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Stochastic Pareto Near-Optimal Strategy for Weakly-Coupled Large-Scale Systems with Imperfect Local State Measurements

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Abstract— **This paper is concerned with the infinite horizon stochastic Pareto-optimal static output feedback control problem for a class of weakly-coupled large-scale systems with state-dependent noise. Necessary conditions, which are related with the solutions of the cross-coupled stochastic algebraic Riccati equations (CSAREs), are given for the existence of a controller that guarantees exponentially mean square stable (EMSS) of the system and minimizes a cost function. After analyzing the asymptotic structure for the solutions of the CSAREs, we will construct a parameter independent Pareto near-optimal controller. We will also propose a new sequential numerical algorithm for solving the reduced-order CSAREs. A numerical example for a practical megawatt-frequency control problem will be solved to show the effeciency of the proposed algorithm.**

I. INTRODUCTION

When we consider optimal control problems of weakly-coupled large-scale interconnected system that are parameterized by a small coupling parameter ε, the algebraic Riccati equations (AREs) play an important role in the design of the controller. We can find various reliable approaches to solve the AREs in literatures (see e.g., [15]). However, a drawback of these approaches is that the small parameter is required to be exactly known. Therefore, these approaches are not applicable to the problems where the small parameter represents the unknown perturbation to a system.

On the other hand, designing a controller for stochastic systems governed by Itô's differential equation has been the subject of many papers during the past few decades [1], [2], [3], [4]. Although many results obtained in these papers are very elegant theoretically, there exist the issues on how to calculate and implement a controller easily. ¿From the viewpoint of implementation, a output feedback controller is extremely desirable since state variables are not always available in practice. Although some results on output feedback designing can be found in the papers [5], [13], the stochastic static output feedback control problem with multiple decision makers has not been considered.

Decisions in large-scale systems are usually made by multiple decision makers who have different information sets. For example, we can consider an optimal megawatt-frequency control of multi-area electric energy systems [6]. This problem has been treated as the Nash games of weakly coupled large scale-systems with multiple decision makers [7]. The study on the linear quadratic Gaussian games in large-sacle population systems is another example [14]. Since it is not easy or even possible to obatin information of other subsystems in a large-scale system, it is common that a local decision maker can only use local information and simplified models to construct his own strategies. Moreover, we can find problems where only the partial information on the systean be utilized through output measurement.

In this paper, we investigate the static output feedback Pareto optimal control problem of stochastic systems governed by Itô differential equations with state-dependent noise. This study is relevant to [5] where only a regular static output feedback optimal control problem is studied. Our problem involves multiple decision makers who use their local information from output measurement of each subsystem in the design of the controller. We extend the existing results [4] to the decentralized stochastic static output feedback problem with multiple decision makers. Moreover, a new stabilization concept called exponentially mean square stable (EMSS) is used in the design of static output feedback Pareto optimal strategies.

The outline of the study is as follows. Firstly, we present the necessary conditions, which are related with the solutions of the cross-coupled stochastic algebraic Riccati equations (CSAREs), for a decentralized controller to be Pareto near-optimal. The boundedness of the solution to the CSAREs [1] and their asymptotic structures are established. Using the obtained asymptotic structure, we construct a parameter independent approximate Pareto strategy. Moreover, a new sequential numerical algorithm to solve the reduced order CSAREs, which are independent of the parameter ε, is developed for the first time. The degradation analysis of the costs by applying the proposed approximate Pareto strategies is provided. It is proved that the proposed strategy achieves $O(\varepsilon)$ approximation of the optimum value. Finally, a numerical example for a two-area electric energy system is solved to show the efficiency of the proposed algorithm.

Notation: The notations used in this paper are fairly standard. I_n denotes an $n \times n$ identity matrix. $\mathbf{block\ diag}$ denotes a block diagonal matrix. $\|\cdot\|$ denotes the Euclidean norm of a matrix. E denotes the expectation. $\otimes$ denotes the Kronecker product. δ_{ij} denotes the Kronecker delta.

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[1]Particularly, it should be noted that this kind of nonlinear algebraic matrix equation contains a stochastic term $A^T PA$, e.g., $PA + A^T P + A^T PA - PSP + Q = 0$, and is called the stochastic algebraic Riccati equation (SARE) [2], [4].

II. DEFINITION AND PRELIMINARY

We first introduce the concept of the exponentially mean square stable (EMSS) and the related facts. These results are essential (see, e.g., [1], [2] and the references therein for more details).

Definition 1: [1] The stochastic system

$$dx(t) = Ax(t)dt + \sum_{k=1}^{N} A_k x(t) dw_k(t) \tag{1}$$

is said to be EMSS if it satisfies the following equation.

$$E\|x(t)\|^2 \le \rho e^{-\psi(t-t_0)} E\|x(t_0)\|^2, \ \exists \rho, \ \psi > 0. \tag{2}$$

Lemma 1: [1], [2] The trivial solution of a stochastic differential equation as follows:

$$dx(t) = f(t, \ x)dt + g(t, \ x)dw(t), \tag{3}$$

where $f(t, \ x)$ and $g(t, \ x)$ sufficiently differentiable maps, is EMSS if there exists a function $V(x(t))$ which satisfies the following inequalities

$$a_1\|x(t)\|^2 \le V(x(t)) \le a_2\|x(t)\|^2, \ a_1, \ a_2 > 0, \tag{4a}$$

$$\begin{aligned} \mathcal{D}V(x(t)) := & \frac{\partial V(x(t))}{\partial x} f(t, \ x) \\ & + \frac{1}{2}\mathbf{Tr}\left[g^T(t, \ x) \frac{\partial^2 V(x(t))}{\partial x^2} g(t, \ x) \right] \\ & \le -c\|x(t)\|^2, \ c > 0 \end{aligned} \tag{4b}$$

for $x(t) \neq 0$.

