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# Stochastic Stability of the Continuous-Time Unscented Kalman Filter

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***Abstract*—The performance of the modified unscented Kalman-Bucy filter (UKBF) for the nonlinear stochastic continuous-time system is investigated. The error behavior of the UKBF is analyzed. It is proved that the estimation error remains bounded if the system satisfies a detectability condition and both the initial estimation error and the disturbing noise terms are small enough. Furthermore, it is shown that the design of noise covariance matrix plays an important role in improving the stability of the algorithm. Moreover, some selected cases with both bounded and unbounded estimation error are demonstrated by numerical simulations.**

## I. Introduction

KALMAN-BUCY filtering (KBF) [1] is an important and widely used tool for the state and parameter estimation of stochastic systems. Although it was originally developed as an optimal filter for linear systems, an application to nonlinear systems is also possible. The usual procedure is to linearize the nonlinear system at the current estimate, leading to the extended Kalman filter (EKF). This technique has turned out to be one of the most useful methods for the state and parameter estimation of nonlinear stochastic systems [2], [3]. However, it has two well known drawbacks [4]: (1) the first-order linearization can introduce large errors in mean and covariance of the state vector, and (2) the derivation of Jacobian matrices is nontrivial in many applications.

The unscented Kalman filter (UKF) [5]-[7] is an efficient derivative free filtering algorithm for computing approximate solutions to nonlinear optimal filtering problems. It has been successfully applied to numerous practical problems and it has been shown to outperform EKF in many cases [8]. However, in its original form, the UKF is a discrete-time algorithm and it cannot be directly applied to continuous-time filtering problems, where the state and measurement processes are modeled as continuous-time stochastic processes. As a result, the differential equations which result in the continuous-time limit of the UKF are derived in [9]. The derived unscented Kalman-Bucy filter (UKBF) equations are similar to the extended Kalman-Bucy filter (EKBF) equations [10] and consist of a pair of differential equations for the mean and covariance of the posterior state process. The UKBF can be widely used in practice, where a continuous-time signal is observed continuously in time. Examples of such applications are GPS and inertial navigation [11], [12], target tracking [13]-[15], and stochastic optimal control [16], [17]. Simo [9] compared the performance of the UKBF, the EKBF and the SLF for an example system and showed that the UKBF performs better than the EKBF.

Despite its superior practical usefulness, the UKBF has not been analyzed in a rigorous mathematical way as the UKF [18]. Some results obtained in the study of the EKBF can be used to treat the stability properties of the UKBF. Motivated by the stability of the usual Kalman-Bucy filter for linear systems [19] and by successful application of the stochastic stability theory to solve nonlinear-estimation problems in [20], [21], analogous results are examined for the nonlinear case in this paper. In order to improve stability, slight modifications of the standard UKBF are performed by introducing an extra positive definite matrix in the noise covariance matrix. The design of the extra additive matrix can be seen as a tradeoff between stability and accuracy. In particular it is shown that the estimation error of the UKBF remains bounded if the initial estimation-error and the disturbing-noise terms are sufficiently small. To carry out the proof, super-martingales are employed. This is a common approach in the stability theory of stochastic differential equations.

## II. The Continuous-Time Unscented Kalman Filter

The considered nonlinear continuous-time system is represented by

$$\begin{aligned} \mathrm{d}x(t) &= f(x(t),t)\mathrm{d}t + L(t)\mathrm{d}w(t) \\ \mathrm{d}y(t) &= h(x(t),t)\mathrm{d}t + V(t)\mathrm{d}v(t) \end{aligned} \tag{1}$$

where $x(t)\in R^n$ is the state process; $y(t)\in R^m$ is the integrated measurement process; $f$ is the drift function; $h$ is the measurement model function; $L(t)$ and $V(t)$ are arbitrary time varying matrices, being independent of $x(t)$ and $y(t)$; $w(t)$ and $v(t)$ are independent Brownian motions with diagonal diffusion matrices $Q_c(t)$ and $R_c(t)$, respectively. The dynamic and measurement models can be equivalently interpreted as Ito or Stratonovich type stochastic differential equations [22].

The filtering model can also be formulated in terms of formal white noises $e_w(t)=\mathrm{d}w(t)/\mathrm{d}t$, $e_v(t)=\mathrm{d}v(t)/\mathrm{d}t$, and differential measurement $z(t)=\mathrm{d}y(t)/\mathrm{d}t$ as follows:

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$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = f(x(t),t) + L(t)e_w(t)$$
$$z(t) = h(x(t),t) + V(t)e_v(t) \tag{2}$$

where the white noise processes $e_w(t)$ and $e_v(t)$ have spectral densities $Q_c(t)$ and $R_c(t)$, respectively.

The procedure for implementing the UKBF for nonlinear stochastic continuous-time systems can be summarized as follows [9].

The $n$-dimensional random variable $x(t)$ with mean $\hat{x}(t)$ and covariance $P(t)$ is approximated by the matrix of sigma points $X(t)$ selected with the following equations

$$\begin{aligned} X^{(0)}(t) &= \hat{x}(t) \\ X^{(i)}(t) &= \hat{x}(t) + \sqrt{cP}\ ,\ i = 1,\ldots,L \\ X^{(i)}(t) &= \hat{x}(t) - \sqrt{cP}\ ,\ i = L+1,\ldots,2L \end{aligned} \tag{3}$$

where $c = \alpha^2(n+l)$ is a tuning parameter. The opposite weight $\omega_m$ is as follows

$$\omega_m = [W_m^{(0)} \ \cdots \ W_m^{(2n)}]^{\mathrm{T}} \tag{4}$$

where

$$W_m^{(0)} = \frac{\lambda}{(n+\lambda)}\ ,\ W_m^{(i)} = \frac{1}{2(n+\lambda)}\ ,\ i = 1,\cdots,2n\ .$$

We define the matrix $W$ as follows

$$W = (I - [\omega_m \cdots \omega_m]) \times diag(W_c^{(0)} \cdots W_c^{(2n)}) \times (I - [\omega_m \cdots \omega_m])^{\mathrm{T}} \tag{5}$$

where

$$W_c^{(0)} = \frac{\lambda}{(n+\lambda) + (1-\alpha^2+\beta)}\ ,\ W_c^{(i)} = \frac{1}{2(n+\lambda)}\ (\ i = 1,\cdots,2n\ ).$$

The parameter $\lambda$ is a scaling parameter defined as $\lambda = \alpha^2(n+l) - n$ . The positive constants $\alpha$ , $\beta$ and $l$ are used as parameters of the method.

