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Singular Perturbation Based Solution to Optimal Microalgal Growth Problem and its Infinite Time Horizon Analysis

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Abstract— **The problem of the optimal microalgal growth is considered here. The objective is to maximize the specific growth rate of microalgae by manipulating the irradiance. The model describing the growth of microalgae is based on the mechanistic description in the form of the so called photosynthetic factory (PSF) resulting into the second order bilinear system which is, nevertheless, known in biotechnological literature to comprise many important features of microalgal growth. To obtain the solution of optimal control problem, the singular perturbation approach is used here to reduce fast components of system dynamics leading to a less dimensional system with more complex performance index which allows a nice analytical solution. Its infinite horizon analysis shows that the optimal solution on large time intervals tends to the optimal steady state of PSF thereby supporting the hypothesis often mentioned in the biotechnological literature. Finally, the numerical algorithm to compute optimal control is applied to the original non-reduced system giving very similar results as the reduction based approach.**

I. INTRODUCTION

The problem of the optimal control of bioreactors operating under high irradiance belongs to intensively studied topics in both biotechnology and mathematical biology literature, see [7] and references within there. It is based on the photosynthetic microorganisms growth modelling reflecting the coupling between photosynthesis and irradiance (being a controlled input), resulting in the steady-state light response curve (so-called *P–I curve*), which represents the microbial kinetics, see e.g. *Monod* or *Haldane* type kinetics [13] and also survey introduction in [10].

Nevertheless, in order to study an optimal control of algae production, a dynamic model should be developed. The model considered later on is the lumped parameter model for photosynthesis and photoinhibition, the so-called model of photosynthetic factory - PSF model [3], [4], [6], [9], [16]. The main difficulty in considering the dynamic behavior of the photosynthetic processes consists in their different time scales. While the characteristic time of microalgal growth (e.g. doubling time) is in order of hours, light and dark reactions occur in milliseconds and photoinhibition in minutes, for more detail see e.g. [12].

The purpose of this paper is to analyze the two time scales phenomena and to use this analysis to compute explicit optimal control law to maximize algal biomass production. Namely, the reduction of the dynamical system to a slow manifold will be developed and then the corresponding less dimensional optimal control problem will be solved analytically. This is the continuation of the efforts made in [10] where further oversimplifying reduction lead to results that did not correspond to non-reduced system numerical analysis. Here, less restrictive reduction will be made resulting into a more complex, but still analytically solvable problem. Moreover, this setting will give mathematical confirmation to the well-known biotechnological experimental observation that for large time intervals optimal solutions tend to be constant.

This paper is organized as follows. Section II presents the dynamic model of the microalgal growth in detail and derives its reduction to a slow manifold. Section III applies Pontryagin's maximum principle to derive analytically optimal irradiance to maximize the average production rate. It also formulates and proves some biotechnological relevant properties of optimal solution. Simulation experiments for the full dimensional non-reduced system are collected in Section IV to support the viability of the reduced system based analysis. Some conclusions and outlooks for further research are drawn in the final section.

II. DYNAMICAL MODEL OF MICROALGAL GROWTH

Microalgal growth has the following important experimentally based properties: (i) the steady state kinetics is of *Haldane* type [8], and (ii) the microalgal culture in suspension has so-called *light integration* property [14], [8], i.e. as the light/dark cycle frequency, [5], is going to infinity, the value of resulting production rate (e.g. oxygen evolution rate) goes to a certain limit value, which depends on average irradiance only [9]. These features are best comprised by the dynamical model described further in detail.

A. Model of photosynthetic factory – dynamical PSF model

The following model, called as the **model of photosynthetic factory** has been recently studied in the biotechnological literature [3], [4], [16], [6]. Its main features are schematically shown on Figure 1 where three states of the photosynthetic factory are: R resting state, A activated state, B inhibited state. Transition rates are αu, βu, γ, δ

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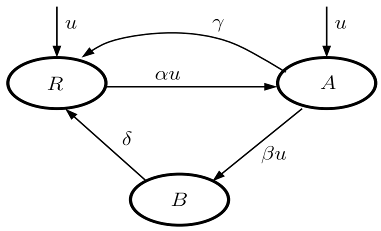

Fig. 1. Scheme of states and transition rates of the photosynthetic factory – Eilers and Peeters PSF model.