Lemma 2: [1] Consider an autonomous stochastic system

$$dx(t) = Ax(t)dt + \sum_{p=1}^{M} A_p x(t) dw_p(t), \ x(0) = x^0 \tag{5}$$

and the corresponding cost function

$$J = E\int_0^\infty x^T(t) Q x(t) dt, \ Q = Q^T \ge 0. \tag{6}$$

For any given positive definite symmetric matrix Q, if there exists a positive definite symmetric matrix X that satisfies the following stochastic algebraic Lyapunov equation (SALE):

$$XA + A^T X + \sum_{p=1}^{M} A_p^T X A_p + Q = 0, \tag{7}$$

then the stochastic system (5) is EMSS. Moreover, $J = E[x^T(0) X x(0)]$.

III. STOCHASTIC PARETO OPTIMAL STATIC OUTPUT FEEDBACK STRATEGY

We now study the static Pareto near-optimal control problem with state dependent noise. Consider linear time-invariant weakly-coupled large-scale stochastic systems.

$$\begin{aligned} dx(t) = & \left[A_\varepsilon x(t) + \sum_{k=1}^{N} B_{k\varepsilon} u_k(t) \right] dt \\ & + \sum_{k=1}^{N} \bar{A}_{k\varepsilon} x(t) dw_k(t), \ x(0) = x^0, \end{aligned} \tag{8a}$$

$$y_i(t) = C_i x(t) = C_{ii} x_i(t), \ i = 1, \ \dots, N, \tag{8b}$$

where

$$x(t) := \begin{bmatrix} x_1^T(t) & \cdots & x_N^T(t) \end{bmatrix}^T,$$

$$A_\varepsilon := \begin{bmatrix} A_{11} & \varepsilon A_{12} & \cdots & \varepsilon A_{1N} \\ \varepsilon A_{21} & A_{22} & \cdots & \varepsilon A_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ \varepsilon A_{N1} & \varepsilon A_{N2} & \cdots & A_{NN} \end{bmatrix},$$

$$\bar{A}_{i\varepsilon} := \begin{bmatrix} \varepsilon^{1-\delta_{i1}} A_{i11} & \varepsilon A_{i12} & \cdots & \varepsilon A_{i1N} \\ \varepsilon A_{i12}^T & \varepsilon^{1-\delta_{i2}} A_{i22} & \cdots & \varepsilon A_{i2N} \\ \vdots & \vdots & \ddots & \vdots \\ \varepsilon A_{i1N}^T & \varepsilon A_{i2N}^T & \cdots & \varepsilon^{1-\delta_{iN}} A_{iNN} \end{bmatrix},$$

$$B_{i\varepsilon} := \begin{bmatrix} \varepsilon^{1-\delta_{1i}} B_{1i}^T & \varepsilon^{1-\delta_{2i}} B_{2i}^T & \cdots & \varepsilon^{1-\delta_{Ni}} B_{Ni}^T \end{bmatrix}^T,$$

$$C_i := \begin{bmatrix} 0 & \cdots & 0 & C_{ii} & 0 & \cdots & 0 \end{bmatrix}.$$

$x_i(t) \in \Re^{n_i}, \ i = 1, \dots, N$ represent the ith state vectors. $u_i(t) \in \Re^{m_i}, \ i = 1, \dots, N$ represent the ith control inputs. $y_i(t) \in \Re^{l_i}, \ i = 1, \dots, N$ represent the ith output measurements vectors. $w_i(t) \in \Re, \ i = i = 1, \dots, N$ is a one-dimensional standard Wiener process defined in the filtered probability space [2], [3], [4]. Here, ε denotes a relatively small coupling parameter that relates the linear system with the other subsystems. The initial state $x(0) = x^0$ is assumed to be a random variable with a covariance matrix $E[x(0)x^T(0)] = I_{\bar{n}}, \ \bar{n} := \sum_{k=1}^{N} n_k$. It should be noted that although $\bar{A}_{i\varepsilon}$ has a special form, it arise in the practical systems [6]. Indeed, it will be demonstrated in the numerical example.

Generally, it is impossible or too costly to incorporate many feedback loops into the controller designing for a large-scale system. These facts motivate the study of decentralized control theory such that each subsystem can be controlled independently by a controller using its locally available information. We now make a realistic assumption that each decision maker can only know the locally simplified model of (8). Moreover, each decision maker can only use the local output feedback information in the design of a controller. In other words, the simplified decomposition system

$$dx_i(t) = [A_{ii} x_i(t) + B_{ii} u_i(t)]dt + A_{iii} x_i(t) dw_i(t), \tag{9a}$$

$$y_i(t) = C_{ii} x_i(t), \ i = 1, \ \dots, N \tag{9b}$$

is only known by the ith decision maker.

The main purpose of this paper is to establish a parameter independent static output feedback strategy and to analyze its reliability. Suppose that the ith decision maker will design a control strategy based on local information and the designing specification of minimizing a cost function J_i. We consider the situation in which decision makers decide their strategies in a cooperative way. This is a Pareto optimal control problem which has the meaning that no variation from Pareto optimal strategy can decrease the costs of all decision makers [10]. It is very important to note that a dynamic multiple decision making problem can be converted to a regular optimal control problem [8].

