Proceedings of the
47th IEEE Conference on Decision and Control
Cancun, Mexico, Dec. 9-11, 2008

Topological Formulation of Discrete-time Switched Linear Systems and Almost Sure Stability

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Abstract— **In this paper, we study the stability of discrete-time switched linear systems via symbolic topology formulation and the multiplicative ergodic theorem. A sufficient and necessary condition for μ_A-almost sure stability is derived, where μ_A is the Parry measure of the topological Markov chain with a prescribed transition (0,1)-matrix A. The obtained μ_A-almost sure stability is invariant under small perturbations of the system. The topological description of stable processes of switched linear systems in terms of Hausdorff dimension is given, and it is shown that our approach captures the maximal set of stable processes for linear switched systems. The obtained results cover the stochastic Markov jump linear systems, where the measure is the natural Markov measure defined by the transition probability matrix.**

Keywords: Discrete-time switched linear system; topological Markov chain, almost sure stability; Lyapunov exponent; Hausdorff dimension.

I. Introduction

A switched linear system consists of a family of linear subsystems and a rule that governs the switching among them. These types of models are found in many practical systems in which switching is necessary and essential as the system dynamics evolve. More specifically, we consider the discrete-time dynamical system in the form of

$$x_{\ell+1} = H_{\omega_\ell} x_\ell, \qquad \ell \geq 0 \tag{1}$$

where $x_\ell \in \mathbb{R}^n$ and $n \geq 2$ is a fixed integer, ω_ℓ takes a value in a given finite-symbolic set, say $\mathcal{A} = \{1, \ldots, \kappa\}$, and $H_i \in \mathbb{R}^{n\times n}$ for $i \in \mathcal{A}$. Let us denote the nonnegative integer set by $\mathbb{Z}_+ = \{0, 1, 2, \ldots\}$ and the set of all mappings $\mathbb{Z}_+ \to \mathcal{A}$ by

$$\Sigma_\kappa = \big\{\omega \,|\, \omega\colon \mathbb{Z}_+ \to \mathcal{A}\big\}. \tag{2}$$

Then switching can be classified into two situations: (i) arbitrary switching, i.e., the switching rule is characterized by Σ_κ defined by (2); (ii) switching is subject to certain constraints, i.e., the switching rule is characterized by a subset of Σ_κ.

Stability is the primary concern for switched systems. The analysis of its stability is much more difficult and challenging than that of linear systems. When arbitrary switching is considered, a switched system is said to be asymptotically stable if *all* its trajectories converge to the origin. This is also called absolute stability and it requires that all infinite products of matrices taken from $\{H_1, H_2, \ldots, H_\kappa\}$ converge to zero. That is, $\lim_{\ell\to\infty} \prod_{j=0}^{\ell-1} H_{\omega_j} = 0$ for any index sequence $\{\omega_0, \omega_1, ...\}$ with $\omega_j \in \mathcal{A}$. This can be equivalently stated by requiring the joint spectral radius of $\{H_1, H_2, \ldots, H_\kappa\}$ to be strictly less than one [2]. To the best of our knowledge, the study of stability of switched linear systems (1) has been focused on absolute stability (for example, see [13], [19], [29], [30], [32], [34], [35], and references therein). Current available approaches for showing absolute stability of switched systems are essentially based on the search of common Lyapunov functions or variations of the same framework. The existence of a common Lyapunov function for a given switched system is quite restrictive, since an expecting common Lyapunov function has to guarantee that the energy of the overall system decreases to zero along all possible state trajectories governed by switches. Moreover, some critical situations are not able to be addressed by using the Lyapunov function approach. For instance, it is well known that the system (1) may not be absolutely stable even if each $H_i, i \in \mathcal{A}$ is asymptotically stable (i.e. all eigenvalues of H_i are inside the unit circle) (f.g., see [8]). A switched system that is not absolutely stable does not imply the end of stability analysis of the system. For example, for the stochastic Markov jump systems, almost sure stability (instead of absolute stability) plays a key role in the study of these types of systems, since it provides important convergence information in an “average” sense (under appropriate probability measures) which has been proved to be very useful and effective in applications (see [4], [5], [15], [16], [17], [21], [23], [22], [26], [24], [27], [31], [36], and references therein). Another situation is when switching is subject to a subset of Σ_κ (so called admissible switching set), how to identify the stability of (1) has not been clearly characterized yet.