(unit: s^{-1}) while the input variable u is the irradiance. The transition from state A to state B models the photoinhibition process, while the transition from state B to state R models the recovery from the photoinhibition. The photosynthetic growth is proportional to the so-called dark reactions modelled as the transition from state A to state R, see equation (3). Light reactions are modelled as the transition from state R to state A. This scheme can be mathematically modelled as follows

$$\dot{x} = \mathcal{A}x + u(t)\mathcal{B}x + u(t)\mathcal{C} \ , \tag{1}$$

where the single scalar input $u(t)$ represents the irradiance in the culture (unit: μE m^{-2} s^{-1}) and $\mathcal{A}, \mathcal{B}, \mathcal{C}$ are matrices and column vector of the appropriate dimensions. The state x of the PSF model is three dimensional, namely, $x = (x_R, x_A, x_B)^\top$, where x_R represents the probability that PSF is in the resting state R, x_A the probability that PSF is in the activated state A, and x_B the probability that PSF is in the inhibited state B, i.e. obviously $x_R + x_A + x_B = 1$. Taking into the account this condition and preferring the states x_A, x_B due to their measurability one has:

$$\begin{aligned} \begin{bmatrix} \dot{x_A} \\ \dot{x_B} \end{bmatrix} &= \begin{bmatrix} -\gamma & 0 \\ 0 & -\delta \end{bmatrix} \begin{bmatrix} x_A \\ x_B \end{bmatrix} + \\ u(t) \begin{bmatrix} -(\alpha+\beta) & -\alpha \\ \beta & 0 \end{bmatrix} & \begin{bmatrix} x_A \\ x_B \end{bmatrix} + u(t) \begin{bmatrix} \alpha \\ 0 \end{bmatrix}, \end{aligned} \tag{2}$$

$$\alpha = 1.935 \times 10^{-3} \mu E^{-1} m^2, \gamma = 1.460 \times 10^{-1} s^{-1},$$
$$\beta = 5.785 \times 10^{-7} \ \mu E^{-1} m^2, \delta = 4.796 \times 10^{-4} s^{-1},$$

where the constants are taken from [16], [11] and $u(t)$ is a known piecewise smooth scalar function. In other words, PSF model is the so-called bilinear controlled system, cf. [2] and references within there. The PSF model has to be completed by an equation connecting the hypothetical states of PSF model with some quantity related to the cell growth. This quantity is the specific growth rate μ. According to [3], [16], the rate of photosynthetic production (proportional to the specific growth rate μ defined as: $\mu := \dot{c_x}/c_x$, where c_x is the microalgal cell density) is proportional to the number of transitions from the activated to the resting state, i.e. $\gamma \ x_A(t)$. Finally, for the average specific growth rate we have the relation:

$$\mu = \kappa\gamma \frac{1}{t_f - t_0} \int_{t_0}^{t_f} x_A(t) \mathrm{d}t \ , \tag{3}$$

where κ is a new dimensionless constant – the fifth PSF model parameter. The quantity in (3) will be further used as the performance index.

For the constant input signal $u \geq 0$ the system of differential equation (2) is linear and its matrix has two distinct negative eigenvalues. Therefore, any solution of (2) with constant $u \geq 0$ globally converges to the following steady state solution depending on that constant $u \geq 0$:

$$x_{Ass} = \frac{\delta \cdot \alpha u}{\lambda_F \lambda_S}, \quad x_{Bss} = \frac{\alpha\beta u^2}{\lambda_F \lambda_S} \ , \tag{4}$$

where $\lambda_{F,S} < 0$ are eigenvalues of the corresponding constant matrix on the right hand side of (2). As already noted, the performance index to be maximized in the sequel is based on quantity defined in (3). If only constant irradiance is considered and steady state transition phenomena are neglected, immediate idea is to maximize the steady state value x_{Ass} with respect to u. Straightforward computations [9], [11] show that such a maximal value exists and is achieved for the unique input denoted as $u_{opt_{ss}}$ and given as follows:

$$u_{opt_{ss}} = \sqrt{\frac{\gamma\delta}{\alpha\beta}}, \quad u^* := u/u_{opt_{ss}}. \tag{5}$$

In the sequel, with a slight abuse of notation, the above $u_{opt_{ss}}$ will be called as *constant optimal input*. The variable u^*, introduced in (5), is a new normalized input variable used in the sequel, with such an input variable, the optimal constant input is simply equal to 1.

Summarizing, the above described PSF model is a convenient modelling framework for lumped parameter model of microalgal growth satisfying two basic properties (i) and (ii) formulated at the beginning of the current section. The latter property is mathematically proved in [9] based on the earlier result on bilinear systems in [2]. For more details, see [10], [9], [11] and further references within there.