The cost function for each strategy subset is defined by

$$J_i = E\int_0^{\infty}\left[x^T(t)Q_{i\varepsilon}x(t) + u_i^T(t)R_iu_i(t)\right]dt, \tag{10}$$

where $i = 1, \dots, N$, $Q_{ii} = Q_{ii}^T \geq 0 \in \Re^{n_i \times n_i}$ with

$$\begin{aligned} Q_{i\varepsilon} &= Q_{i\varepsilon}^T \\ &= \begin{bmatrix} \varepsilon^{1-\delta_{i1}}Q_{i1} & \varepsilon Q_{i12} & \cdots & \varepsilon Q_{i1N} \\ \varepsilon Q_{i12}^T & \varepsilon^{1-\delta_{i2}}Q_{i2} & \cdots & \varepsilon Q_{i2N} \\ \vdots & \vdots & \ddots & \vdots \\ \varepsilon Q_{i1N}^T & \varepsilon Q_{i2N}^T & \cdots & \varepsilon^{1-\delta_{iN}}Q_{iN} \end{bmatrix} \\ &\geq 0 \in \Re^{n \times n}, \end{aligned}$$

and $R_i = R_i^T > 0 \in \Re^{m_i \times m_i}$.

A Pareto solution is a set u_i, $i = 1, \dots, N$ which minimizes

$$J = \sum_{k=1}^{N}\gamma_kJ_k,\ 0 < \gamma_k < 1,\ \sum_{k=1}^{N}\gamma_k = 1, \tag{11}$$

for some γ_k, $k = 1, \dots, N$ [10], [11].

The optimal linear quadratic regulator problem is a special case of this problem when the decision makers agree on a choice of γ_k, $k = 1, \dots, N$ as weight factors.

It should be noted that in this study, the strategies $u_i(t) := F_iC_ix(t) = \tilde{u}_i(t) := F_iC_{ii}x_i(t)$ are restricted as the linear feedback strategies [9].

To develop necessary conditions for this problem, F_i, $i = 1, \dots, N$ must be restricted to the following set

$\mathbf{F}_i := \Big\{ F_i \in \Re^{m_i \times l_i} \mid$ There exists a positive definite symmetric matrix X_{ii} that satisfies the following parameter independent SALE:

$$\begin{aligned} &X_{ii}(A_{ii} + B_{ii}F_iC_{ii}) + (A_{ii} + B_{ii}F_iC_{ii})^TX_{ii} \\ &\quad + A_{iii}^TX_{ii}A_{iii} + \gamma_i(C_{ii}^TF_i^TR_iF_iC_{ii} + Q_{ii}) = 0. \end{aligned} \tag{12}$$

Moreover, $I_{n_i} \otimes (A_{ii} + B_{ii}F_iC_{ii})^T + (A_{ii} + B_{ii}F_iC_{ii})^T \otimes I_{n_i} + A_{iii}^T \otimes A_{iii}^T$ is nonsingular. $\Big\}$.

Using Lemma 2 and the assumption of $E[x(0)x^T(0)] = I_{\bar{n}}$, it is immediately obtained that the closed-loop stochastic system is EMSS and the integral portion of J satisfies the relation

$$J = \mathbf{Tr}[P_\varepsilon], \tag{13}$$

if there exists a solution to the following SALE.

$$\begin{aligned} &\mathcal{F}(\varepsilon,\ P_\varepsilon,\ F_1,\ \dots, F_N) \\ &= P_\varepsilon\left(A_\varepsilon + \sum_{k=1}^{N}B_{k\varepsilon}F_kC_k\right) + \left(A_\varepsilon + \sum_{k=1}^{N}B_{k\varepsilon}F_kC_k\right)^TP_\varepsilon \\ &\quad + \sum_{k=1}^{N}\bar{A}_{k\varepsilon}^TP_\varepsilon\bar{A}_{k\varepsilon} + \sum_{k=1}^{N}\gamma_kC_k^TF_k^TR_kF_kC_k + Q_\varepsilon = 0, \end{aligned} \tag{14}$$

where $Q_\varepsilon := \sum_{k=1}^{N}\gamma_kQ_{k\varepsilon}$.

In order to clarify the existence of P_ε of (14), we now investigate the asymptotic structure of the solution and establish the existence condition that is confirmed by the reduced-order and the parameter independent calculation.

Since A_ε, $A_{i\varepsilon}$ and $B_{i\varepsilon}$ contain the parameter ε, the solutions P_ε of CSARE (14) - if it exists - should contain the parameter ε. Therefore, we assume that the solutions of SALE (14) have the following structure [15].

$$P_\varepsilon := \begin{bmatrix} P_{11} & \varepsilon P_{12} & \cdots & \varepsilon P_{1N} \\ \varepsilon P_{12}^T & P_{22} & \cdots & \varepsilon P_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ \varepsilon P_{1N}^T & \varepsilon P_{2N}^T & \cdots & P_{NN} \end{bmatrix} \in \Re^{\bar{n} \times \bar{n}}. \tag{15}$$

Substituting these matrices into SALE (14), letting $\varepsilon = 0$, and partitioning SALE (14), the following reduced-order SALE (14) is obtained, where $\bar{P}_{ii}$ and $\bar{F}_i$, $i = 1, \dots, N$ are the 0-order solutions of SALE (14) as $\varepsilon = 0$.

The asymptotic expansion of CSARE (14) for $\varepsilon = 0$ is described by the following Lemma.

Lemma 3: Suppose that $\bar{F}_i \in \mathbf{F}_i$. There exists a small constant σ_1^* such that for all $\varepsilon \in (0,\ \sigma_1^*)$, SALE (14) admits the unique positive definite solution P_ε^* that can be expressed as

$$P_\varepsilon := P_\varepsilon^* = \bar{P} + O(\varepsilon), \tag{16}$$

where $\bar{P} = \mathbf{block\ diag}\begin{pmatrix} \bar{P}_{11} & \cdots & \bar{P}_{NN} \end{pmatrix}$,

$$\begin{aligned} &\bar{P}_{ii}(A_{ii} + B_{ii}\bar{F}_iC_{ii}) + (A_{ii} + B_{ii}\bar{F}_iC_{ii})^T\bar{P}_{ii} \\ &\quad + A_{iii}^T\bar{P}_{ii}A_{iii} + \gamma_i(C_{ii}^T\bar{F}_i^TR_i\bar{F}_iC_{ii} + Q_{ii}) = 0. \end{aligned} \tag{17}$$

Proof: This can be proved by performing the implicit function theorem on SALE (14). To do so, it is sufficient to show that the corresponding Jacobian is nonsingular at $\varepsilon = 0$. Since this follows the same lines of [15], it is omitted. ■

It follows from Lemma 2 that the closed-loop stochastic system (8a) with $\tilde{u}_i(t) = F_iC_{ii}x_i(t)$ is EMSS because SALE (14) admits the unique positive definite solution. Moreover, it is easy to verify that the behavior of the closed-loop stochastic system (8a) for small value of ε can be stated as the following observation.