The main challenge for the study of switched systems results from the switched paths that are arbitrary, although it may be subject to some constraints. The switched mechanism basically is uncertain, and the stability analysis has to cover all possible switchings (jumps). With such an uncertainty, the condition of absolute stability of switched systems is hardly met as the number of switches increases (i.e. κ is getting large). Thus, to look for condition(s) of “almost” stability instead of absolute stability becomes more realistic in real applications.

In this paper we apply the ergodic theorem from topology to discuss the stability of (1), an approach that is not available in the current literature. More specifically, we translate the

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problem (1) into a finite state topological Markov chain setting under the framework of symbolic topology formulation. Then we study the dynamics (1) by the Lyapunov exponents based on the corresponding topological Markov chain. The main mathematical tool is the Multiplicative Ergodic Theorem, which is a fundamental theory for describing the qualitative behaviors of dynamical systems from the topological point of view. We derive a necessary and sufficient condition for μ_A-almost sure stability of (1) in which μ_A is the Parry measure, i.e., the unique measure with maximal entropy for the underlying setting. Moreover we have shown that the almost sure stability is not altered under small linear perturbation of the system (1), which is important and critical for real applications. Furthermore, a topological description of stable processes of (1) in terms of Hausdorff dimension is given and this finding illustrates the significance for choosing the measure μ_A (the Parry measure). Our obtained results cover the stochastic Markov jump linear systems, where the measure is the natural Markov measure defined by the transition probability matrix. The proposed approach and the obtained results of this paper provide a fresh point of view for the study of switched systems.

The paper is organized as follows. In section 2, we transform the switched system (1) to a symbolic dynamical system under the framework of topology. Then system (1) with constraints is expressed equivalently as a one-sided topological Markov chain with a prescribed transition (0,1)-matrix A. The concept of almost sure stability is introduced and two preliminary propositions are provided in this section. In section 3, a necessary and sufficient condition for μ_A-almost sure stability of (1) is presented, and a topological description of stable processes of (1) in terms of Hausdorff dimension is given. Section 4 addresses the connection between our obtained results and those for stochastic Markov jump linear systems. The paper ends with concluding remarks.

Let $H = (h_{ij})$ be an $n \times n$ matrix of real numbers. Throughout this paper the *norm*, $\|H\|$, of H can be either $\|H\|_F$, or $\|H\|_1$, or $\|H\|_\infty$ whose definitions are respectively

$$\|H\|_F = \sqrt{\sum_{i,j=1}^{n} |h_{ij}|^2}, \ \|H\|_1 = \sum_{i,j=1}^{n} |h_{ij}|,$$

and $\|H\|_\infty = \max_{1\le i,j\le n} |h_{ij}|$.

II. Symbolic Topology Formulation

We use a symbolic string $\omega = (\omega_0\omega_1\cdots)$ with $\omega_j \in \mathcal{A} = \{1,\ldots,\kappa\}$ to represent a specific switching path of (1). The set of all possible switching paths $\omega = (\omega_0\omega_1\cdots)$ is the κ-dimensional one-sided symbolic space given by (2), that is,

$$\Sigma_\kappa = \{\omega = (\omega_0\omega_1\cdots)\,|\,\omega_i \in \mathcal{A} \text{ for } i = 0,1,2,\ldots\},$$

which is a compact metric space endowed with the usual distance function

$$\mathrm{dist}(\omega,\omega') = \rho^{-n(\omega,\omega')}, \qquad \forall\,\omega,\omega' \in \Sigma_\kappa, \qquad (3)$$

where $\rho > 1$ is any prescribed constant and

$$n(\omega,\omega') = \inf\{\ell \in \mathbb{Z}_+\,|\,\omega_\ell \ne \omega'_\ell\}.$$

If $\omega_\ell = \omega'_\ell$ for all nonnegative integers ℓ, then $n(\omega,\omega') := +\infty$. In order to include those cases in which switching constraints exist, let $A = [a_{ij}]$ be an irreducible $(0,1)$-matrix of size $\kappa \times \kappa$, which is predefined. That is, a_{ij} equals either 0 or 1, and for any pair (i,j) there is some integer $n > 0$ such that $a_{ij}^{(n)} > 0$ where $a_{ij}^{(n)}$ is the (i,j)-th element of A^n. The *admissible set* Σ_A is defined to be

$$\Sigma_A = \left\{\omega = (\omega_0\omega_1\cdots) \in \Sigma_\kappa\,|\,a_{\omega_\ell\omega_{\ell+1}} = 1,\ \ell = 0,1,\ldots\right\}.$$