B. PSF model re-parametrization and reduction

The aim of this short subsection is to rewrite the model (2),(3) introducing a more convenient parametrization. Namely, consider new parameters $q_i, \ i = 1, .., 5$, defined as

$$q_1 := \sqrt{\frac{\gamma\delta}{\alpha\beta}}, \ q_2 := \sqrt{\frac{\alpha\beta\gamma}{\delta}} \ \frac{1}{\alpha+\beta}, \ q_3 := \kappa\gamma\sqrt{\frac{\alpha\delta}{\beta\gamma}} \ , \tag{6}$$

$$q_4 := \alpha \ q_1, \ q_5 := \beta/\alpha, \tag{7}$$

together with earlier introduced dimensionless irradiance $u^* := u/u_{opt_{ss}}$ giving the re-parameterized model

$$\begin{aligned} \tfrac{1}{q_4} \begin{bmatrix} \dot{x}_A \\ \dot{x}_B \end{bmatrix} &= - \begin{bmatrix} q_2(1+q_5) & 0 \\ 0 & \frac{q_5}{q_2(1+q_5)} \end{bmatrix} \begin{bmatrix} x_A \\ x_B \end{bmatrix} \\ &+ u^* \begin{bmatrix} -(1+q_5) & -1 \\ q_5 & 0 \end{bmatrix} \begin{bmatrix} x_A \\ x_B \end{bmatrix} + u^* \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \end{aligned} \tag{8}$$

$$\mu = q_2 q_3 (1+q_5) \frac{1}{t_f - t_0} \int_{t_0}^{t_f} x_A(t) \mathrm{d}t \ . \tag{9}$$

Notice that q_1 units are those of irradiance (μE m^{-2} s^{-1}), q_2, q_5 are dimensionless, q_3, q_4 are in s^{-1}. The reason to introduce such a re-parameterization is that the role of each new parameter is now much more clearly visible. Namely, parameters q_1, q_2, q_3 correspond to the steady state properties of the PSF, while $q_1 := u_{opt_{ss}}$ by definition. Furthermore, q_4 influence the overall dynamics through constant time scaling only, while q_5 is a small parameter quantifying the separation between the fast and slow dynamic; $q_5 \approx 10^{-4}$. More specifically, based on (2) and [16], the following values of PSF re-parameterized model parameters were calculated for the microalga *Porphyridium* sp.: $q_1 := 250.106$ μE m^{-2}, $q_2 := 0.301591$, $q_3 := 0.176498e-3$ s^{-1}, $q_4 := 0.483955$ s^{-1}, $q_5 := 0.298966e-3$.

Finally, the expressions for the steady states depending on constant inputs given by (4) has after the above re-parameterization the following simpler form

$$x_{Bss} = \frac{u^{*2}}{u^{*2} + u^*/q_2 + 1}, \; x_{Ass} = \frac{x_{Bss}}{q_2(1+q_5)u^*}. \quad (10)$$

In particular, by (8-10) the constant input $u^* = 1$ maximizes value of both μ and x_A among all constant inputs $u^* \geq 0$:

$$\mu_{max} = \frac{q_3}{2 + q_2^{-1}}, \quad x_{Ass}^{max} \cong \frac{1}{2q_2 + 1}. \quad (11)$$

As stated above, the system (8) is a stiff system, i.e., roughly saying, its first equation contains coefficients that are several order higher than those of the second one. To make advantage of that, one can reduce the dynamics to the one dimensional one using the singular perturbation approach with respect to the small parameter $q_5 \approx 10^{-4}$ [15]. This is done in the following way. First, introduce a new faster time scale $\tau = q_5^{-1} t$, so that the system (8) takes the form

$$\frac{q_5}{q_4}\frac{\mathrm{d}}{\mathrm{d}\tau}\begin{bmatrix} x_A \\ x_B \end{bmatrix} = -\begin{bmatrix} q_2(1+q_5)x_A \\ \frac{q_5}{q_2(1+q_5)}x_B \end{bmatrix} + u^*\begin{bmatrix} -(1+q_5) & -1 \\ q_5 & 0 \end{bmatrix}\begin{bmatrix} x_A \\ x_B \end{bmatrix} + u^*\begin{bmatrix} 1 \\ 0 \end{bmatrix}. \quad (12)$$

Now, after dividing the second equation by q_5 one obtains the singularly perturbed system with respect to the small parameter q_5. This system thanks to the properties of its right hand side clearly satisfies the sufficient condition for the convergence of the singular perturbation and therefore one can consider the reduction obtained by setting $q_5 = 0$ to obtain

$$x_A = (1 - x_B)\frac{u^*}{(u^* + q_2)(1+q_5)},$$
$$\frac{1}{q_4}\frac{\mathrm{d}}{\mathrm{d}\tau}x_B = -\frac{1}{q_2(1+q_5)}x_B + \frac{(1-x_B)u^{*2}}{(u^*+q_2)(1+q_5)}.$$