Corresponding to the stochastic differential equations of the UKF in the continuous-time, the UKBF can be derived as

$$\frac{\mathrm{d}\hat{x}(t)}{\mathrm{d}t} = f(X(t),t)\omega_m + K(t)[z(t) - h(X(t),t)\omega_m] \tag{6}$$

$$K(t) = X(t)Wh^{\mathrm{T}}(X(t),t)[V(t)R_c(t)V^{\mathrm{T}}(t)]^{-1} \tag{7}$$

$$\begin{aligned} \frac{\mathrm{d}P(t)}{\mathrm{d}t} &= X(t)Wf^{\mathrm{T}}(X(t),t) + f(X(t),t)WX^{\mathrm{T}}(t) + L(t)Q_c(t)L^{\mathrm{T}}(t) \\ &\quad - K(t)V(t)R_c(t)V^{\mathrm{T}}(t)K^{\mathrm{T}}(t) \end{aligned} \tag{8}$$

And the predicted covariance $P^-(t)$ , the measurement covariance $S(t)$ and the cross-covariance of the state and measurement $C(t)$ can be written in matrix form as follows.

$$\frac{\mathrm{d}P^-(t)}{\mathrm{d}t} = X(t)Wf^{\mathrm{T}}(X(t),t) + f(X(t),t)WX^{\mathrm{T}}(t) + L(t)Q_c(t)L^{\mathrm{T}}(t) \tag{9}$$

$$\frac{\mathrm{d}S(t)}{\mathrm{d}t} = h(X(t),t)Wh^{\mathrm{T}}(X(t),t) + V(t)R_c(t)V^{\mathrm{T}}(t) \tag{10}$$

$$\frac{\mathrm{d}C(t)}{\mathrm{d}t} = X(t)Wh^{\mathrm{T}}(X(t),t) + V(t)R_c(t)V^{\mathrm{T}}(t) \tag{11}$$

If we assume that the posterior mean and covariance of $x(t)$ are $\hat{x}(t)$ and $P(t)$ , respectively, the unscented transform-based approximations to the expectations and covariances can be formed as follows:

$$\mathrm{E}[f(x,t)] \approx f(X(t),t)\omega_m\ ,\ \mathrm{E}[h(x,t)] \approx h(X(t),t)\omega_m$$

$$\mathrm{Cov}[x, f(x,t)] \approx X(t)Wf^{\mathrm{T}}(X(t),t)\ ,\ \mathrm{Cov}[f(x,t),x] \approx f(X(t),t)WX^{\mathrm{T}}(t)$$

$$\mathrm{Cov}[x, h(x,t)] \approx X(t)Wh^{\mathrm{T}}(X(t),t)\ ,\ \mathrm{Cov}[h(x,t),x] \approx h(X(t),t)WX^{\mathrm{T}}(t)$$

The computational complexity of the UKBF can be seen to be two to three times the computational complexity of the EKBF, when compared in terms of number of multiplications and additions. When the state dimension is $n$, the UKBF needs $2n+1$ evaluations of $f$ and $h$, while the EKBF needs only one. However, in addition to that, the EKBF needs evaluations of the Jacobian matrices of both the functions.

## III. Stability Analysis of the UKBF for Nonlinear Stochastic Continuous-time System

### *A. Instrumental Diagonal Matrix and Extra Positive Definite Matrix*

In this section, a simple approach to present error of the UKBF is given. First the nonlinear functions $f$ and $h$ are expanded up to first order via

$$f(x(t),t) - f(\hat{x}(t),t) = F(t)[x(t) - \hat{x}(t)] + \phi(x(t),\hat{x}(t),t) \tag{12}$$

and

$$h(x(t),t) - h(\hat{x}(t),t) = H(t)[x(t) - \hat{x}(t)] + \chi(x(t),\hat{x}(t),t) \tag{13}$$

where

$$F(t) = \frac{\partial f}{\partial x}(\hat{x}(t),t)\ ,\ H(t) = \frac{\partial h}{\partial x}(\hat{x}(t),t) \tag{14}$$

are matrix-valued stochastic processes, and $\phi(x(t),\hat{x}(t),t)$ and $\chi(x(t),\hat{x}(t),t)$ are the remaining nonlinear terms. The estimation error is defined by

$$\zeta(t) = x(t) - \hat{x}(t) \tag{15}$$

Subtracting (6) from (2) and using (12) and (13) leads to the error evolution

$$\begin{aligned} \frac{\mathrm{d}\zeta(t)}{\mathrm{d}t} &= f(x(t),t) + L(t)\mathrm{d}w(t) - f(X(t),t)\omega_m - K(t)[z(t) - h(X(t),t)\omega_m] \\ &= [f(x(t),t) - f(\hat{x}(t),t)] - K(t)[h(x(t),t) - h(\hat{x}(t),t)] \\ &\quad + L(t)e_w(t) - K(t)V(t)e_v(t) \\ &= (F(t) - K(t)H(t))\zeta(t) + \theta(t) + \Gamma(t)\rho(t) \end{aligned} \tag{16}$$

with

$$\theta(t) = \phi(x(t),\hat{x}(t),t) - K(t)\chi(x(t),\hat{x}(t),t) \tag{17}$$

$$\Gamma(t) = L(t) - K(t)V(t) \tag{18}$$

$$\rho(t) = [e_w(t) \quad e_v(t)] \tag{19}$$

From (6)-(11) and (14), the relationship between the UKBF and EKBF approximations [9] can be seen to be

$$\begin{aligned} K(t) &= X(t)Wh^{\mathrm{T}}(X(t),t)[V(t)R_c(t)V^{\mathrm{T}}(t)]^{-1} \\ &\Leftrightarrow P(t)H(\hat{x}(t),t)[V(t)R_c(t)V^{\mathrm{T}}(t)]^{-1} \end{aligned} \tag{20}$$