Thus it is clear that in general $\Sigma_A \subset \Sigma_\kappa$. For the arbitrary switching case where the matrix A satisfies $a_{ij} \equiv 1$ for any $1 \le i,j \le \kappa$, we have $\Sigma_A = \Sigma_\kappa$. Clearly the matrix A contains transition information of all admissible paths, and thus it is usually called a transition matrix. It is not difficult to see that if $(\omega_0\omega_1\omega_2\cdots) \in \Sigma_A$ then we have $(\omega_1\omega_2\cdots) \in \Sigma_A$. For a given Σ_A, the mapping

$$\sigma_A\colon \Sigma_A \to \Sigma_A;\ (\omega_0\omega_1\ldots) \mapsto (\omega_1\omega_2\ldots)$$

is called the one-sided shift defined by the transition matrix A. The dynamical system (Σ_A,σ_A) is said to be *one-sided topological Markov chain* with the transition matrix A, which is a compact subsystem of the one-sided full-shift dynamical system (Σ_κ,σ), where $\sigma\colon \Sigma_\kappa \to \Sigma_\kappa$ is defined by $(\omega_0\omega_1\cdots) \mapsto (\omega_1\omega_2\cdots)$ for any $\omega = (\omega_0\omega_1\cdots) \in \Sigma_\kappa$.

We next define a random matrix over Σ_A associated with the system (1) by

$$S\colon \Sigma_A \to \{H_1,\ldots,H_\kappa\};\ \omega \mapsto S(\omega) = H_{\omega_0}\ \forall\,\omega \in \Sigma_A.$$

Let us denote for any $t \in \mathbb{N}$ and for any $\omega \in \Sigma_A$

$$\sigma_A^t = \overbrace{\sigma_A \circ \cdots \circ \sigma_A}^{t \text{ times}} \text{ and } \sigma_A^0 = \textit{id}\colon \Sigma_A \to \Sigma_A \qquad (4)$$

and

$$S(\omega,t) = S\left(\sigma_A^{t-1}\omega\right)\cdots S(\omega)\colon \mathbb{R}^n \to \mathbb{R}^n.$$

Here $S(\omega,t)$ is called a *linear cocycle* based on (Σ_A,σ_A). Notice $S\left(\sigma_A^\ell\omega\right) = H_{\omega_\ell}$ for any $\omega = (\omega_0\omega_1\cdots)$. Then the system

$$x_{\ell+1} = S\left(\sigma_A^\ell\omega\right)x_\ell, \quad \text{where } \omega \in \Sigma_A, \qquad (5)$$

can be regarded as a hybrid linear system with Markovian switchings.

For a given one-sided topological Markov chain (Σ_A,σ_A) with a transition matrix A, one always can define an invariant measure μ under the one-sided shift σ_A from the classical Krylov-Bogolioubov theorem [37]. Now we are ready to introduce the definition of μ-almost sure stability of (1).

Definition 1: Let A be an irreducible transition matrix and μ be an ergodic σ_A-invariant Borel probability measure on Σ_A; namely, $\mu(\sigma_A^{-1}B) = \mu(B)$ for any Borel subset B of Σ_A and $\mu(B) = 0$ or 1 whenever $\sigma_A^{-1}B = B$ holds μ-mod

0.[1] The switched linear system (1) is said to be "μ-almost sure stable" with respect to an admissible set Σ_A if (1) is exponentially stable for μ-almost all switching sequences ω in Σ_A. This is equivalent to saying, for μ-a.e. $\omega \in \Sigma_A$ we have

$$\lim_{\ell\to\infty} \frac{1}{\ell} \ln \|x_\ell\| = \lim_{\ell\to\infty} \frac{1}{\ell} \ln \|S(\omega,\ell)x_0\| < 0 \quad \forall x_0 \in \mathbb{R}^n.$$