Now, changing the time scale back to the real time variable t, one has finally the following reduced system

$$x_A = \frac{u^*(1-x_B)}{(u^*+q_2)(1+q_5)}, \quad (13)$$

$$\frac{\mathrm{d}x_B}{\mathrm{d}t} = -\frac{q_4 q_5 x_B}{q_2(1+q_5)} + \frac{q_4 q_5 (1-x_B)u^{*2}}{(1+q_5)(u^*+q_2)}. \quad (14)$$

Roughly saying, any solution of the system, no matter what the initial conditions are, quickly satisfies the above relations (13,14) with some precision. The rate of the error decay and its steady state estimate has been obtained as well. They show that crucial issue is actually estimate of the input derivative. Details of this analysis will be presented in the future publication. The set of all states satisfying (13) is called as the **slow manifold** while the relation (14) is called as the **slow dynamics**.

Notice, that the performance index (9) is computed via x_A while reduced dynamics is in terms of x_B. Contrary to [10] making further simplifications to keep functional simple, the present paper takes a different approach. Namely, the slow dynamics (14) is considered in terms of x_B and functional (9) is re-computed via (13) to obtain functional depending on x_B and u^*. The resulting complicated functional is now depending both on state and input leading to a still quite complicated optimization problem. Nevertheless, the explicit analytical solution to this problem is possible and it is provided in the next section in detail.

III. OPTIMAL CONTROL - MAXIMUM PRINCIPLE FOR REDUCED SYSTEM

In this section, the optimal control problem for the system (14) with the performance index obtained by (9),(13) is considered and solved analytically. Recall that initial state is assumed to be given and fixed, the final state is free and time interval is fixed. Summarizing, for given **fixed** $T > 0, U > 0, x^0 \in \mathbb{R}^2$, the following optimal control problem[1] is to be solved: find measurable on $[0,T]$ function $u(t)$ such that (denote in the sequel $x_1 := x_B$):

$$J = \int_0^T (x_1 - 1)\frac{u(t)}{u(t)+L}\,\mathrm{d}t \;\mapsto\; \min, \; u(t) \in [0,U], \quad (15)$$

$$\dot{x_1} = -\frac{K}{L}x_1 + u(1-x_1)\frac{u}{u+L}K, \quad x_1(0) = x_1^0, \quad (16)$$

$$\text{where } K := q_4 q_5 (1+q_5)^{-1}, \; L := q_2. \quad (17)$$

Maximum Principle can be adapted for this case as follows.

Proposition 3.1: Consider the following system

$$\dot{x} = f(x,u), x = [x_1, \ldots, x_n]^\top \in \mathbb{R}^n, u \in U \subset \mathbb{R}^m, \quad (18)$$

$$J = \int_{t_0}^{t_f} f_0(x,u)\mathrm{d}t, \; x(t_0) = x^0, \; x(t_f) \in \mathbb{R}^n, \quad (19)$$

J to be minimized choosing a measurable function $u(t)$ where $x^0 \in \mathbb{R}^n, 0 \leq t_0 < t_f$ and compact U are given. Suppose u^{opt} is an optimal control minimizing performance index in problem (18,19) and let $x^{opt}(t), x^{opt}(0) = x^0$, be the corresponding state trajectory. Then there exists non-trivial solution $\psi(t) = [\psi_1(t), \ldots, \psi_n(t)]^\top$ of the following adjoint equation

$$\dot{\psi} = \frac{\partial f_0}{\partial x}(u^{opt}, x^{opt})^\top - \frac{\partial f}{\partial x}(u^{opt}, x^{opt})^\top \psi, \; \psi(t_f) = 0,$$

[1] We are replacing maximization of algae production by minimization of its amount with minus sign added, without any loss of generality putting $t_0 = 0, t_f = T$, omitting all constants before integral of the performance index and for shortness, where no confusion arises, putting $u := u^*$.

such that for all $t \in [0,T]$ it holds

$$\max_{u\in U} \mathcal{H}(x^{opt}(t), u(t), \psi(t)) = \mathcal{H}(x^{opt}(t), u^{opt}(t), \psi(t)),$$

where $\mathcal{H}(x,u,\psi) := -f_0(x,u) + \psi^\top f(x,u)$ is the so-called Hamiltonian for the optimal control problem (18,19).