In order to take these residuals between the two expression into account and obtain an exact equality, an unknown instrumental diagonal matrix $\Xi(t) = diag(\xi_1,\xi_2,\cdots,\xi_n)$ is introduced, so that

$$K(t) \doteq \hat{K}(t) = \Xi(t)P(t)H(\hat{x}(t),t)[V(t)R_c(t)V^{\mathrm{T}}(t)]^{-1} \tag{21}$$

In the modified form of the algorithm, the predicted covariance matrix $\mathrm{d}P^-(t)/\mathrm{d}t$ is calculated with enlarged $Q_c(t)$ ,

$$\begin{aligned} \frac{\mathrm{d}\hat{P}^-(t)}{\mathrm{d}t} &= X(t)Wf^{\mathrm{T}}(X(t),t) + f(X(t),t)WX^{\mathrm{T}}(t) \\ &\quad + L(t)Q_c(t)L^{\mathrm{T}}(t) + \Delta Q_c(t) \end{aligned} \tag{22}$$

and

$$\begin{aligned} \frac{\mathrm{d}\hat{P}(t)}{\mathrm{d}t} &= X(t)Wf^{\mathrm{T}}(X(t),t) + f(X(t),t)WX^{\mathrm{T}}(t) + L(t)Q_c(t)L^{\mathrm{T}}(t) \\ &\quad + \Delta Q_c(t) - K(t)V(t)R_c(t)V^{\mathrm{T}}(t)K^{\mathrm{T}}(t) \\ &= X(t)Wf^{\mathrm{T}}(X(t),t) + f(X(t),t)WX^{\mathrm{T}}(t) + \hat{Q}_c(t) \\ &\quad - K(t)V(t)R_c(t)V^{\mathrm{T}}(t)K^{\mathrm{T}}(t) \end{aligned} \tag{23}$$

where

$$\hat{Q}_c(t) = L(t)Q_c(t)L^{\mathrm{T}}(t) + \Delta Q_c(t)$$

and $\Delta Q_c(t)$ is an extra positive definite matrix introduced in the calculated covariance matrix as a slight modification of the UKBF, so that the stability will be improved and the difference between $\hat{K}(t)V(t)R_c(t)V^{\mathrm{T}}(t)\hat{K}^{\mathrm{T}}(t)$ and $K(t)V(t)R_c(t)V^{\mathrm{T}}(t)K^{\mathrm{T}}(t)$ in the covariance matrix $\mathrm{d}P(t)/\mathrm{d}t$ will be compensated. Then the Kalman gain matrix shown in (21) becomes

$$K(t) \doteq \hat{K}(t) = \Xi(t)\hat{P}(t)H(\hat{x}(t),t)[V(t)R_c(t)V^{\mathrm{T}}(t)]^{-1} \quad (24)$$

### B. Stochastic Boundedness of Estimation Error

To examine the dynamics of the estimation error the following concepts of boundedness for solutions of stochastic differential equations are used in this paper:

**Definition 3.1:** The stochastic process $\zeta(t)$ is said to be stochastically sample path bounded, if for every $a>0$ there is a $\varphi(a)>0$ such that

$$P\{\sup_{t\geq 0}\|\zeta(t)\|\leq\varphi(a)\}\geq 1-a_j \quad (25)$$

**Definition 3.2:** The stochastic process $\zeta(t)$ is said to be exponentially bounded in mean square, if there are real numbers $\eta$, $\vartheta$, $\upsilon>0$ such that

$$\mathrm{E}\{\|\zeta(t)\|^2\}\leq\eta\|\zeta(0)\|^2\exp(-\vartheta t)+\upsilon \quad (26)$$

holds for every $t\geq 0$.

**Lemma 3.1:** Assume there is a stochastic process $V(\zeta(t),t)$ and real numbers $\upsilon_{\min}$, $\upsilon_{\max}$, $\gamma$, $\mu>0$ such that

$$\upsilon_{\min}\|\zeta(t)\|^2\leq V(\zeta(t),t)\leq\upsilon_{\max}\|\zeta(t)\|^2 \quad (27)$$

and

$$\mathfrak{L}V(\zeta(t),t)\leq-\gamma V(\zeta(t),t)+\mu \quad (28)$$

are fulfilled. Then the stochastic process $\zeta(t)$ is exponentially bounded in mean square,

$$\mathrm{E}\{\|\zeta(t)\|^2\}\leq\frac{\upsilon_{\max}}{\upsilon_{\min}}\|\zeta(0)\|^2\exp(-\gamma t)+\frac{\mu}{\upsilon_{\min}\gamma} \quad (29)$$

for every $t\geq 0$. Moreover the stochastic process $\zeta(t)$ is sample-path bounded.

With these lemmas and formulations shown in (12), (13) and (16), it is able to state a main result of this paper.