III. Almost Sure Stability

Following convention, we now define a *canonical Markov measure* generated by an irreducible $(0,1)$-matrix A of size $\kappa \times \kappa$. Let us denote the spectral radius of the nonnegative matrix A by ρ_A. Then from the Perron-Frobenius theorem it follows that there are two positive vectors

$$v = (v_1, \ldots, v_\kappa)^{\mathrm{T}} \text{ and } u = (u_1, \ldots, u_\kappa) \tag{6}$$

in $\mathbb{R}^\kappa$ such that

$$Av = \rho_A v \text{ and } uA = \rho_A u \quad \text{with} \sum_{i=1}^{\kappa} u_i v_i = 1. \tag{7}$$

Let

$$p_A = (p_1, \ldots, p_\kappa) \text{ with } p_i = u_i v_i \text{ for } 1 \le i \le \kappa \tag{8}$$

and

$$P_A = [p_{ij}], \text{ where } p_{ij} = \frac{a_{ij} v_j}{\rho_A v_i} \text{ for } 1 \le i, j \le \kappa. \tag{9}$$

Then we have $p_A P_A = p_A$. The matrix $P_A = [p_{ij}]$ can be viewed as a transition probability matrix with $p_{ij} = 0$ if and only if $a_{ij} = 0$. The canonical σ_A-invariant Markov measure μ_A on Σ_A is derived as follows:

$$\mu_A([i_0 \cdots i_\ell]_A) = p_{i_0} p_{i_0 i_1} \cdots p_{i_{\ell-1} i_\ell} \tag{10}$$

where

$$[i_0 \cdots i_\ell]_A = \{\omega \in \Sigma_A \,|\, \omega_0 = i_0, \ldots, \omega_\ell = i_\ell\} \tag{11}$$

is the *cylinder* defined by the word of length $\ell + 1$ for $(i_0 \cdots i_\ell) \in \mathcal{A}^{\ell+1}$ with any $\ell + 1 \in \mathbb{N}$. One can verify that

(1) μ_A is supported on Σ_A with *supp*$(\mu_A) = \Sigma_A$, where *supp*(μ_A) means the minimal σ_A-invariant closed subset of Σ_A with μ_A-measure 1.
(2) μ_A is an ergodic σ_A-invariant Borel probability measure on Σ_A.

Remark 1: Compared to the stochastic Markovian, the above formulation is more general since the transition rates (or called the generator) of the Markov chain can be arbitrary due to it is not necessary for the transition matrix A to be a probability matrix.

We now derive the first main result of this paper. All proofs of theorems and corollaries can be found in the full version of our paper [11].

[1]Two Borel sets "$B = C$ μ-mod 0" means that $\mu((B\backslash C) \cup (C\backslash B)) = 0$. In addition, the σ_A-invariance of μ is equivalent to

$$\int_{\Sigma_A} f\, d\mu = \int_{\Sigma_A} f \circ \sigma_A \, d\mu \quad \forall f \in C(\Sigma_A)$$

.

Theorem 1: Consider the switched linear system (1) with the switching sequence belonging to a topological Markov shift (Σ_A, σ_A). For any $t \in \mathbb{N}$, we write

$$\lambda_{i_0 i_1 \cdots i_{t-1}} = \|H_{i_{t-1}} \cdots H_{i_0}\| \quad \forall (i_0 \cdots i_{t-1}) \in \mathcal{A}^t, \tag{12}$$

where $\|\cdot\|$ is the matrix norm and where $\mathcal{A}^t = \overbrace{\mathcal{A} \times \ldots \times \mathcal{A}}^{t \text{ times}}$. Let $p_A = (p_1, \ldots, p_\kappa)$, $P_A = [p_{ij}]$ and the measure μ_A be defined by (8) and (10), respectively. Then (1) is μ_A-almost sure stable if and only if there is at least one $\hat{t} \in \mathbb{N}$ such that

$$\prod_{(i_0 \cdots i_{\hat{t}-1}) \in \mathcal{A}^{\hat{t}}} \lambda_{i_0 i_1 \cdots i_{\hat{t}-1}}^{p_{i_0} p_{i_0 i_1} \cdots p_{i_{\hat{t}-2} i_{\hat{t}-1}}} < 1. \tag{13}$$

Moreover, if (1) is μ_A-almost sure stable, then there exists some $\varepsilon > 0$ such that every switched linear system

$$x_{\ell+1} = H'_{\omega_\ell} x_\ell \qquad \ell \ge 0$$

is also μ_A-almost sure stable on (Σ_A, σ_A) whenever

$$\|H_i - H'_i\| \le \varepsilon \quad 1 \le i \le \kappa. \tag{14}$$

Remark 2: The ergodic σ_A-invariant measure μ_A defined by (8)-(10) is called the "Parry measure" of the topological Markov chain (Σ_A, σ_A). It is a Gibbs measure which has the maximal entropy, namely $h_{top}(\sigma_A) = h_{\mu_A}(\sigma_A)$, and such a property can characterize other measures in a "maximal" way.

