$$\max_{v\in\mathcal{K}[f](t,x)} \langle\zeta, v\rangle \leq 0. \qquad (12)$$

Finally, we present sufficient conditions for semistability and uniform semistability for (1).

Theorem 3.2: Assume that there exists $M > 0$ such that for almost every $(t,x) \in \mathbb{R}^{q+1}$, (11) holds. Furthermore, assume that there exist a continuously differentiable function $V(\cdot)$ and a continuous function $W(\cdot)$ such that *i)* of Proposition 3.2 holds, $\mathcal{E} \subseteq W^{-1}(0)$, and

$$\max_{v\in\mathcal{K}[f](t,x)} \langle\nabla V(x), v\rangle \leq -W(x) \qquad (13)$$

for every $x \in \mathcal{S}$ and $t \in \mathbb{R}$. If every point in $W^{-1}(0)$ is a Lyapunov stable equilibrium of (1), then (1) is semistable. Alternatively, if every point in $W^{-1}(0)$ is a uniformly Lyapunov stable equilibrium of (1), then (1) is uniformly semistable.

Proof. Using similar arguments as in the proof of Proposition 3.2 and Proposition 3.4 it can be shown that every Filippov solution $x(\cdot)$ of (1) satisfies $x(t) \to W^{-1}(0)$ as $t \to \infty$. Since every point in $W^{-1}(0)$ is a Lyapunov stable equilibrium of (1), it follows that $W^{-1}(0) \subseteq \mathcal{E}$. Since, by assumption, $\mathcal{E} \subseteq W^{-1}(0)$, it follows that $W^{-1}(0) = \mathcal{E}$. Hence, $x(t) \to \mathcal{E}$ as $t \to \infty$ and every point in $\mathcal{E}$ is Lyapunov stable. Now, it follows from Proposition 3.3 that $x(t) \to z$ as $t \to \infty$, where $z \in \mathcal{E}$. By definition, (1) is semistable. To prove the second assertion, note that since $\dot{x}(t)$ is uniformly bounded for almost all $t \geq t_0$, it follows using similar arguments as in the proof of Proposition 3.2 that $x(t) \rightrightarrows W^{-1}(0)$ as $t \to \infty$. Now, using similar arguments as above, it can be shown that (1) is uniformly semistable. □

Example 3.2: Consider the time-varying discontinuous dynamical system given by

$$\begin{aligned}\dot{x}_1(t) &= (2-\cos t)\mathrm{sign}(x_2(t)-x_1(t)),\\ &\qquad x_1(t_0)=x_{10},\quad t\geq t_0, \qquad (14)\\ \dot{x}_2(t) &= (2-\cos t)\mathrm{sign}(x_1(t)-x_2(t)),\\ &\qquad x_2(t_0)=x_{20}, \qquad (15)\end{aligned}$$

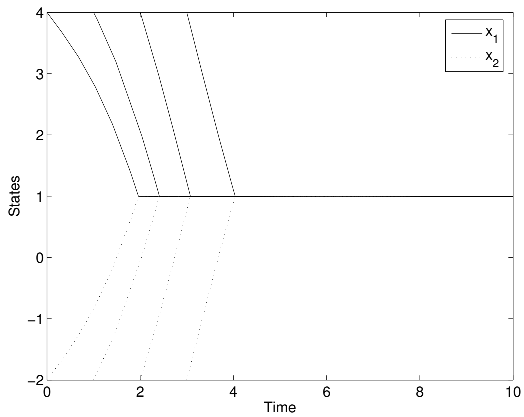

Fig. 2. State trajectories versus time for Example 3.2

where $x_1, x_2 \in \mathbb{R}$. Clearly, $\|f(t,x)\| \leq 3\sqrt{2}$ for almost all $t \geq t_0$ and $x \in \mathbb{R}^2$. Next, consider $V(x_1,x_2) = \frac{1}{2}(x_1-\alpha)^2 + \frac{1}{2}(x_2-\alpha)^2$, where $\alpha \in \mathbb{R}$. Then it follows from the time-dependent version of Theorem 1 of [20] that

$$\begin{aligned}
&[x_1-\alpha, x_2-\alpha]^{\mathrm{T}}\mathcal{K}[f](t,x)\\
&\quad= \mathcal{K}[[x_1-\alpha, x_2-\alpha]^{\mathrm{T}} f](t,x)\\
&\quad= \mathcal{K}\left[-(2-\cos t)(x_1-x_2)\mathrm{sign}(x_1-x_2)\right](t,x)\\
&\quad= -(2-\cos t)(x_1-x_2)\mathcal{K}[\mathrm{sign}(x_1-x_2)](x)\\
&\quad= -(2-\cos t)(x_1-x_2)\mathrm{SGN}(x_1-x_2)\\
&\quad= -(2-\cos t)|x_1-x_2|\\
&\quad\leq -|x_1-x_2|, \quad t \in \mathbb{R}, \quad (x_1,x_2) \in \mathbb{R}^2, \qquad (16)
\end{aligned}$$