Proof: See [10]. □

The Hamiltonian and the adjoint system for (15,16) are:

$$\mathcal{H} = -\frac{u(x_1-1)}{u+L} + \psi_1 K\left(\frac{(1-x_1)u^2}{u+L} - \frac{1}{L}x_1\right), \quad (20)$$

$$\dot{\psi}_1 = \frac{u}{u+L} + \psi_1\left(\frac{K}{L} + u\frac{u}{u+L}K\right), \quad \psi_1(T) = 0. \quad (21)$$

Suppose $u^o(t), t \in [0,T]$, solves the optimal control problem (15-16), then by Proposition 3.1 it holds for all $t \in [0,T]$:

$$\mathcal{H}(\psi(t), u^o(t), x(t)) = \max_{u\in[0,U]} \mathcal{H}(\psi(t), u, x(t)) \equiv 0, \quad (22)$$

for some solution of (21) $\psi(t) := [\psi_0(t), \psi_1(t)]^\top \not\equiv 0$, i.e.

$$\phi(u^0) = \max_{u\in[0,U]} \phi(u), \quad \phi(u) := \frac{u(1-x_1)(1+\psi_1 K u)}{u+L},$$

where $\psi_1(t)$ is the uniquely given solution of (21). To determine u^0, compute $\phi'(u)$ to obtain

$$\frac{\partial\phi(u)}{\partial u} = \frac{1-x_1}{(u+L)^2}\left(K\psi_1 u^2 + 2KL\psi_1 u + L\right). \quad (23)$$

First of all, it is obvious from (15,16) that $x_1(t) < 1\ \forall t > 0$. As the co-state ψ_1 is given by (21), it is easy to see that $\psi_1(t) \leq 0\ \forall t < T$. Actually, assuming $\psi_1(t') > 0$ for some $t' < T$ one has by (recall that $K, L > 0$)

$$\frac{u}{u+L} \geq 0, \quad \frac{K}{L} + u\frac{u}{u+L}K > 0, \ \forall u \in [0,U],$$

that $\dot{\psi}_1(t) > 0,\ \forall t \geq t'$, i.e. $\psi_1(t) > 0,\ \forall t \geq t'$ what contradicts to the condition $\psi_1(T) = 0$. From the same equation one can see that $\psi_1 \equiv 0$ on some $[t', T]$ if and only if $u(t) \equiv 0,\ \forall t \in [t', T]$. Nevertheless, on such a time interval the derivative in (23) equals to $L > 0$, i.e. maximum of ϕ can not be achived at $u = 0$. Therefore, only case $\psi_1(t) < 0\ \forall t < T$ is possible.

Now, notice that in (21), when $\psi_1 < 0$, derivative is equal to zero at the interior point of $[0,U]$ and changes from the positive to the negative one (notice, that in system (15-16) it obviously holds $x_1(0) \in [0,1] \Rightarrow x_1(t) \in [0,1], \forall t \geq 0$ and $x_1(0) \in [0,1]$ is inevitable by biological meaning of $x_1 := x_B$). This point can be computed by solving the quadratic equation and is given as

$$\tilde{u}(\psi_1) = -L + \sqrt{L^2 - \frac{L}{K\psi_1}}. \quad (24)$$

Summarizing, the only possible optimal control $u^o(t)$ is given by the following formula

$$u^o(t) = \alpha(\psi_1(t)),$$
$$\alpha(\psi_1) = \min\left\{ -L + \sqrt{L^2 - L\big(K\psi_1\big)^{-1}}\ ,\ U\right\}, \quad (25)$$

where $\psi_1(t)$ is the solution of (21) with $u = \alpha(\psi_1)$.

As matter of fact, to obtain the optimal control (25) one has first to solve a nonlinear differential equation, i.e. (21) with $u = \alpha(\psi_1)$ and then substitute this solution to the above $\alpha(\cdot)$. Nevertheless, some interesting properties of this solution can be obtained by analysis of that nonlinear equation (21) with $u = \alpha(\psi_1)$. First, the full qualitative description of the above optimal control is formulated and proved as the following

Proposition 3.2: Optimal control given in (25) is strictly increasing on time interval $[0, T - T^{sat}]$ while on $[T - T^{sat}, T]$ it holds $u(t) \equiv U$. Moreover, the length T^{sat} of the interval where the optimal control is saturated does not depend on T, namely, it equals to

$$T^{sat} = \frac{L(U+L)}{K(U+L+LU^2)} \log\left(\frac{U^2(U+2L)}{(U^2-1)(U+L)}\right).$$