Observation 1: If $\tilde{u}_i(t) = F_iC_{ii}x_i(t)$, $i = 1, \dots, N$ are designed subject to $F_i \in \mathbf{F}_i$, then, for all t, there exists a positive scalar δ^* such that for all $\varepsilon \in (0,\ \delta^*)$, the following approximations hold.

$$E[x_i^T(t)x_i(t)] = E[\tilde{x}_i^T(t)\tilde{x}_i(t)] + O(\varepsilon), \tag{18}$$

where $d\tilde{x}_i(t) = [A_{ii} + B_{ii}F_iC_{ii}\tilde{x}_i(t)]dt + A_{iii}\tilde{x}_i(t)dw_i(t)$.

Necessary condition for Pareto optimality will be obtained in term of the CSAREs.

Theorem 1: Suppose that $F_i \in \mathbf{F}_i$ forms the gain of the static output feedback Pareto near-optimal strategies. Then, it is necessary that there exist the symmetric positive definite solutions P_ε and S_ε that satisfy the SALE (14) and the following SALE (19a), respectively, such that F_i is obtained by (19b).

$$\begin{aligned}&\mathcal{G}(\varepsilon,\ S_\varepsilon,\ F_1,\ \ldots,F_N)\\&= S_\varepsilon\left(A_\varepsilon+\sum_{k=1}^N B_{k\varepsilon}F_kC_k\right)^T+\left(A_\varepsilon+\sum_{k=1}^N B_{k\varepsilon}F_kC_k\right)S_\varepsilon\\&\quad+\sum_{k=1}^N \bar{A}_{k\varepsilon}S_\varepsilon\bar{A}_{k\varepsilon}^T+I_{\bar{n}}=0, \qquad (19a)\\&\mathcal{H}_i(\varepsilon,\ P_\varepsilon,\ S_\varepsilon,\ F_1,\ \ldots,F_N)\\&=\gamma_iR_iF_iC_iS_\varepsilon C_i^T+B_{i\varepsilon}^TP_\varepsilon S_\varepsilon C_i^T=0, \qquad (19b)\end{aligned}$$

where $i=1,\ \ldots,N$.

Proof: The result can be proved by using a Lagrange multiplier approach. First, the closed-loop cost with the static output feedback controller $\tilde{u}_i(t)=F_iC_{ii}x_i(t)$ can be obtained by $J=\mathbf{Tr}[P_\varepsilon]$, where P_ε is the solution of the SALE (14). Let us consider the Hamiltonian $\mathcal{L}$

$$\begin{aligned}&\mathcal{L}(\varepsilon,\ P_\varepsilon,\ S_\varepsilon,\ F_1,\ \ldots,F_N)\\&=\mathbf{Tr}\ [P_\varepsilon]+\mathbf{Tr}\ [\mathcal{F}(\varepsilon,\ P_\varepsilon,\ F_1,\ \ldots,F_N)S_\varepsilon], \qquad (20)\end{aligned}$$

where S_ε is a symmetric positive definite matrix of Lagrange multipliers. Necessary conditions for a F_i to be optimal can be found by setting $\dfrac{\partial\mathcal{L}}{\partial P_\varepsilon}$ and $\dfrac{\partial\mathcal{L}}{\partial F_i}$ equal to zero, and solving the resulting equations (19b) simultaneously for F_i. ■

Remark 1: It should be noted that Theorem 1 only gives the necessary conditions for a controller to be optimal. However, it is quite possible that the solutions of (14) and (19) will not lead to a Pareto optimal controller.

Remark 2: It is obvious that there will be many Pareto solutions. Different criteria are required to make the choice of multiple Pareto solutions.

Remark 3: The stochastic static output feedback Pareto optimal problem in this paper cannot be treated using the technique of [5] because the multiple decision makers exist. In fact, the obtained CSAREs (14) and (19) are quite different from the results of [5].

Observation 2: If full state information is available, i.e., $C_i:=I_{n_i}$ and S_ε is nonsingular, then, according to (19b),

$$F_i=-(\gamma_iR_i)^{-1}B_{i\varepsilon}^TP_\varepsilon, \qquad (21)$$

and, with this F_i, it is possible to show that (14) implies

$$P_\varepsilon A_\varepsilon+A_\varepsilon^TP_\varepsilon+\sum_{k=1}^N\bar{A}_{k\varepsilon}^TP_\varepsilon\bar{A}_{k\varepsilon}-P_\varepsilon U_\varepsilon P_\varepsilon+Q_\varepsilon=0, \qquad (22)$$

where $U_\varepsilon:=\sum_{k=1}^N\gamma_k^{-1}B_{k\varepsilon}R_k^{-1}B_{k\varepsilon}^T$.

The discussion on the uniqueness and the stabilizing solution of (22) will be given in a later section.

If $C_iS_\varepsilon C_i^T$ is nonsingular then (19b) may be solved for F_i to obtain

$$F_i=-(\gamma_iR_i)^{-1}B_{i\varepsilon}^TP_\varepsilon S_\varepsilon C_i^T(C_iS_\varepsilon C_i^T)^{-1}. \qquad (23)$$

In the remaining part of the section, in order to propose a new concept of the parameter independent Pareto near-optimal strategy set, we will discuss the asymptotic structure of S_ε and F_i.