which implies that $\langle \nabla V(x_1,x_2), v\rangle \leq -|x_1-x_2|$ for every $v \in \mathcal{K}[f](t,x)$. Now, it follows from Theorem 1 of [15, p. 153] that $x_1 = x_2 = \alpha$ is Lyapunov stable. In fact, it can be shown that $x_1 = x_2 = \alpha$ is uniformly Lyapunov stable. Next, let $W(x_1,x_2) = |x_1-x_2|$ and note that $W^{-1}(0) = \{(x_1,x_2) \in \mathbb{R}^2 : x_1 = x_2\} = \mathcal{E}$. Now, it follows from Theorem 3.2 that (14) and (15) is semistable. Moreover, it can be shown that (14) and (15) is uniformly semistable. Figure 2 shows the solutions of (14) and (15) for $x_{10} = 4$, $x_{20} = -2$, and $t_0 = 0, 1, 2, 3$. $\triangle$

IV. CONSENSUS PROBLEMS IN SWITCHING DYNAMICAL NETWORKS

In this section, we use the semistability theory developed in Section III to analyze stability of consensus protocols with time-dependent and state-dependent communication topologies. Specifically, we use undirected graphs to represent a dynamical network and present solutions to the consensus problem for networks with undirected graph *topologies* (or information flow) [21]. Specifically, let $\mathfrak{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ be a weighted *directed graph* (or digraph) denoting the dynamical network (or dynamic graph) with the set of *nodes* (or vertices) $\mathcal{V} = \{1,\ldots,n\}$ involving a finite nonempty set denoting the agents, the set of *edges* $\mathcal{E} \subseteq \mathcal{V}\times\mathcal{V}$ involving a set of ordered pairs denoting the direction of information flow, and an *adjacency* matrix $\mathcal{A} \in \mathbb{R}^{n\times n}$ such that $\mathcal{A}_{(i,j)} = 1$, $i,j = 1,\ldots,n$, if $(j,i) \in \mathcal{E}$, and 0 otherwise. The edge $(i,j) \in \mathcal{E}$ denotes that agent $\mathcal{G}_j$ can obtain information from agent $\mathcal{G}_i$, but not necessarily vice versa. Moreover, we assume that $\mathcal{A}_{(i,i)} = 0$ for all $i \in \mathcal{V}$. A *graph* or *undirected graph* $\mathfrak{G}$ associated with the adjacency matrix $\mathcal{A} \in \mathbb{R}^{q\times q}$ is a directed graph for which the *arc set* is symmetric, that is, $\mathcal{A} = \mathcal{A}^{\mathrm{T}}$. A graph $\mathfrak{G}$ is *balanced* if $\sum_{j=1}^{n}\mathcal{A}_{(i,j)} = \sum_{j=1}^{n}\mathcal{A}_{(j,i)}$ for all $i = 1,\ldots,n$. Finally, we denote the *value* of the node i, $i = 1,\ldots,n$, at time k by $x_i(k) \in \mathbb{R}$. The consensus problem involves the design of a dynamic algorithm that guarantees information state equipartition, that is, $\lim_{k\to\infty} x_i(k) = \alpha \in \mathbb{R}$ for $i = 1,\ldots,n$.

In this section, we consider a discontinuous consensus protocol $\mathcal{G}$ with time-dependent and state-dependent communication links given by

$$\begin{aligned}
\dot{x}_i(t) &= \sum_{j=1, j\neq i}^{q} \mathcal{C}_{(i,j)}(x_i(t), x_j(t)) a_{ij}(t, x_i(t), x_j(t))\\
&\quad \cdot\mathrm{sign}(x_j(t) - x_i(t)),\\
&\quad x_i(t_0) = x_{i0}, \quad t \geq t_0, \quad i = 1,\ldots,q, \qquad (17)
\end{aligned}$$

where $t \geq t_0$, $x_i(t) \in \mathbb{R}$, $a_{ij} : \mathbb{R}^3 \to \mathbb{R}$ satisfies $a_{ij}(t,x_i,x_j) = a_{ji}(t,x_j,x_i)$ and $m \leq a_{ij}(t,x_i,x_j) \leq M$, $a_{ij}(t,x_i,x_j) \not\equiv 0$, $i,j = 1,\ldots,q$, $i \neq j$, $0 < m < M$ is a constant, and $\mathcal{C}_{(i,j)} : \mathbb{R}^2 \to \mathbb{R}$ satisfies the following assumption:

Assumption 1: For the *connectivity matrix*[1] $\mathcal{C}(x) \in \mathbb{R}^{q\times q}$, $x \triangleq [x_1,\ldots,x_q]^{\mathrm{T}} \in \mathbb{R}^q$, associated with $\mathcal{G}$ defined by

$$\mathcal{C}_{(i,j)}(x_i,x_j) \triangleq \begin{cases} 0, & \text{if } (j,i) \in \mathcal{E},\\ 1, & \text{otherwise},\end{cases}$$
$$i \neq j, \quad i,j = 1,\ldots,q, \qquad (18)$$

and $\mathcal{C}_{(i,i)}(x_i,x_i) = -\sum_{k=1,\,k\neq i}^{q}\mathcal{C}_{(i,k)}(x_i,x_k)$, $i = 1,\ldots,q$, $\mathrm{rank}\,\mathcal{C}(x) = q-1$, $x \in \mathbb{R}^q$, and $\mathcal{C}^{\mathrm{T}}(x) = \mathcal{C}(x)$, $x \in \mathbb{R}^q$.