Proof: The first part of the proposition follows from the fact that the right hand side of the adjoint equation is always strictly positive, so that $\psi_1(t) < 0\ \forall t < T$ and strictly increases, while in (25) u^o depends on ψ_1 in strictly increasing way, unless the saturation occurs. To obtain the formula for T^{sat}, for $\tilde{u}$ given by (24) consider the costate ψ_1^{sat} where $\tilde{u}(\psi_1^{sat}) = U$:

$$\psi_1^{sat} := \frac{-L}{KU(U+2L)}. \quad (26)$$

As $\psi_1(T) = 0$, $\psi_1(T^{sat}) = \psi_1^{sat}$ and $\psi_1(t)$ is strictly increasing, T^{sat} should obviously satisfy the following relation

$$0 = \mathrm{e}^{K(\frac{1}{L}+\frac{U^2}{U+L})T^{sat}}\left(\psi_1^{sat} + \int_0^{T^{sat}} \frac{U\mathrm{e}^{-K(\frac{1}{L}+\frac{U^2}{U+L})s}}{U+L}\mathrm{d}s\right),$$

i.e. after integration, re-grouping and cancelling some terms

$$\left(\mathrm{e}^{(\frac{K}{L}+U\frac{U}{U+L}K)T^{sat}} - \frac{U^2(U+2L)}{(U^2-1)(U+L)}\right) = 0,$$

giving easily the above formula for T^{sat}. □

The proposition just proved shows that the optimal control course depends only on the input saturation and does not depend on initial condition $x_1(0)$. Besides, for the same U, and two different $T_1 > T_2$ the optimal control on $[0, T_1]$ coincides on subinterval $[T_1 - T_2, T_1]$ with the optimal control on interval $[0, T_2]$. Moreover, for $U \geq 1$ with increasing T, the optimal control converges to the constant input $u \equiv 1$ known to maximize the performance index within constant inputs. More precisely, it holds the following

Proposition 3.3: Denote $u^o_T(t)$ the optimal control (25) corresponding to the fixed time interval $[0,T]$ and assume $U \geq 1$. Then

$$\forall\, \epsilon, \tilde{T} > 0,\ \exists\, T(\epsilon, \tilde{T}) > 0 :\ |u^o_{T(\epsilon,\tilde{T})}(t) - 1| \leq \epsilon,\ \forall\, t \in [0, \tilde{T}].$$

Proof: Consider the following relations

$$\dot{\psi}_1 = \tfrac{\alpha(\psi_1)}{\alpha(\psi_1)+L} + \psi_1\left(\tfrac{K}{L} + \tfrac{K\alpha^2(\psi_1)}{\alpha(\psi_1)+L}\right),\ \psi_1(T) = 0,$$
$$\alpha(\psi_1) = \min\left\{ -L + \sqrt{L^2 - L\big(K\psi_1\big)^{-1}}\ ,\ U\right\}, \quad (27)$$

$$\psi_1^e := \frac{-L}{K(1+2L)}, \quad \alpha(\psi_1^e) = 1. \quad (28)$$

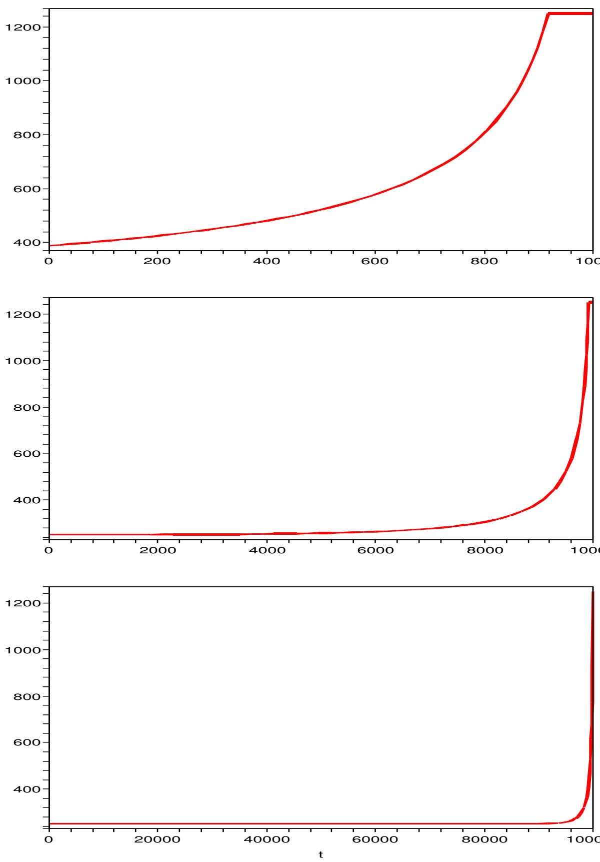

Fig. 2. Singular perturbation based reduction - optimal control for $U = 1250$ and $T = 10^3, 10^4, 10^5$ (from top to bottom, correspondingly).