Lemma 4: Suppose that $\bar{F}_i\in\mathbf{F}_i$. There exists a small constant σ_2^* such that for all $\varepsilon\in(0,\ \sigma_2^*)$, SALE (19a) and the linear equation (19b) admit a positive definite solution S_ε^* and a feedback gain F_i^* that can be expressed as

$$S_\varepsilon:=S_\varepsilon^*=\bar{S}+O(\varepsilon), \qquad (24a)$$
$$F_i:=F_i^*=\bar{F}_i+O(\varepsilon), \qquad (24b)$$

where $\bar{S}=\mathbf{block\ diag}\begin{pmatrix}\bar{S}_{11} & \cdots & \bar{S}_{NN}\end{pmatrix}$,

$$\begin{aligned}&\bar{S}_{ii}(A_{ii}+B_{ii}\bar{F}_iC_{ii})^T+(A_{ii}+B_{ii}\bar{F}_iC_{ii})\bar{S}_{ii}\\&\quad+A_{iii}^T\bar{S}_{ii}A_{iii}+I_{n_i}=0, \qquad (25a)\\&\gamma_iR_iF_iC_{ii}\bar{S}_{ii}C_{ii}^T+B_{ii}^T\bar{P}_{ii}\bar{S}_{ii}C_{ii}^T=0. \qquad (25b)\end{aligned}$$

Without loss of generality, as an additional technical assumption, we suppose that F_i is confined to the following set.

$\mathbf{L}_i:=\{F_i\in\mathbf{F}_i\ |\ \ C_{ii}\bar{S}_{ii}C_{ii}^T>0$, where $\bar{S}_{ii}$ satisfies (25a). $\}$.

The positive definiteness condition holds, for example, when $\bar{S}_{ii}$ is positive definite and C_{ii} has full row rank. In this case, $\bar{F}_i$ can be written as

$$\bar{F}_i=-(\gamma_iR_i)^{-1}B_{ii}^T\bar{P}_{ii}\bar{S}_{ii}C_{ii}^T(C_{ii}\bar{S}_{ii}C_{ii}^T)^{-1}. \qquad (26)$$

IV. PARAMETER INDEPENDENT PARETO NEAR-OPTIMAL STRATEGY WITH LOCAL OUTPUT MEASUREMENTS

We now propose a new design approach for constructing Pareto near-optimal strategy. The new ε-independent Pareto near-optimal strategy $\bar{F}_i$ of (26) can be obtained by solving reduced-order algebraic equations (17) and (25). The ε-independent Pareto near-optimal strategy is obtained by neglecting the term of $O(\varepsilon)$ of the full-order strategy (23). The main result of this paper is as follows.

Theorem 2: The approximate Pareto near-optimal strategy $\bar{u}_i(t):=\bar{F}_iC_{ii}x_i(t)$ that is based on (26) results in the following relation.

$$\bar{J}_i-J_i^*=O(\varepsilon), \qquad (27)$$

where

$$\bar{J}_i:=\mathbf{Tr}[M_{i\varepsilon}], \qquad (28a)$$
$$J_i^*:=\mathbf{Tr}[N_{i\varepsilon}], \qquad (28b)$$
$$\begin{aligned}&M_{i\varepsilon}\left(A_\varepsilon+\sum_{k=1}^NB_{k\varepsilon}\bar{F}_kC_k\right)+\left(A_\varepsilon+\sum_{k=1}^NB_{k\varepsilon}\bar{F}_kC_k\right)^TM_{i\varepsilon}\\&\quad+\sum_{k=1}^N\bar{A}_{k\varepsilon}^TM_{k\varepsilon}\bar{A}_{k\varepsilon}+C_i^T\bar{F}_i^TR_i\bar{F}_iC_i+Q_{i\varepsilon}=0, \qquad (28c)\end{aligned}$$
$$\begin{aligned}&N_{i\varepsilon}\left(A_\varepsilon+\sum_{k=1}^NB_{k\varepsilon}F_kC_k\right)+\left(A_\varepsilon+\sum_{k=1}^NB_{k\varepsilon}F_kC_k\right)^TN_{i\varepsilon}\\&\quad+\sum_{k=1}^N\bar{A}_{k\varepsilon}^TN_{k\varepsilon}\bar{A}_{k\varepsilon}+C_i^TF_i^TR_iF_iC_i+Q_{i\varepsilon}=0. \qquad (28d)\end{aligned}$$

Proof: Subtracting (28d) from (28c) and using the result of (24b), $L_{i\varepsilon} = M_{i\varepsilon} - N_{i\varepsilon}$ satisfies the following SALE

$$L_{i\varepsilon}\left(A_\varepsilon + \sum_{k=1}^{N} B_{k\varepsilon}\bar{F}_k C_k\right) + \left(A_\varepsilon + \sum_{k=1}^{N} B_{k\varepsilon}\bar{F}_k C_k\right)^T L_{i\varepsilon} + \sum_{k=1}^{N} \bar{A}_{k\varepsilon}^T L_{k\varepsilon}\bar{A}_{k\varepsilon} + O(\varepsilon) = 0. \quad (29)$$

Without loss of generality, it is supposed that SALE (29) has the following structure [15].

$$L_{i\varepsilon} := \begin{bmatrix} L_{i11} & \varepsilon L_{i12} & \cdots & \varepsilon L_{i1N} \\ \varepsilon L_{i12}^T & L_{i22} & \cdots & \varepsilon L_{i2N} \\ \vdots & \vdots & \ddots & \vdots \\ \varepsilon L_{i1N}^T & \varepsilon L_{i2N}^T & \cdots & L_{iNN} \end{bmatrix} \in \Re^{\bar{n}\times\bar{n}}. \quad (30)$$