Theorem 4.1: Consider the discontinuous consensus protocol $\mathcal{G}$ given by (17). Assume that Assumption 1 holds. Then $\mathcal{G}$ is uniformly semistable and $x_i(t) \rightrightarrows \frac{1}{q}\sum_{i=1}^{q} x_{i0}$ as $t \to \infty$, $i = 1,\ldots,q$.

Proof. First, note that $\|f(t,x)\| \leq M(q-1)\sqrt{q}$ for almost all $t \geq t_0$ and $x \in \mathbb{R}^q$. Next, consider the Lyapunov function candidate

$$V(x) = \frac{1}{2}(x-\alpha\mathbf{e})^{\mathrm{T}}(x-\alpha\mathbf{e}), \qquad (19)$$

where $x \triangleq [x_1,\ldots,x_q]^{\mathrm{T}} \in \mathbb{R}^q$, $\mathbf{e} \triangleq [1,\ldots,1]^{\mathrm{T}}$, and $\alpha \in \mathbb{R}$, and note that

$$\begin{aligned}
&(x-\alpha\mathbf{e})^{\mathrm{T}}\mathcal{K}[f](t,x)\\
&= \mathcal{K}[(x-\alpha\mathbf{e})^{\mathrm{T}} f](t,x)\\
&= \mathcal{K}[x^{\mathrm{T}} f](t,x)\\
&= \mathcal{K}\left[\sum_{i=1}^{q} x_i \sum_{j=1,j\neq i}^{q} \mathcal{C}_{(i,j)} a_{ij}\mathrm{sign}(x_i-x_j)\right](t,x)\\
&= \mathcal{K}\left[-\sum_{i=1}^{q}\sum_{j=1,j\neq i}^{q} \mathcal{C}_{(i,j)} a_{ij}(x_i-x_j)\mathrm{sign}(x_i-x_j)\right](t,x)\\
&\subseteq -\sum_{i=1}^{q}\sum_{j=1,j\neq i}^{q} \mathcal{C}_{(i,j)} a_{ij}(x_i-x_j)\mathcal{K}[\mathrm{sign}(x_i-x_j)](x)\\
&= -\sum_{i=1}^{q}\sum_{j=1,j\neq i}^{q} \mathcal{C}_{(i,j)} a_{ij}(x_i-x_j)\mathrm{SGN}(x_i-x_j)
\end{aligned}$$

[1]The negative of the connectivity matrix, that is, $-\mathcal{C}$, is known as the Laplacian of the directed graph $\mathfrak{G}$ in the literature.

$$= -\sum_{i=1}^{q}\sum_{j=1,j\neq i}^{q}\mathcal{C}_{(i,j)}a_{ij}|x_i-x_j|$$
$$\leq -\sum_{i=1}^{q}\sum_{j=1,j\neq i}^{q}m\mathcal{C}_{(i,j)}|x_i-x_j|,\quad (t,x)\in[t_0,\infty)\times\mathbb{R}^q,$$

which implies that $\langle\nabla V(x),v\rangle \leq -\sum_{i=1}^{q}\sum_{j=1,j\neq i}^{q}m\mathcal{C}_{(i,j)}$ $|x_i-x_j|$ for every $v\in\mathcal{K}[f](t,x)$. Now, it follows from Theorem 1 of [15, p. 153] that $x_1=\cdots=x_q=\alpha$ is Lyapunov stable. In fact, it can be shown that $x_1=\cdots=x_q=\alpha$ is uniformly Lyapunov stable. Next, let $W(x)=\sum_{i=1}^{q}\sum_{j=1,j\neq i}^{q}m\mathcal{C}_{(i,j)}|x_i-x_j|$ and note that $W^{-1}(0)=\{x\in\mathbb{R}^q : x_1=\cdots=x_q\}=\mathcal{E}$. Now, it follows from Theorem 3.2 that $\mathcal{G}$ is uniformly semistable. Finally, since $\sum_{i=1}^{q}\dot{x}_i(t)=0$, $t\geq t_0$, it follows that $x_i(t)\rightrightarrows\frac{1}{q}\sum_{i=1}^{q}x_{i0}$ as $t\to\infty$, $i=1,\ldots,q$. □

Note that Example 3.2 serves as a special case of Theorem 4.1.

V. CONCLUSION

This paper extends the notions of semistability to nonlinear dynamical systems involving discontinuous time-varying vector fields. In particular, Lyapunov-based tests for semistability, weak semistability, as well as uniform semistability are established. These results are used to develop a framework for information consensus algorithms in dynamical networks with switching topologies involving time-dependent and state-dependent communication links for addressing communication link failures, communication dropouts, and time-varying information exchange.

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