Chaos and entropy are characteristics of the complexity of system (X, f) from two different viewpoints. Chaos is closely related to system behavior, while entropy focuses on "physical principles." In general, they have the following relationships:

(i) $h_{top}(f) > 0$ implies that (X, f) is Li–Yorke chaotic [3].
(j) $h_{top}(\sigma_A) > 0$ if and only if (Σ_A, σ_A) is Li–Yorke chaotic [38].

The following theorem provides a topological description of the set of stable processes.

Theorem 2: We consider the topological Markov jump linear system (5). Let

$$\Sigma_{\text{stab}}(S; A) = \Big\{\omega \in \Sigma_A \,|\, x_{\ell+1} = S(\sigma_A^\ell \omega) x_\ell \text{ is exponentially stable}\Big\}.$$

If system (1) is μ_A-almost sure stable , then we have

$$HD_\rho(\Sigma_{\text{stab}}(S; A)) = HD_\rho(\Sigma_A) = \frac{h_{\text{top}}(\sigma_A)}{\ln \rho}. \tag{15}$$

Here $HD_\rho(\cdot)$ means the Hausdorff dimension under the metric $\rho(\cdot,\cdot)$ defined by (3). Moreover, $HD_\rho(\Sigma_{\text{stab}}(S; A)) > 0$ if and only if (Σ_A, σ_A) is Li-Yorke chaotic.

Remark 3: Identity (15) implies that the μ_A measure defined in (10) (which is the Parry measure of (Σ_A, σ_A)) is a desirable measure from the topological point of view since the "size" of the set of all stable paths is the same as the set of all admissible paths in the sense of Hausdorff dimension.

The largest topological entropy of (Σ_A, σ_A) is attained when $\Sigma_A = \Sigma_\kappa$ and $\sigma_A = \sigma$. We thus consider the switched system (1) with the switching paths allowed to be the whole symbol space Σ_κ. The corresponding transition matrix $A = [a_{ij}]$ satisfies $a_{ij} = 1$ for all $1 \le i,\ j \le \kappa$. In this case, the unique maximal entropy measure μ_A is the $(\frac{1}{\kappa}, \ldots, \frac{1}{\kappa})$-product measure μ_κ with $h_{top}(\sigma) = \ln\kappa$ (Theorem 8.9 in [37]). The following corollary characterizes the μ_κ-almost sure stability of (1) without constraints.

Corollary 1: Let us consider the switched linear system (1) with the switching sequences belonging to the whole symbol space Σ_κ. Then we have

(1) the system is μ_κ-almost stable if and only if there is at least one $\hat{t} \in \mathbb{N}$ such that

$$\prod_{(i_0 \cdots i_{\hat{t}-1}) \in \mathcal{A}^{\hat{t}}} \lambda_{i_0 i_1 \cdots i_{\hat{t}-1}} < 1$$

where $\lambda_{i_0 i_1 \cdots i_{\hat{t}-1}}$ is defined by (12);

(2) Let $\Sigma_{stab}(S; \Sigma_\kappa) = \{\omega \in \Sigma_\kappa \,|\, x_{\ell+1} = S(\sigma^\ell \omega) x_\ell$ is exponentially stable$\}$. Then we have

$$HD_\rho(\Sigma_{\text{stab}}(S; \Sigma_\kappa)) = \text{HD}_\rho(\Sigma_\kappa) = \frac{\ln\kappa}{\ln\rho},$$

provided that the system (1) is μ_κ-almost sure stable.

IV. Stochastic Markov Jump Linear Systems

In this section, we consider the discrete-time system (1), where ω_k is a discrete-time Markovian stochastic process taking value in $\mathcal{A} = \{1, \ldots, \kappa\}$, with transition probabilities $p_{ij} = Pr\{\omega_{k+1} = j | \omega_k = i\}$. The matrix $P = [p_{ij}]$ is called the transition probability distribution. Now the switched system given by

$$x_{\ell+1} = H_{\omega_\ell} x_\ell, \quad \ell \ge 0 \tag{16}$$

is a standard stochastic Markov jump linear system where ω_ℓ is a random variable. The almost sure stability of this type of systems has been discussed by many authors. Recently a new necessary and sufficient condition for almost sure stability by the approach of using a so called "lifted" version of the system is proposed [4, Proposition 3.4]. We shall show that our results obtained in the previous section can cover such a case.