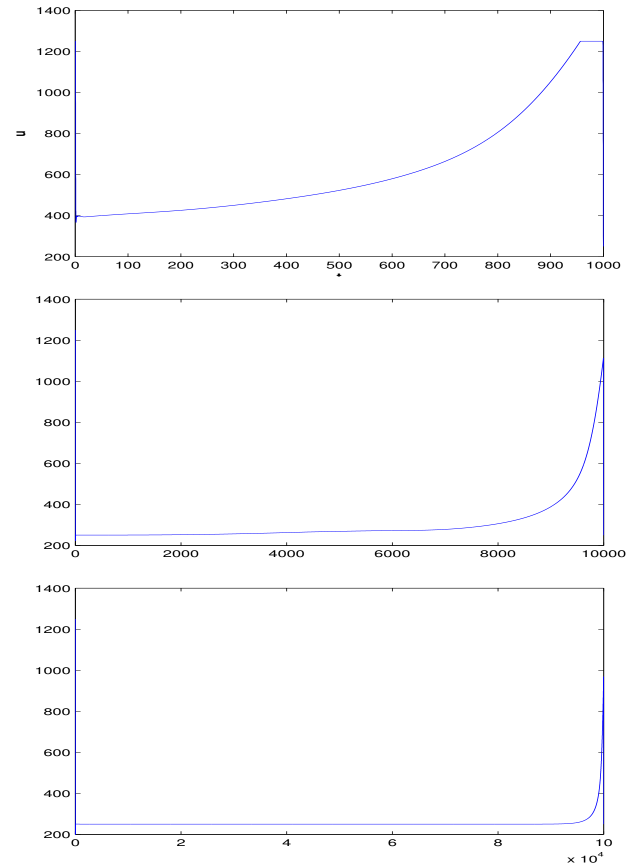

Fig. 3. Gradient algorithm - computed optimal control for $U = 1250$ and $T = 10^3, 10^4, 10^5$ (from top to bottom, correspondingly).

Straightforward, though laborious computations show that

$$\alpha\left(\tfrac{-L}{K(1+2L)}\right) = \min\left\{ -L + \sqrt{L^2 + 2L + 1}\ ,\ U \right\} = \min\{1, U\} = 1, \quad \tfrac{1}{1+L} + \tfrac{-L}{K(1+2L)}\left(\tfrac{K}{L} + \tfrac{K}{1+L}\right) = 0.$$

Therefore, ψ_1^e given by (28) is the equilibrium of (27) being, in turn, the co-state equation (23) with $u = \alpha(\psi_1)$. Further,

$$\frac{\alpha(\psi_1)}{\alpha(\psi_1) + L} + \psi_1\left(\frac{K}{L} + \frac{K\alpha^2(\psi_1)}{\alpha(\psi_1) + L}\right) \begin{cases} < 0 \text{ for } \psi_1^e > \psi \\ = 0 \text{ for } \psi_1^e = \psi \\ > 0 \text{ for } \psi_1^e < \psi \end{cases}$$

giving by the simple Lyapunov-like function $V = (\psi - \psi^{eq})^2/2$ argument that the equilibrium (28) is actually the unique and globally asymptotically stable one for the system (27) in reversed time. The last fact obviously implies that

$$\forall \tilde{T} > 0, \forall \epsilon > 0\ \exists T = T(\tilde{T}, \epsilon) :\ |\psi_1(t) - \psi_1^e| < \epsilon, t \in [0, \tilde{T}],$$

where $\psi_1(t)$ is the solution of (27). Now, the claim of the proposition to be proved follows by the second equality of (28) and by (25). □

Remark 3.4: The last proposition actually confirms widely spread in biotechnological literature but unproved conjecture that on large time intervals the so-called constant optimal control is actually close to the general optimal control, that does not depend on initial condition. All properties are nicely demonstrated on Fig. 2, where the input is depicted in the original biologically relevant non-scaled units, i.e. the constant optimal control input scaled dimensionless value $u = 1$ corresponds there to the non-scaled value $u = 250\ \mu E m^{-2} s^{-1}$.

IV. Optimal control - numerical gradient algorithm for non-reduced system

The aim of this section is to compute for the same data as in previous section the optimal control for non-reduced system numerically to demonstrate that results do not differ from those provided by the analytical ones for the singular perturbation based reduction. To this end, the well-known gradient algorithm is applied here to compute numerical approximation to the optimal control problem defined by (2, 3). Here we take advantage of the fact, that the gradient in the optimal control problem with fixed initial condition, free end state condition and fixed time interval is quite easily computable [1]. The only tricky practical problem is to create a sophisticated step adjustment, as our optimum appears to be quite flat. As a consequence, to proceed with a fixed step is practically impossible.