**Theorem 3.1:** Consider a nonlinear stochastic system given by (2) and UKBF for nonlinear stochastic continuous-time systems as stated by (3)-(8), (23) and (24). Let the following assumptions hold:

(1) There are real numbers $h_{\max}$, $\xi_{\max}$, $\hat{p}_{\min}$, $\hat{p}_{\max}$, $\hat{q}_{\min}$, $r_{\min}>0$ such that the following bounds are satisfied for every $t\leq 0$:

$$\|H(t)\|\leq h_{\max} \quad (30)$$

$$\|\Xi(t)\|\leq\xi_{\max} \quad (31)$$

$$\hat{p}_{\min}I\leq\hat{P}(t)\leq\hat{p}_{\max}I \quad (32)$$

$$\hat{q}_{\min}I\leq\hat{Q}_c(t) \quad (33)$$

$$r_{\min}I\leq R_c(t) \quad (34)$$

(2) There are real numbers $\varepsilon_\phi$, $\varepsilon_\chi$, $\kappa_\phi$, $\kappa_\chi>0$ such that the nonlinear functions $\phi$, $\chi$ in (17) are bounded by

$$\|\phi(x(t),\hat{x}(t),t)\|\leq\kappa_\phi\|x(t)-\hat{x}(t)\|^2 \quad (35)$$

$$\|\chi(x(t),\hat{x}(t),t)\|\leq\kappa_\chi\|x(t)-\hat{x}(t)\|^2 \quad (36)$$

for $x(t)$, $\hat{x}(t)\in R^n$ with $|x(t)-\hat{x}(t)|\leq\varepsilon_\phi$ and $|x(t)-\hat{x}(t)|\leq\varepsilon_\chi$, respectively.

Then there exist real numbers $\delta$, $\varepsilon>0$ such that the estimation error $\zeta(t)$ given by (16) is exponentially bounded in mean square and stochastically sample-path bounded, if the initial estimation error satisfies

$$\|\zeta(0)\|\leq\sigma \quad (37)$$

and the covariance matrices of the noise terms are bounded via

$$L(t)L^{\mathrm{T}}(t)\leq\delta I \quad (38)$$

$$V(t)V^{\mathrm{T}}(t)\leq\delta I \quad (39)$$

for every $t\geq 0$.

**Proof:** Choose

$$V(\zeta(t),t)=\zeta^{\mathrm{T}}(t)\hat{P}^{-1}(t)\zeta(t) \quad (40)$$

$\hat{P}(t)$ is defined with positive definite and probability 1 since (32) holds. From (32) it follows that

$$\frac{1}{\hat{p}_{\max}}I\leq\hat{P}^{-1}(t)\leq\frac{1}{\hat{p}_{\min}}I \quad (41)$$

and with (40) $V(\zeta(t),t)$ is bounded by

$$\frac{1}{\hat{p}_{\max}}\|\zeta(t)\|^2\leq V(\zeta(t),t)\leq\frac{1}{\hat{p}_{\min}}\|\zeta(t)\|^2 \quad (42)$$

Consider the following stochastic system

$$d\zeta(t)=\tilde{f}(\zeta(t),t)dt+\tilde{L}(t)\tilde{e}_w(t) \quad (43)$$

For a given stochastic process $V(\zeta(t),t)$ the differential generator can be defined by

$$\mathfrak{L}V(\zeta,t)=\frac{\partial V}{\partial t}(\zeta,t)+\frac{\partial V}{\partial\zeta}(\zeta,t)\tilde{f}(\zeta,t)+\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\partial^2 V}{\partial\zeta_i\partial\zeta_j}(\zeta,t)[\tilde{L}(t)\tilde{L}^{\mathrm{T}}(t)]_{i,j} \quad (44)$$

with $\zeta=[\zeta_1,\cdots,\zeta_n]^{\mathrm{T}}$, and $[\tilde{L}(t)\tilde{L}^{\mathrm{T}}(t)]_{i,j}$ denoting the matrix element of $\tilde{L}(t)\tilde{L}^{\mathrm{T}}(t)$ in the *i*th row and the *j*th column. Comparing (16) and (43), and using (44) yields

$$\begin{aligned}\mathfrak{L}V(\zeta,t)=&\frac{\partial V}{\partial t}(\zeta,t)+\frac{\partial V}{\partial\zeta}(\zeta,t)[(F(t)-K(t)H(t))\zeta(t)+\theta(t)]\\&+\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\partial^2 V}{\partial\zeta_i\partial\zeta_j}(\zeta,t)[\Gamma(t)\Gamma^{\mathrm{T}}(t)]_{i,j}\end{aligned} \quad (45)$$

with $[\Gamma(t)\Gamma^{\mathrm{T}}(t)]_{i,j}$ denoting the matrix element of $\Gamma(t)\Gamma^{\mathrm{T}}(t)$ in the *i*th row and the *j*th coltunn.

Furthermore, the sum in (45) can be written as

$$\begin{aligned}\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\partial^2 V}{\partial\zeta_i\partial\zeta_j}(\zeta,t)[\Gamma(t)\Gamma^{\mathrm{T}}(t)]_{i,j}&=\mathrm{tr}(\Gamma(t)\Gamma^{\mathrm{T}}(t)\mathrm{Hess}[V(\zeta,t)])\\&=\mathrm{tr}(\mathrm{Hess}[V(\zeta,t)]\Gamma(t)\Gamma^{\mathrm{T}}(t))\end{aligned} \quad (46)$$

where $\mathrm{Hess}[\bullet]$ denotes the Hessian matrix.