Assume that the transition probability matrix P is irreducible. According to the Perron-Frobenius theorem [6], there exists a unique (invariant) distribution $p = (p_1, \ldots, p_\kappa)$ such that

(a) $0 < p_i < 1$, $\sum_{i=1}^{\kappa} p_i = 1$;
(b) (invariance)$pP = p$.

Now a natural *Markov measure* $\mu_{p,P}$ on Σ_κ defined by (*p*,*P*) can be obtained as follows:

$$\mu_{p,P}([i_0 \cdots i_\ell]) = p_{i_0} p_{i_0 i_1} \cdots p_{i_{\ell-1} i_\ell}. \tag{17}$$

Next, we define an irreducible $(0,1)$-matrix $A_P = [a_{ij}]_{\kappa\times\kappa}$ associated with *P* as follows:

$$a_{ij} = \begin{cases} 1 & \text{if } p_{ij} > 0, \\ 0 & \text{if } p_{ij} = 0. \end{cases} \tag{18}$$

Let $(\Sigma_{A_P}, \sigma_{A_P})$ be the one-side Markov shift defined by the transition matrix A_P. One can verify directly that

- $\mu_{p,P}$ is supported on Σ_{A_P} with *supp*$(\mu_{p,P}) = \Sigma_{A_P}$.
- $\mu_{p,P}$ is an ergodic σ_{A_P}-invariant Borel probability measure on Σ_{A_P} [37, Theorem 1.13].

Thus, according to the proofs of Theorem 1 and Theorem 2, we immediately have

Corollary 2: Consider the stochastic Markov jump linear system (16) with the probability transition matrix P. Let the corresponding ergodic σ_{A_P}-invariant Borel probability measure $\mu_{p,P}$ be defined as (17). Then the system is $\mu_{p,P}$-almost sure stable if and only if there is at least one $\hat{t} \in \mathbb{N}$ such that

$$\prod_{(i_0 \cdots i_{\hat{t}-1}) \in \mathcal{A}^{\hat{t}}} \lambda_{i_0 i_1 \cdots i_{\hat{t}-1}}^{p_{i_0} p_{i_0 i_1} \cdots p_{i_{\hat{t}-2} i_{\hat{t}-1}}} < 1, \tag{19}$$

where $\lambda_{i_0 i_1 \cdots i_{\hat{t}-1}}$ is given by (12). Moreover, if (16) is $\mu_{p,P}$-almost sure stable, then there exists some $\varepsilon > 0$ such that every switched linear system

$$x_{\ell+1} = H'_{\omega_\ell} x_\ell \qquad \ell \ge 0$$

is also $\mu_{p,P}$-almost sure stable based on $(\Sigma_{A_P}, \sigma_{A_P})$ whenever

$$\|H_i - H'_i\| \le \varepsilon \quad 1 \le i \le \kappa.$$

Corollary 3: Over a (*p*,*P*)-Markov shift $(\Sigma_{A_P}, \sigma_{A_P})$, denote

$$\Sigma_{\text{stab}}(\text{p,P}) = \Big\{\omega \in \Sigma_{A_P} \,|\, x_{\ell+1} = S(\sigma_{A_P}^\ell \omega) x_\ell \text{ with } \lim_{\ell\to\infty} \frac{1}{\ell} \ln\|x_\ell\| < 0 \,\forall\, x_0 \in \mathbb{R}^n\Big\}.$$

If (19) holds for some $\hat{t} \in \mathbb{N}$, then

$$HD_\rho(\Sigma_{A_P}) \ge HD_\rho(\Sigma_{\text{stab}}(\text{p,P})) \ge HD_\rho(\mu_{\text{p,P}}). \tag{20}$$

(21)

Remark 4: Criterion (19) for almost sure stability is an extension of the Fang-Loparo-Feng condition [15]. In fact, if $\hat{t} = 1$, then we have

$$\prod_{i \in \mathcal{A}} \|S_i\|^{p_i} < 1,$$

which coincides with the Fang-Loparo-Feng sufficient criterion.