Using the implicit function theorem under the condition of $\bar{F}_i \in \mathbf{F}_i$, it can be shown that there exists a neighbourhood of $\varepsilon = 0$ and a function $L_{i\varepsilon} := \bar{L}_i + O(\varepsilon)$, where $\bar{L}_i = \mathbf{block\ diag}\begin{pmatrix} \bar{L}_{i11} & \cdots & \bar{L}_{iNN} \end{pmatrix}$. Substituting $\bar{L}_i$ into (29) and letting $\varepsilon = 0$, $\bar{L}_{ijj}$, $j = 1, \ \dots \ , N$ satisfies the reduced-order parameter independent SALE (31).

$$\bar{L}_{ijj}(A_{ii} + B_{ii}\bar{F}_i C_{ii}) + (A_{ii} + B_{ii}\bar{F}_i C_{ii})^T \bar{L}_{ijj} + A_{iii}^T \bar{L}_{ijj} A_{iii} = 0. \quad (31)$$

Then, since $\bar{F}_i \in \mathbf{F}_i$, $I_{n_i} \otimes (A_{ii} + B_{ii}F_iC_{ii})^T + (A_{ii} + B_{ii}F_iC_{ii})^T \otimes I_{n_i} + A_{iii}^T \otimes A_{iii}^T$ is nonsingular. Hence, $\bar{L}_{ijj} = 0$, $j = 1, \ \dots \ , N$ and for all i. Consequently,

$$L_{i\varepsilon} = O(\varepsilon) \quad (32)$$

results in (27) because $\bar{L}_i = 0$. ■

The proposed Pareto near-optimal strategy brings about the following reliability and usefulness. The strategy set can be computed with the reduced order dimension even though the weakly coupled parameter is unknown. Particularly, it is worth pointing out that the design of the strategy can be solved for each subsystem independently. Moreover, the feedback information only rely on the local output measurement.

Solving the reduced-order CSAREs (17) and (25) is not an easy task in general even though each problem can be solved independently. In the rest of this section, we propose some numerical techniques for solving the reduced-order CSAREs (17) and (25).

A proposed approach is to use the following algorithm.

Step 1. Choose a matrix $\bar{F}_i^{(0)}$, $i = 1, \ \dots \ , N$ such that there exists a positive definite symmetric matrix $\bar{P}^{(0)}$ that satisfy $\bar{P}_{ii}^{(0)}(A_{ii} + B_{ii}\bar{F}_i^{(0)}C_{ii}) + (A_{ii} + B_{ii}\bar{F}_i^{(0)}C_{ii})^T\bar{P}_{ii}^{(0)} + A_{iii}^T\bar{P}_{ii}^{(0)}A_{iii} + \gamma_i(C_{ii}^T\bar{F}_i^{(0)T}R_i\bar{F}_i^{(0)}C_{ii} + Q_{ii}) = 0$. That is, the closed-loop system $dx_i(t) = [A_{ii} + B_{ii}\bar{F}_i^{(0)}C_{ii}]x_i(t) + A_{iii}x_i(t)dw_i(t)$ is EMSS.

Step 2. Set $n = 0$, and solve the following SALEs for $\bar{P}_{ii}^{(n+1)}$ and $\bar{S}_{ii}^{(n+1)}$.

$$\bar{P}_{ii}^{(n+1)}\left(A_{ii}+B_{ii}\bar{F}_i^{(n)}C_{ii}\right) + \left(A_{ii}+B_{ii}\bar{F}_i^{(n)}C_{ii}\right)^T \bar{P}_{ii}^{(n+1)} + \bar{A}_{iii}^T\bar{P}_{ii}^{(n+1)}\bar{A}_{iii} + \gamma_i(C_{ii}^T\bar{F}_i^{(n)T}R_i\bar{F}_i^{(n)}C_{ii}+Q_{ii}) = 0, \quad (33a)$$

$$\bar{S}_{ii}^{(n+1)}\left(A_{ii}+B_{ii}\bar{F}_i^{(n)}C_{ii}\right)^T + \left(A_{ii}+B_{ii}\bar{F}_i^{(n)}C_{ii}\right)\bar{S}_{ii}^{(n+1)} + \bar{A}_{iii}\bar{S}_{ii}^{(n+1)}\bar{A}_{iii}^T + I_{n_i} = 0. \quad (33b)$$

Step 3. Compute

$$\tilde{F}_i^{(n+1)} = -(\gamma_i R_i)^{-1}B_{ii}^T\bar{P}_{ii}^{(n+1)}\bar{S}_{ii}^{(n+1)}C_{ii}^T \times (C_{ii}\bar{S}_{ii}^{(n+1)}C_{ii}^T)^{-1}. \quad (34)$$

Step 4. If the closed-loop system is EMSS with $\tilde{F}_i^{(n+1)}$, compute

$$\bar{F}_i^{(n+1)} = \bar{F}_i^{(n)} + \alpha(\tilde{F}_i^{(n+1)} - \bar{F}_i^{(n)}), \quad (35)$$

where $\alpha \in (0,\ 1]$ is chosen to ensure the minimum is not overshot, that is,

$$J^{(n+1)} = \mathbf{Tr}[P_\varepsilon^{(n+1)}] < J^{(n)} = \mathbf{Tr}[P_\varepsilon^{(n)}]. \quad (36)$$

Moreover, set $n \to n+1$ and return to Step 1; otherwise STOP.

Step 5. Pareto near-optimal static output feedback gain is $\bar{F}_i = \lim_{n\to\infty}\bar{F}_i^{(n)}$

It should be noted that convergence of the above algorithm can be guaranteed by using the similar proof in [13]. However, the convergence rate is unclear even though this algorithm work well.