Using (24), we yields

$$\begin{aligned}&\|2(x-\hat{x})^T P^{-1}\theta\|\\&=\|2(x-\hat{x})^T\hat{P}^{-1}\phi(x,\hat{x},t)-\|2(x-\hat{x})^T\hat{P}^{-1}\hat{K}\chi(x,\hat{x})\|\\&\leq\|2(x-\hat{x})^T\hat{P}^{-1}\phi(x,\hat{x},t)\|+\|2(x-\hat{x})^T\Xi H^{\mathrm{T}}[V(t)R_c(t)V^{\mathrm{T}}(t)]^{-1}\chi(x,\hat{x},t)\|\end{aligned}$$

Set $\varepsilon'=\min(\varepsilon_\phi,\varepsilon_\chi)$. Using (30)-(36) we obtain

$$\begin{aligned}&\|2(x-\hat{x})^T\hat{P}^{-1}\theta\|\\&\leq 2\|x-\hat{x}\|\frac{\kappa_\phi}{\hat{p}_{\min}}\|x-\hat{x}\|^2+2\|x-\hat{x}\|\frac{\xi_{\max}h_{\max}\kappa_\chi}{r_{\min}}\|x-\hat{x}\|^2\end{aligned}$$

for $|x(t)-\hat{x}(t)|\leq\varepsilon'$. Defining

$$\kappa_{nonl}=\frac{\kappa_\phi}{\hat{p}_{\min}}+\frac{\xi_{\max}h_{\max}\kappa_\chi}{r_{\min}} \quad (47)$$

and using (17), we can obtain that there is a positive real number $\kappa_{nonl}>0$ such that

$$2(x(t)-\hat{x}(t))^T\hat{P}^{-1}(t)\theta(t)\leq\kappa_{nonl}\|x(t)-\hat{x}(t)\|^3 \quad (48)$$

holds for $|x(t)-\hat{x}(t)|\leq\varepsilon'$ with $\varepsilon'=\min(\varepsilon_\phi,\varepsilon_\chi)$.

From (40), (45), (46) and (47), it can be obtained

$$\begin{aligned}\mathfrak{L}V(\zeta,t)=&\zeta^{\mathrm{T}}(t)\frac{\partial\hat{P}^{-1}(t)}{\partial t}\zeta(t)+\zeta^{\mathrm{T}}(t)(F(t)-K(t)H(t))^{\mathrm{T}}\hat{P}^{-1}(t)\zeta(t)\\&+\zeta^{\mathrm{T}}(t)\hat{P}^{-1}(t)(F(t)-K(t)H(t))^{\mathrm{T}}\zeta(t)+2\zeta^{\mathrm{T}}(t)\hat{P}^{-1}(t)\theta(t)\\&+tr[(L(t)L^{\mathrm{T}}(t)+K(t)D(t)D^{\mathrm{T}}(t))\hat{P}^{-1}(t)]\end{aligned} \quad (49)$$

With (24), (30)-(32) and (34) one can obtain

$$\begin{aligned}\|K(t)\|&=\|\Xi(t)\|\|\hat{P}(t)\|\|H(\hat{x}(t),t)\|\|[V(t)R_c(t)V^{\mathrm{T}}(t)]^{-1}\|\\&\leq\frac{\xi_{\max}\hat{p}_{\max}h_{\max}}{r_{\min}}=\hat{k}_{\max}\end{aligned} \quad (50)$$

Moreover, from (38) and (39) one obtains

$$\mathrm{tr}[L(t)L^{\mathrm{T}}(t)]\leq\delta\,\mathrm{tr}[I]\leq d_l\delta \quad (51)$$

$$\mathrm{tr}[V(t)V^{\mathrm{T}}(t)]\leq\delta\,\mathrm{tr}[I]\leq d_v\delta \quad (52)$$

where $d_l$ and $d_v$ are the number of the rows of $L(t)$ and $V(t)$, respectively. Furthermore, using (32) and (50)-(52) it follows that

$$tr[L(t)L^{\mathrm{T}}(t)+K(t)D(t)D^{\mathrm{T}}(t)]\hat{P}^{-1}(t) \le \frac{1}{\hat{p}_{\min}}\mathrm{tr}[L(t)L^{\mathrm{T}}(t)]+\frac{\hat{k}_{\max}^2}{\hat{p}_{\min}}\mathrm{tr}[V(t)V^{\mathrm{T}}(t)] \le \kappa_{noise}\delta \quad (53)$$

with

$$\kappa_{noise}=\frac{d_l}{\hat{p}_{\min}}+\frac{\hat{k}_{\max}^2 d_v}{\hat{p}_{\min}} \quad (54)$$

Using (24) and (48) leads to

$$\mathfrak{L}V(\zeta,t)\le \zeta^{\mathrm{T}}(t)[\frac{\partial \hat{P}^{-1}(t)}{\partial t}+F^{\mathrm{T}}(t)\hat{P}^{-1}(t)+\hat{P}^{-1}(t)F(t) -2H^{\mathrm{T}}(t)R^{-1}(t)H(t)]\zeta(t)+\kappa_{nonl}\|\zeta(t)\|^3+\kappa_{noise}\delta \quad (55)$$

for $\|\zeta(t)\|\le\varepsilon'$ with $\varepsilon'=\min(\varepsilon_\phi,\varepsilon_\chi)$ . Calculate $\mathrm{d}\hat{P}^{-1}(t)$ by the formula

$$\mathrm{d}\hat{P}^{-1}(t)=-\hat{P}^{-1}(t)\mathrm{d}\hat{P}(t)\hat{P}^{-1}(t) \quad (56)$$

where (56) is given by the Riccati differential equation (25), and insert (56) into (55). This leads to

$$\mathfrak{L}V(\zeta,t)\le -\zeta^{\mathrm{T}}(t)[\hat{P}^{-1}(t)\hat{Q}_c(t)\hat{P}^{-1}(t)+H^{\mathrm{T}}(t)R^{-1}(t)H(t)]\zeta(t) +\kappa_{nonl}\|\zeta(t)\|^3+\kappa_{noise}\delta \quad (57)$$

and, with (32) and (33) and $H^{\mathrm{T}}(t)R^{-1}(t)H(t)\ge 0$ , we obtain

$$\mathfrak{L}V(\zeta,t)\le -(\frac{\hat{q}_{\min}}{\hat{p}_{\max}^2}-\kappa_{nonl}\|\zeta(t)\|)\|\zeta(t)\|^2+\kappa_{noise}\delta \quad (58)$$