V. Illustrative examples

In this section, we give two examples (with $\kappa = 2$) to show how to apply the criteria obtained in this paper. In what follows, we denote by μ_2 the Parry measure of the full shift system (Σ_2, σ), and $\|\cdot\|$ stands for $\|\cdot\|_F$, which is defined in section 1.3.

Example 1: Consider the switched system (1) with $\mathcal{A} = \{1, 2\}$, and

$$H_1 = \begin{bmatrix} 0.2 & 1 \\ 0 & 0.2 \end{bmatrix}, \qquad H_2 = \begin{bmatrix} 0.9 & 0.4 \\ 0.5 & 0.2 \end{bmatrix}.$$

It is obvious that this system is not stable for all switching sequences $\omega \in \Sigma_2$ since the spectral radius of H_2 is greater than 1. However, it is μ_2-almost surely stable, where μ_2 is the maximal entropy measure of (Σ_2, σ), since

$$\lambda_{11} = \|H_1H_1\| = 0.4040,\ \lambda_{12} = \|H_1H_2\| = 0.7432,$$
$$\lambda_{21} = \|H_2H_1\| = 1.1377,\ \lambda_{22} = \|H_2H_2\| = 1.2545,$$

$$\prod_{(i_0i_1)\in\{1,2\}\times\{1,2\}} \lambda_{i_0i_1} = 0.4285 < 1.$$

Hence, this demonstrates that the system is μ_2-almost surely stable and the Hausdorff dimension of the set of all stable sequences $\omega \in \Sigma_2$ equals 1 (under the metric constant $\rho = 2$) by Corollary 1.

In [4], the system with stochastic Markov chains was considered, where the transition probability distribution was

$$P = \begin{bmatrix} 0.6 & 0.4 \\ 0.1 & 0.9 \end{bmatrix}.$$

The unique invariant distribution in this case is $p = (0.2, 0.8)$. It is not difficult to see that condition (19) holds true for $\hat{t} = 3$. Hence, the system is $\mu_{p,P}$-almost surely stable by Corollary 2. However, one can directly verify that the Hausdorff dimension of the set of all stable sequences in the sense of $\mu_{p,P}$-almost sure stability is strictly less than 1.

We have known that the system is both μ_2-almost surely stable and $\mu_{p,P}$-almost surely stable. Nevertheless, to check $\mu_{p,P}$-almost surely stable, one needs to use Corollary 2 since it requires the information of transition probability distribution. It is possible for a stochastic Markov jump system to be $\mu_{p,P}$-almost surely stable but not to be μ_2-almost surely stable since the maximal entropy measure μ_2 and the ergodic invariant measure $\mu_{p,P}$ are mutually singular.

According to Theorem 1, we know that the almost sure stability of (1) is robust; that is, there exists $\varepsilon > 0$ such that whenever $\|H_i' - H_i\| \le \varepsilon$, $i = 1, 2$, the switched system

$$x_{\ell+1} = H'_{\omega_\ell} x_\ell, \quad \omega_\ell \in \{1, 2\}, \qquad (22)$$

is also μ_2-almost surely stable and the Hausdorff dimension of the set of all stable sequences $\omega \in \Sigma_2$ equals 1, too. Now as an example we estimate the upper bound of admissible perturbation constant ε. Let

$$\lambda_{ij} = \|H_iH_j\| \quad \forall\, i, j \in \{1, 2\}.$$

We first solve the inequality

$$\begin{aligned}&(\lambda_{11}+\delta)(\lambda_{12}+\delta)(\lambda_{21}+\delta)(\lambda_{22}+\delta)\\ &\le \lambda_{11}\lambda_{12}\lambda_{21}\lambda_{22} + |\delta|(\lambda_{11}\lambda_{12}\lambda_{21} + \lambda_{11}\lambda_{12}\lambda_{22} + \lambda_{11}\lambda_{21}\lambda_{22}\\ &+\lambda_{12}\lambda_{21}\lambda_{22}) + \delta^2(\lambda_{11}\lambda_{12} + \lambda_{11}\lambda_{21} + \lambda_{11}\lambda_{22} + \lambda_{12}\lambda_{21}\\ &+\lambda_{12}\lambda_{22} + \lambda_{21}\lambda_{22})|\delta|^3(\lambda_{11} + \lambda_{12} + \lambda_{21} + \lambda_{22}) + \delta^4\\ &< 1.\end{aligned}$$