Observation 3: If the small parameter ε is known, the full-order static output feedback gain F_i, $i = 1, \ \dots \ , N$ can be obtained by using the following algorithm directly.

$$P_\varepsilon^{(n+1)}\left(A_\varepsilon + \sum_{k=1}^{N} B_{k\varepsilon}F_k^{(n)}C_k\right) + \left(A_\varepsilon + \sum_{k=1}^{N} B_{k\varepsilon}F_k^{(n)}C_k\right)^T P_\varepsilon^{(n+1)} + \sum_{k=1}^{N}\bar{A}_{k\varepsilon}^T P_\varepsilon^{(n+1)}\bar{A}_{k\varepsilon} + \sum_{k=1}^{N}\gamma_k C_k^T F_k^{(n)T}R_kF_k^{(n)}C_k + Q_\varepsilon = 0, \quad (37a)$$

$$S_\varepsilon^{(n+1)}\left(A_\varepsilon + \sum_{k=1}^{N} B_{k\varepsilon}F_k^{(n)}C_k\right)^T + \left(A_\varepsilon + \sum_{k=1}^{N} B_{k\varepsilon}F_k^{(n)}C_k\right)S_\varepsilon^{(n+1)} + \sum_{k=1}^{N}\bar{A}_{k\varepsilon}S_\varepsilon^{(n+1)}\bar{A}_{k\varepsilon}^T + I_{\bar{n}} = 0, \quad (37b)$$

$$F_i^{(n+1)} = -(\gamma_i R_i)^{-1}B_{i\varepsilon}^T P_\varepsilon^{(n+1)}S_\varepsilon^{(n+1)}C_i^T \times (C_iS_\varepsilon^{(n+1)}C_i^T)^{-1}. \quad (37c)$$

It should be noted that $F_i^{(0)}$ is chosen such that the closed-loop stochastic systems (8a) with $u_i(t) = \tilde{u}_i^{(0)} := F_i^{(0)} C_{ii} x_i(t)$ are EMSS.

V. UNIQUENESS OF STOCHASTIC PARETO NEAR OPTIMAL STRATEGY

In this section, the uniqueness of the stochastic Pareto near-optimal strategy is discussed as a special case of the state feedback problems. Consider stochastic linear time-invariant weakly coupled large-scale systems with the state feedback strategy for the stochastic systems (8), where $C_i := I_{n_i}$. The following conditions are assumed.

Assumption 1: The following matrix is nonsingular.

$$\left(\bar{A} - \sum_{k=1}^{N} \bar{U}_k \hat{P}\right)^T \otimes I_{\bar{n}} + I_{\bar{n}} \otimes \left(\bar{A} - \sum_{k=1}^{N} \bar{U}_k \hat{P}\right)^T + \sum_{k=1}^{N} \bar{A}_k^T \otimes \bar{A}_k^T, \quad (38)$$

where

$$\begin{aligned}
\hat{P} &= \textbf{block diag}\left(\hat{P}_{11} \ \cdots \ \hat{P}_{NN} \right), \\
\bar{A} &:= \textbf{block diag}\left(A_{11} \ \cdots \ A_{NN} \right), \\
\bar{A}_i &:= \textbf{block diag}\left(0 \ \cdots \ 0 \quad A_{iii} \quad 0 \ \cdots \ 0 \right), \\
\bar{B}_i &= \left[0 \ \cdots \ 0 \quad B_{ii}^T \quad 0 \ \cdots \ 0 \right]^T, \\
\bar{U}_i &:= \gamma_i^{-1} \bar{B}_i R_i^{-1} \bar{B}_i^T \\
&= \textbf{block diag}\left(0 \ \cdots \ 0 \quad U_{ii} \quad 0 \ \cdots \ 0 \right), \\
U_{ii} &:= \gamma_i^{-1} B_{ii} R_i^{-1} B_{ii}^T,
\end{aligned}$$

and

$$\hat{P}_{ii} A_{ii} + A_{ii}^T \hat{P}_{ii} + A_{iii}^T \hat{P}_{ii} A_{iii} - \hat{P}_{ii} U_{ii} \hat{P}_{ii} + \gamma_i Q_{ii} = 0. \quad (39)$$

Assumption 2: $(A_{ii}, \ B_{ii})$ is stabilizable, $(\sqrt{Q_{ii}}, \ A_{ii})$ is detectable, and $\inf_{F_i} \left\| \int_0^{\infty} \exp[(A_{ii} - B_{ii}F_i)^T t] A_{iii}^T A_{iii} \exp[(A_{ii} - B_{ii}F_i)t] dt \right\| < 1$.

The asymptotic expansion of stochastic algebraic Riccati equation (SARE) (22) at $\varepsilon = 0$ is described by the following theorem.

Theorem 3: Under Assumptions 1 and 2, there exists a small constant ρ^* such that for all $\varepsilon \in (0, \ \rho^*)$, SARE (22) admits the unique positive semidefinite solution P_ε^* that can be expressed as

$$P_\varepsilon := P_\varepsilon^* = \hat{P} + O(\varepsilon). \quad (40)$$

In order to prove Theorem 3, the following lemma is used [12].

Lemma 5: Let us consider the following SARE

$$XA + A^T X + \Pi(X) - XBR^{-1}B^T X + C^T C = 0, \quad (41)$$

where Π denotes a positive linear map of the class of symmetric matrices into itself, i.e., $\Pi(X) \geq 0$ whenever $X \geq 0$.

If $(A, \ B)$ is stabilizable, $(C, \ A)$ is detectable, and $\inf_K \left\| \int_0^{\infty} \exp[(A - BK)^T t] \Pi(I_{\bar{n}}) \times \exp[(A - BK)t] dt \right\| < 1$, then SARE (41) has a unique positive semidefinite solution such that $A - BR^{-1}B^T X$ is stable.