Defining

$$\varepsilon=\min(\varepsilon',\frac{\hat{q}_{\min}}{2\kappa_{nonl}\hat{p}_{\max}^2}) \quad (59)$$

and using (42) one obtains

$$\mathfrak{L}V(\zeta(t),t)\le -\frac{\hat{q}_{\min}\hat{p}_{\min}}{2\hat{p}_{\max}^2}V(\zeta(t),t)+\kappa_{noise}\delta \quad (60)$$

for $\|\zeta(t)\|\le\varepsilon$ . With (42) and (60), the requirements can be satisfied applying Lemma 3.1, where $\|\zeta(0)\|\le\varepsilon$ , $\delta\le\mu/\kappa_{noise}$ , $\upsilon_{\min}=1/\hat{p}_{\max}$ , $\upsilon_{\max}=1/\hat{p}_{\min}$ , and establish mean- square exponential boundedness as well as stochastic sample- path boundedness of the estimation error under the conditions of (37)-(38).

For later use it is necessary to establish some estimates for $\varepsilon$ and $\delta$ . From (59) we obtain immediately

$$\varepsilon=\min(\varepsilon_\phi,\varepsilon_\chi,\frac{\hat{q}_{\min}}{2\kappa_{nonl}\hat{p}_{\max}^2}) \quad (61)$$

with $\kappa_{nonl}$ given by (47). For the evaluation of $\delta$ , it is necessary to take care that, for $\tilde{\varepsilon}\le\|\zeta(t)\|\le\varepsilon$ with some $\tilde{\varepsilon}$ the inequality (60)

$$\mathfrak{L}V(\zeta(t),t)\le -\frac{\hat{q}_{\min}\hat{p}_{\min}}{2\hat{p}_{\max}^2}V(\zeta(t),t)+\kappa_{noise}\delta\le 0 \quad (62)$$

is fulfilled to guarantee the boundedness of the estimation error. Choosing

$$\delta=\frac{\hat{q}_{\min}\hat{p}_{\max}\varepsilon_{\max}^2}{2\hat{p}_{\max}^3\kappa_{noise}} \quad (63)$$

with some $\tilde{\varepsilon}<\varepsilon$ one has, for $|\zeta(t)|\ge\varepsilon$ ,

$$\kappa_{noise}\delta\le\frac{\hat{q}_{\min}\hat{p}_{\max}}{2\hat{p}_{\max}^3}\|\zeta(t)\|^2\le\frac{\hat{q}_{\min}\hat{p}_{\max}}{2\hat{p}_{\max}^2}V[\zeta(t),t] \quad (64)$$

as (62) holds. □

**Remark 3.1:** $\Xi_k$ is unknown instrumental diagonal matrices introduced to evaluate the difference between $\hat{K}(t)V(t)R_c(t)V^{\mathrm{T}}(t)\hat{K}^{\mathrm{T}}(t)$ and $K(t)V(t)R_c(t)V^{\mathrm{T}}(t)K^{\mathrm{T}}(t)$ . From (50), (54) and (60) it is shown that the stability of the algorithm depends on the magnitude of $\Xi_k$ because different $\Xi_k$ may change the value of $\kappa_{noise}$ through $\hat{k}_{\max}$ .

**Remark 3.2:** To ensure the stability of UKBF, the matrices $\hat{Q}_c(t)$ need to be positive definition. From (23), as $\mathrm{d}\hat{P}(t)/\mathrm{d}t$ may be not positive definite matrices, extra additive matrix $\Delta Q_c(t)$ should be introduced as a modification to the UKBF so that $\hat{Q}_c(t)\ge\hat{q}_{\min}I$ will be satisfied. Obviously, if $\Delta Q_c(t)$ is sufficiently large, condition (33) can always be fulfilled.

**Remark 3.3:** To obtain $\kappa_\phi$ and $\kappa_\chi$ in (35) and (36), a compact subset $\mathrm{K}$ of $R^n$ was considered. The bounds defined by (35) and (36) for *x*, $\hat{x}\in\mathrm{K}$ can be calculated by a standard estimation via an integral formula. Let $f_i$ , $h_i$ be the components of *f* and *h*, respectively. If *f* and *h* are twice differentiable for every $x\in\mathrm{K}$ , it follows that the Hessian matrices of $f_i$ and $h_i$ are bounded with respect to the Euclidian norm of matrices. The constants $\kappa_\phi$ and $\kappa_\chi$ are then given by

$$\kappa_\phi=\max_{1\le i\le m_l}\sup_{x\in\mathrm{K}}\|\mathrm{Hess}f_i(x(t),t)\|\ ,\ \kappa_\chi=\max_{1\le i\le m_v}\sup_{x\in\mathrm{K}}\|\mathrm{Hess}h_i(x(t),t)\|\ .$$

**Remark 3.4:** Regarding the restrictiveness of condition (30) for many applications the state variables are bounded inside reasonable limits. Outside this operating area the form of the function *h* has no influence on the behavior of the system. If the function *h* satisfies $|(\partial h/\partial x)(x)|\le\varepsilon_{\max}$ for every physical reasonable value of the state vector *x*, it can be assumed without loss for generality that $|H(t)|\le h_{\max}$ holds.

**Remark 3.5:** Eqn.(32) in assumption (1) of Theorem 3.1 is closely related to the observability and detectability properties of the system. These relations are discussed in Section 4.

## IV. The Significance of Nonlinear Observability of the UKBF for Continuous-time Systems

Proving the boundedness of the estimation error bounds is required for the solution $\hat{P}(t)$ of the Riccati differential equation (23) according to (6). This condition is closely related to observability and detectability properties of the system to be observed. The classical treatment for linear systems with deterministic state matrices has been given in [19], and generalizations to stochastic state matrices have been proposed. Firstly, the following definition for nonlinear stochastic systems are introduced for the main result in this section.