Substituting the values of λ_{ij} into the above inequality, one can get approximately

$$|\delta| < 0.1542.$$

Next, we denote $G_i = H_i' - H_i$, $i = 1, 2$. A sufficient condition for the μ_2-almost sure stability of the perturbation system (22) is

$$\begin{aligned}&\prod_{(i_0i_1)\in\{1,2\}\times\{1,2\}} \|(H_{i_0}+G_{i_0})(H_{i_1}+G_{i_1})\|\\ &\le \left[\|H_1H_1\| + 2\varepsilon\|H_1\| + \varepsilon^2\right]\left[\|H_1H_2\| + \varepsilon(\|H_1\| + \|H_2\|)\right.\\ &\left.+\varepsilon^2\right]\left[\|H_2H_1\| + \varepsilon(\|H_1\| + \|H_2\|) + \varepsilon^2\right]\\ &\cdot\left[\|H_2H_2\| + 2\varepsilon\|H_2\| + \varepsilon^2\right] < 1,\end{aligned}$$

where

$$\varepsilon = \max\{\|G_1\|, \|G_2\|\}.$$

Since

$$\|H_1\| = 1.0392, \qquad \|H_2\| = 1.1225,$$

it follows that a sufficient condition for the inequality

$$\begin{aligned}&\prod_{(i_0i_1)\in\{1,2\}^2} \|(H_{i_0}+G_{i_0})(H_{i_1}+G_{i_1})\|\\ &\le \prod_{(i_0i_1)\in\{1,2\}^2} (\lambda_{i_0i_1} + 2.245\varepsilon + \varepsilon^2) < 1\end{aligned}$$

to hold is

$$2.245\varepsilon + \varepsilon^2 = \delta < 0.1542,$$

which yields

$$\varepsilon < 0.0667.$$

Example 2: Let us consider the system given in Example 1 with the switching sequences ω belonging to the topological Markov chain (Σ_A, σ_A) with the topological transition matrix A as

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}.$$

This means that during the switching process the subsystem H_2 cannot be allowed to follow itself.

Direct computations show that the spectral radius of A is given by

$$\rho_A = \frac{1+\sqrt{5}}{2},$$

and the Perron vectors are

$$v^T = u = \left(\frac{1+\sqrt{5}}{\sqrt{10+2\sqrt{5}}}, \sqrt{\frac{2}{5+\sqrt{5}}}\right).$$

Thus the Parry distribution is

$$p_A = (0.7236,\ 0.2764),$$

and the transition probability matrix is given by

$$P_A = \begin{bmatrix} \frac{2}{1+\sqrt{5}} & \frac{2}{3+\sqrt{5}} \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0.6180 & 0.3820 \\ 1 & 0 \end{bmatrix}.$$

So we have

$$\prod_{(i_0i_1)\in\{1,2\}^2} \lambda_{i_0i_1}^{p_{i_0}p_{i_0i_1}} = 0.6366 < 1.$$

This verifies that condition (13) holds for $\hat{t} = 2$. The system thus is μ_A-almost surely stable, where μ_A is the Parry measure of $(\Sigma_A,\ \sigma_A)$, and

$$HD_\rho(\Sigma_{stab}(S; A)) = HD_\rho(\Sigma_A) = \frac{\ln(1+\sqrt{5})}{\ln 2} - 1 > 0,$$

where $\rho = 2$ (see (3)).

VI. Concluding Remarks

By viewing switching sequences as the elements in symbolic topology space, we have established a necessary and sufficient condition for almost sure stability of discrete-time switched linear system by using the multiplicative ergodic theorem. Among all ergodic probability measures, Parry measure has been shown to be able to capture the maximal set of stable processes for linear switched systems in the sense of Hausdorff dimension. The μ_A-almost sure stability is unchanged under small linear perturbations of the system. Furthermore, a connection between the switched system (1) and its corresponding symbolic dynamical system (Σ_A, σ_A) is identified, that is, the more Li-Yorke chaotic (Σ_A, σ_A) behaves, the larger set of μ_A almost sure stable paths (1) has. Some recent results for the stochastic Markov jump linear systems can be adopted in our framework. Future research will be concentrated on the continuous-time case as well as nonlinear switched systems.

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