First, let briefly recall the gradient algorithm. Recall, that the performance index J to be minimized is chosen as the integral in (3) with minus sign subject to (2). Denote there $x = (x_1, x_2)^\top := (x_A, x_B)^\top$ giving the Hamiltonian $H = -x_2 + \psi(Ax + Bxu + cu)$, where the matrices A, B and vector c are visible from right hand side of (2). The co-state

$\psi(t)$ is given by the following adjoint equation

$$\dot{\psi} = -\frac{\partial H}{\partial x} = \left[\begin{array}{c} 1 \\ 0 \end{array} \right] - \psi A^\top - \psi B^\top u, \quad \psi(t_f) = 0. \quad (29)$$

Algorithm 4.1: <u>*Gradient algorithm with variable step*</u>

1) Suppose initial state of the system $x(0) = x^0 \in \mathbb{R}^2$, input saturation limit $U > 0$ and the terminal time $T > 0$ are given. Choose an initial iteration of the control law u_0, the initial value of the step constant k_0, fix some $\kappa > 0$ and some suitable small $\epsilon_1 > 0, \epsilon_2 > 0$.
2) Suppose an iteration u_i of the optimal control, the step constant k_i and the corresponding value of the performance index J_i are known. Compute using (2) and (29) the state and co-state trajectories $x_i(t), \psi_i(t)$ corresponding to the initial condition x^0 and the input $u_i(t)$.
3) Now, compute the gradient $\mathcal{G}$ of J at u^i as follows:
$\mathcal{G}^i(t) := \frac{\partial H}{\partial u}\left(x^i(t), \psi^i(t)\right)$.
If $\int_0^T \mathcal{G}^i(t)^2 \mathrm{d}t \leq \epsilon_2$ put $u_{opt} := u_i, J_{opt} := J_i$, **end**.
4) Compute the next iteration of the optimal control u^{i+1} and the next step constant k_{i+1} as follows:
a) Set $k := k_i$, $J := J(u_i)$, $k_{pr} := k$, $J_{pr} := J$.
b) Set $\tilde{u}(t) = \max\left\{u_i + k\mathcal{G}^i(t), U\right\}$ and compute value of functional $J(\tilde{u})$.
c) If$\Big[J(\tilde{u}) < J(1-\epsilon_1)\Big]$, put $J_{pr} := J$, $J := J(\tilde{u}), k_{pr} := k, k := k_{pr} + \kappa$, $\kappa := 2\kappa$, go to b), else go to d).
d) If$\Big[J(\tilde{u}) \geq J\Big]$, put $k_{pr} := k, k := (k+k_{pr})/2, \kappa := \kappa/8$, go to b), else go to e).
e) Put $i := i+1, k_i := k, u_i = \tilde{u}, J_i = J$, go to 2.

Once the above algorithm ends, the resulting u_{opt}, J_{opt} are the approximations with precision of order of $\epsilon_2 > 0$ of the optimal control law and the performance index value, correspondingly, maximizing the production rate of the system.

Algorithm 4.1 was implemented in MATLAB and applied to the model of the system given by (2,3), with real biological system parameters, the same as in the previous section. Here, only most typical from numerous and extensive numerical experiments will be described. It turns out that the initial condition x_0 affects the value of performance index but does not affect the resulting optimal control, therefore this value is fixed in all simulations here as $x_0 = [0, 0.5]^T$. The input saturation limit was set to $U = 1250$, the same as in the previous sections. Fig. 3 shows the computed optimal control u for a time interval $T = 1000, 10000, 100000$ sec, respectively. One can see a great deal of coincidence with the same time courses computed in the previous section by reduction method, cf. Fig. 2. We have also compared the resulting values of the performance index with the production provided by constant optimal input $u = 250$ showing that, indeed, with increasing $T > 0$, the average performance converges to that of constant $u = 250$, as predicted in the previous section, see the table below which also shows perfect matching of gradient algorithm based results with those based on singular perturbation reduction.

T	Constant $u \equiv 250$	Reduced	Gradient
1000	442	479	479.6
10 000	5830	5893	5892.7
100 000	61951	62 020	62013

V. Conclusion

It has been demonstrated that on a sufficiently large time interval the optimal irradiation is closed to the constant one maximizing the appropriate component of the system steady state. This fact has been confirmed both analytically based on singular perturbation reduction and numerically via gradient optimization algorithm. Though the extensive numerical experiments gave the same solutions as singular perturbation based reduction, theoretically, one can not completely rule out the possible existence of the fast oscillating optimal solution. This remains an open question subjected to further research.

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