**Definition 4.1:** The pair

$$\left[\frac{\partial f}{\partial x}(x,t),\frac{\partial h}{\partial x}(x)\right],\ x\in R^n \quad (65)$$

is called uniformly detectable if there is a bounded matrix valued function $\Lambda(x)$ and a real number $\gamma>0$ such that

$$\varpi^{\mathrm{T}}\left[\frac{\partial f}{\partial x}(x,t)+\Lambda(x)\frac{\partial h}{\partial x}(x)\right]\varpi\le-\gamma\|\varpi\|^2 \quad (66)$$

holds for every $\varpi$ , $x\in R^n$ .

**Lemma 4.1:** Assume that the pair

$$\left[\frac{\partial f}{\partial x}(x,t),\frac{\partial h}{\partial x}(x)\right],\ x\in R^n \quad (67)$$

is uniformly detectable according to Definition 4.1. Then the solution $\hat{P}(t)$ of the Riccati differential equation (24) satisfies the bound

$$p_{\min}I\le\hat{P}(t)\le\hat{p}_{\max}I \quad (68)$$

Now it is possible to state the following result.

**Theorem 4.1:** Consider a nonlinear stochastic system with state differential equations (1) or (2) and UKBF for nonlinear stochastic continuous-time system as stated by (3)-(8), (23) and (24). Let the following assumptions hold:

(1) There are positive real numbers $h_{\max}$, $\hat{q}_{\min}$, $r_{\min} > 0$ such that the following bounds are satisfied for every $t \geq 0$:

$$\| H(t) \| \leq h_{\max} \tag{69}$$

$$\hat{q}_{\min} I \leq \hat{Q}(t) \tag{70}$$

$$r_{\min} I \leq R(t) \tag{71}$$

(2) The pair

$$\left[\frac{\partial f}{\partial x}(x,t), \frac{\partial h}{\partial x}(x)\right],\ x \in R^n \tag{72}$$

is uniformly detectable according to Definition 4.1.

(3) There are real numbers $\varepsilon_\phi$, $\varepsilon_\chi$, $\kappa_\phi$, $\kappa_\chi > 0$ such that the nonlinear functions $\phi$, $\chi$ in (17) are bounded by

$$\| \phi(x(t), \hat{x}(t), t) \| \leq \kappa_\phi \| x(t) - \hat{x}(t) \|^2 \tag{73}$$

$$\| \chi(x(t), \hat{x}(t), t) \| \leq \kappa_\chi \| x(t) - \hat{x}(t) \|^2 \tag{74}$$

for $x(t)$, $\hat{x}(t) \in R^n$ with $| x(t) - \hat{x}(t) | \leq \varepsilon_\phi$ and $| x(t) - \hat{x}(t) | \leq \varepsilon_\chi$, respectively.

Then there exist real numbers $\delta$, $\varepsilon > 0$ such that the estimation error $\zeta(t)$ given by (15) is exponentially bounded in mean square and stochastically sample-path bounded, if the initial estimation error satisfies

$$\| \zeta(0) \| \leq \sigma \tag{75}$$

and the covariance matrices of the noise terms are bounded via

$$L(t) L^{\mathrm{T}}(t) \leq \delta I \tag{76}$$

$$V(t) V^{\mathrm{T}}(t) \leq \delta I \tag{77}$$

for every $t \geq 0$.

**Proof:** According to Lemma 4.1 the solution $\hat{P}(t)$ of the Riccati differential equation satisfies the bounds of (68). Therefore all requirements are satisfied to apply Theorem 3.1, and the boundedness of the estimation error can be proved under the stated conditions.

## V. Numerical Simulations

In the preceding two sections, it has been shown that, under certain conditions the estimation error for the UKBF for nonlinear stochastic continuous-time system remains bounded. To obtain the error bounds one requires especially a sufficiently small initial estimation error and sufficiently small noise. In this section the numerical simulations are presented, which indicate that the estimation error is bounded for small initial estimation errors and small noise, while divergent for large initial estimation errors or large noise. For this purpose, consider a nonlinear stochastic example system with the state differential equations (1) or (2), where

$$f(x(t),t) = \begin{bmatrix} x_2(t) \\ -x_1(t) + (x_1^2(t) + x_2^2(t) - 1) x_2(t) \end{bmatrix} \tag{77}$$

$$h(x(t),t) = \exp(c - x_2(t)) \tag{78}$$

From (79) and (80) we obtain

$$\frac{\partial f}{\partial x}(x,t) = \begin{bmatrix} 0 & 1 \\ -1 + 2x_1x_2 & x_1^2 + 3x_2^2 - 1 \end{bmatrix},\ \frac{\partial h}{\partial x}(x,t) = [0 \quad -\exp(-x_2)]$$

It can be checked that the matrices fulfill the uniform detectability condition of Definition 4.1 with

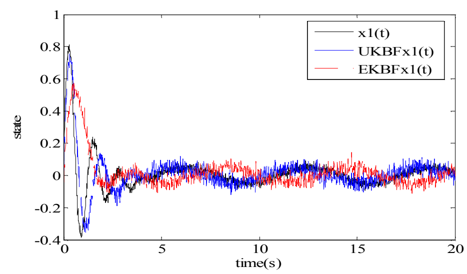
$$\Lambda(x) = \begin{bmatrix} 2x_1x_2 \exp(-x_2) + 1 \\ \exp(-x_2)(x_1^2 + 3x_2^2 - 1) + 1 \end{bmatrix}$$

Therefore, according to Lemma 4.1 the Riccati differential equation (23) has a bounded solution.