Proof: By using the implicit function theorem, it is clear that there exists a neighbourhood of $\varepsilon = 0$ and a unique continuous function $P_\varepsilon := P_\varepsilon^* = \Psi(\varepsilon)$. Moreover, it should be noted that the asymptotic structure of solution (40) can also be obtained by applying the Newton-Kantorovich theorem [15]. On the other hand, the use of Assumption 2 yields a unique positive semidefinite solution $\hat{P}_{ii}$. Therefore, there exists a small constant ρ^* such that for all $\varepsilon \in (0, \ \rho^*)$, SARE (22) admits the unique positive semidefinite solution P_ε^*. ■

VI. NUMERICAL EXAMPLE

In order to demonstrate the efficiency of the stochastic Pareto near-optimal strategies, we present results for the megawatt-frequency control problem of multiarea electric energy systems. The model is based on the multi-stage decomposition of two interconnected areas [6]. The system matrices are given as follows.

$$A_{11} = \begin{bmatrix} 0 & 0.315 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & -1.888 & -0.0498 & 6 & 0 \\ 0 & 0 & 0 & -3.333 & 3.333 \\ 0 & 0 & -13.9 & 0 & -33.333 \end{bmatrix},$$

$$A_{12} = \begin{bmatrix} -3.15 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 18.88 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \ A_{21} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 18.88 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$A_{22} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1.888 & -0.0498 & 6 & 0 \\ 0 & 0 & -3.333 & 3.333 \\ 0 & -13.9 & 0 & -33.333 \end{bmatrix},$$

$$\begin{aligned}
A_{111} &= \textbf{block diag}\left(0 \ \ 0 \ \ 0.00249 \ \ 0 \ \ 0 \right), \\
A_{222} &= \textbf{block diag}\left(0 \ \ 0.00249 \ \ 0 \ \ 0 \right), \\
A_{112} &= A_{121} = A_{122} = A_{211} = A_{212} = A_{221} = 0, \\
B_{11}^T &= \begin{bmatrix} 0 & 0 & 0 & 0 & 33.333 \end{bmatrix}, \ B_{22}^T = \begin{bmatrix} 0 & 0 & 0 & 33.333 \end{bmatrix}, \\
B_{12} &= B_{21} = 0,
\end{aligned}$$

$$C_{11} = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \ C_{22} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$\begin{aligned}
Q_1 &= \textbf{block diag}\left(I_5 \ \ \varepsilon I_4 \right), \\
Q_2 &= \textbf{block diag}\left(\varepsilon I_5 \ \ I_4 \right), \\
R_1 &= R_2 = 0.1, \ \gamma_1 = \gamma_2 = 0.5.
\end{aligned}$$

Referring to the design procedure, Pareto near-optimal strategies are given by

$$\bar{F}_1 = \left[\ -1.5084 \quad -2.1493 \quad -4.6392e-001 \ \right], \quad (42a)$$

$$\bar{F}_2 = \left[\ -2.4749 \quad -4.4046 \quad -1.9703 \ \right], \quad (42b)$$

where "$e - f$" stands for "$\times 10^f$".

TABLE I
DEGRADATION OF COST.

ε	$\bar{J}_1$	J_1^*	ϕ_1	$\bar{J}_2$	J_2^*	ϕ_2
$1.0e-2$	$1.4466e+002$	$1.4493e+002$	$2.7574e+001$	$6.2568e+002$	$6.3271e+002$	$7.0337e+002$
$1.0e-3$	$1.4281e+002$	$1.4284e+002$	$2.1209e+001$	$6.2474e+002$	$6.2552e+002$	$7.7613e+002$
$1.0e-4$	$1.4263e+002$	$1.4263e+002$	$2.0551e+001$	$6.2465e+002$	$6.2473e+002$	$7.8321e+002$
$1.0e-5$	$1.4261e+002$	$1.4261e+002$	$2.0485e+001$	$6.2464e+002$	$6.2465e+002$	$7.8391e+002$

On the other hand, letting $\varepsilon = 0.01$, Pareto optimal strategies are given by

$$F_1 = \begin{bmatrix} -1.5102 & -2.1539 & -4.6490e-001 \end{bmatrix}, \quad (43a)$$
$$F_2 = \begin{bmatrix} -2.4886 & -4.4284 & -1.9800 \end{bmatrix}. \quad (43b)$$

We evaluate the costs using Pareto near-optimal strategy (42). For the first decision maker, the average values of the performance index are $\bar{J}_1 = 1.4466e+002$, $J_1^* = 1.4493e+002$, where $\varepsilon = 0.01$. Hence, the loss of performance $\bar{J}_1$ is less than 0.19025% compared with J_1^*. The values of the cost functional for various ε are given in Table 1, where $\phi_i = |\bar{J}_i - J_i^*|/\varepsilon$, $i = 1, 2$.

It is easy to verify that $\bar{J}_i = J_i^* + O(\varepsilon)$ which is given by (27) because of $\phi_i < \infty$.

VII. CONCLUSION

In this paper, the static output feedback Pareto near-optimal strategy to the stochastic system governed by Itô differential equations where only the local output measurements are available has been developed. Firstly, we have derived the necessary conditions for a decentralized controller to be Pareto optimal strategy. The uniqueness and boundedness of the solution to the CSAREs and their asymptotic structures have been established. Using the obtained asymptotic structure, we have developed a new parameter independent approximation Pareto strategy. Secondly, a new sequential numerical algorithm for solving the reduced order CSAREs has been described for the first time. As the summary, the following appearing properties can be stated: 1) The strategy set can be computed with the reduced order dimension even though the weakly coupled parameter is unknown; 2) Particularly, the design of the strategies can be decentralized to each subsystem; 3) Since the near-optimal strategy can be implemented using the local output measurements, the design can be applied to practical situations more easily.

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