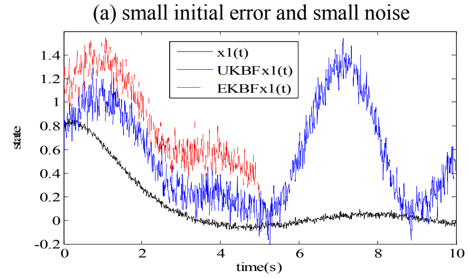
For a numerical solution of the stochastic differential equations (1), (2) and (23), the stochastic version of Heun's method is employed. For the numerical simulations, one case with bounded estimation error and two cases with divergent estimation error were considered. For all three cases, we simply select as $Q(t) = I$, $R(t) = 1$, $P(0) = I$, $x(0) = [0.8 \quad 0.2]^{\mathrm{T}}$.

TABLE I
INITIAL VALUES AND NOISE-WEIGHTING MATRICES FOR THE NUMERICAL SIMULATION

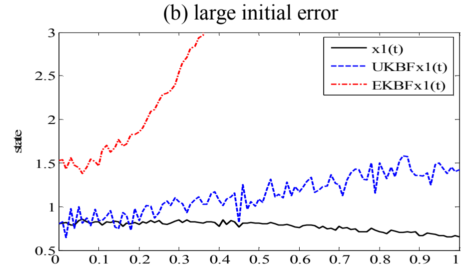
| | Small initial error and small noise | Large noise | Large initial error |
|---|---|---|---|
| $\hat{x}(0)$ | $[0.5 \quad 0.5]^{\mathrm{T}}$ | $[0.5 \quad 0.5]^{\mathrm{T}}$ | $[1.5 \quad 1.5]^{\mathrm{T}}$ |
| $L(t)$ | 0.1 | 0.1 | 0.1 |
| $V(t)$ | 0.1 | $\sqrt{2}$ | 0.1 |
| Error Behavior | Bounded | Divergent | Divergent |




(a) small initial error and small noise




(b) large initial error



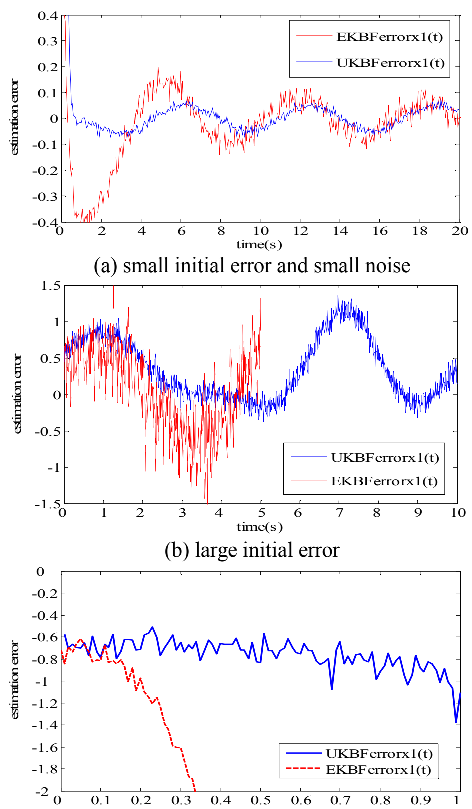

(c) large initial error

Fig.1. The state component $x_1(t)$ and its estimation with the EKBF and the UKBF for the example system.

The stochastic differential equations considered are solved numerically using the Heun discretization with step size $\Delta t = 10^{-3}$. The remaining matrices $L(t)$ and $V(t)$, as well as the initial value $\hat{x}(0)$, are chosen particularly for each of the three cases and are shown in Table I. The following cases were considered: small initial error and small noise, large noise as well as large initial error. The simulation results are depicted in Figs. 1-2, where sample paths for the unknown state $x_1(t)$ and the estimated state $\hat{x}_1(t)$, as well as for the estimation error $\zeta_1(t)$, are plotted against time *t*.

It can be seen in Figs. 1 and 2 that for small initial error and small noise (37)-(39) or (75)-(77), respectively, the estimation error remains bounded. However, if the initial estimation error or the disturbing noise is large, (37)-(39) or (75)-(77) are violated respectively, then the estimation error is no longer bounded, as verified in Figs. 1(a) and 2(a). Because of the high nonlinearities of the example system considered, the error is divergent as verified in Figs. 1 (b)-(c) and 2 (b)-(c).

The numerical simulations have shown that the estimation error is bounded if $\|\zeta(0)\|\leq 0.4$ and $L(t)L^{\mathrm{T}}(t)\leq 0.01I$, $V(t)V^{\mathrm{T}}(t)\leq 0.01I$ is fulfilled. However, estimating $\varepsilon$ and $\delta$ via (61) and (63) yields much smaller values for the bounds. For a considered compact subset $\mathrm{K}\in R^n$ with $\mathrm{K}=\{x\in R^n \mid \|x\|\leq 3\}$ one obtains $\varepsilon\leq 1.4\times 10^{-4}$ and $\delta\leq 5\times 10^{-14}$, respectively.




(a) small initial error and small noise

(b) large initial error

(c) large initial error

Fig.2. The estimation error $\xi_1(t)$ with the EKBF and the UKBF for the example system.

## VI. Conclusion

In this paper the behavior of the estimation error for the derived nonlinear continuous-time UKBF has been examined. It has been shown that, under certain conditions, the estimation error is bounded in mean square and stochastically sample-path bounded. This fact is embodied in the theorems 3.1 and 4.1 in Sections 3 and 4. To obtain the error bounds, a good initial guess is required, along with small noise terms, and the original nonlinear stochastic system must be uniformly detectable, from which we can get a bounded solution for the Riccati differential equation. In Section 5 it has been shown by numerical simulations that, for systems with severe nonlinearities, these assumptions are, although restrictive, often necessary. The numerical simulations verify that the estimation error remains bounded for small initial errors and small noise terms, moreover they indicate that the error is divergent for large initial errors or large noise power.

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