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Stabilization of a Class of Non-holonomic Systems by Means of Switching Control Laws

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Abstract— **This paper deals with the stabilization problem for nonlinear systems: it provides a sufficient condition for the existence of a time-varying switching control scheme which globally asymptotically stabilizes the zero equilibrium. The sufficient condition is proven to hold for a class of non-holonomic systems and the corresponding switching control law is described in detail.**

I. INTRODUCTION

In recent years non-holonomic systems have been widely analyzed since they represent a paradigm for a number of mechanical systems. As shown by Brockett's theorem [1], these systems are not asymptotically stabilizable by means of smooth (or "mildly" discontinuous [2]) control laws. A general treatment of the stabilizability problem has been mainly addressed for non-holonomic systems belonging to specific classes, i.e. systems in "chained" (or "power") form (see, for instance, [3], [4], [5], [6], [7], [8]). For systems not in these forms (or not feedback-equivalent to these forms) very general results exist [9], [10], [11], [12], [13], [14], [15], most of which, however, cannot be easily exploited in order to design explicit control laws.

The approach followed in the present paper is analogous to the one proposed in [14]: stability is achieved by the iterative application of an open-loop control law in a closed-loop strategy. However, the novelty of this paper consists in the specification of sufficient conditions for the existence of a stabilizing control law for a wide class of systems.

We show that a switching control scheme can be designed according to the value of a function in such a way that along the trajectories of the closed loop system the function itself behaves as a strictly decreasing Lyapunov function, thus providing asymptotic stability to the equilibrium.

The paper is organized as follows. In Section II some definitions and notions are introduced which are subsequently used to prove the main stability theorem. In Section III a class of n-dimensional non-holonomic systems is described and the corresponding stabilizing switching control scheme is presented. Concluding remarks are drawn in Section IV.

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II. A STABILITY THEOREM FOR NONLINEAR SYSTEMS

The paper focuses on *switched systems* [16] which, in qualitative terms, can be interpreted as a family of dynamic systems whose continuous state $\mathbf{x} \in \mathcal{X} \subset \mathbb{R}^n$ evolves in time according to a law depending on the value of the discrete state q of a finite state automaton (FSA) which can assume value on a finite set $\mathcal{Q}$ of positive integers:

$$\dot{\mathbf{x}} = \mathbf{f}_q(\mathbf{x}), \qquad q \in \mathcal{Q} = \{1, 2, \ldots, N\}. \tag{1}$$

In turn, q varies according to some *switching law* which is completely determined by a sequence of time-instants $\{\tau_k\}_{k\in\mathbb{Z}^+}$, $\tau_k \in \mathbb{R}^+$, and by a sequence of pairs of values of q, $\left\{(q_k^-, q_k^+)\right\}_{k\in\mathbb{Z}^+}$, where q_k^- and q_k^+ denote the values of q before and after the k-th switching, respectively. Since $q_k^+ = q_{k+1}^-$, for all $k \in \mathbb{Z}^+$, a switching sequence σ may be defined as[1] $\sigma \triangleq (\tau_1, q_1), \ldots, (\tau_k, q_k), \ldots$, where $q_k \triangleq q_k^+$. If we denote with S_σ the set of all possible sequences σ, then the switched system can be denoted by

$$\dot{\mathbf{x}} = \mathbf{f}_\sigma(\mathbf{x}), \qquad \sigma \in S_\sigma. \tag{2}$$

For the sake of simplicity, in what follows we suppose that for all sequences σ the vector field $\mathbf{f}_\sigma$ is forward complete. Now, by introducing in a natural way the piecewise-constant function $q(t)$: $q(t) \equiv q_k$ for all $t \in [\tau_k, \tau_{k+1})$, we can define the concept of solution for a switched system.

Definition 2.1: Suppose that there exists a $\kappa > 0$ such that[2] for all $k \in \mathbb{Z}^+$, $\tau_{k+1} - \tau_k \geqslant \kappa$. For a given switching sequence σ we say that $\mathbf{x}(t)$ is a *solution* of (2) starting from $\mathbf{x}_0$ if

- $\mathbf{x}(t)$ is right-continuous at $t = 0$, continuous for all $t > 0$ and $\lim_{t\to 0^+} \mathbf{x}(t) = \mathbf{x}_0$,
- for all $k \in \mathbb{Z}^+$, $\mathbf{x}(t)$ is right- and left-differentiable at τ_k and differentiable for all $t \in (\tau_k, \tau_{k+1})$,
- for all $t \in [\tau_k, \tau_{k+1})$ $\dot{\mathbf{x}}(t) = \mathbf{f}_{q_k}(\mathbf{x}(t))$. ◊

Note that in Definition 2.1 some regularity conditions of the trajectory of the hybrid state $(\mathbf{x}, q)$ are required. The switching strategy and the vector fields that are considered in the remainder of the paper are such that these conditions are guaranteed; thus in all the results it is understood that the solution is such that Definition 2.1 holds.

In what follows, we consider a switched system characterized by a set of N smooth vector fields $F = \{\mathbf{f}_1, \ldots, \mathbf{f}_N\}$:

$$\left\{ \begin{array}{rcl} \dot{\mathbf{x}}(t) &=& \mathbf{f}_{q_k}(\mathbf{x}(t)), \\ \tau_{k+1} &=& \tau_k + \sigma_\tau(q_k, \mathbf{x}(\tau_k)), \\ q_{k+1} &=& \sigma_q(q_k, \mathbf{x}(\tau_{k+1})), \end{array} \right. \tag{3}$$

[1] We suppose $\tau_0 = 0$ and that the initial value q_0 of q is assigned.

[2] Note that this requirement implies the absence of Zeno behaviour and chattering, a constraint that a switched system should fulfill in practice.

where $\mathbf{f}_{q_k}:\mathbb{R}^n\to\mathbb{R}^n$ is the vector field associated to q_k, with $\mathbf{f}_{q_k}\in F$, and $\sigma_\tau(q_k,\mathbf{x}(\tau_k))\geqslant\kappa>0$. Moreover, we assume $\mathbf{f}_q(\mathbf{0})=\mathbf{0}$, for all $q\in\mathcal{Q}$; hence, $\mathbf{x}=\mathbf{0}$ is an equilibrium point of the continuous part of the hybrid system (3).

The concept that we want to formalize, already used in [17], [18], is the following.

It is possible to design a switching strategy based on the value of a positive definite function $V(\mathbf{x})$ in such a way that the trajectory of the state of the closed loop tends to zero.

To this aim, not only the first-order time-derivative will be considered but also the time-derivatives of higher order (see a previous result in [19]).

A first result can be immediately deducted from Lyapunov's theory[3]. In fact, consider a non-linear system $\dot{\mathbf{x}}(t)=\mathbf{f}(\mathbf{x}(t),\mathbf{w}(t))$, such that the origin of the state space is an equilibrium state associated with zero value of the control variable: $\mathbf{f}(\mathbf{0},\mathbf{0})=\mathbf{0}$. Suppose that $\mathbf{w}_1^*(\mathbf{x}(t)),\ldots,\mathbf{w}_N^*(\mathbf{x}(t))$ are control laws such that $\mathbf{w}_j^*(\mathbf{0})=\mathbf{0}$ for all $j=1,\ldots,N$ and define $\tilde{\mathbf{f}}_{q_k}(\mathbf{x}(t))\triangleq\mathbf{f}(\mathbf{x}(t),\mathbf{w}_{q_k}^*(\mathbf{x}(t)))$. Finally, suppose that the switching law σ originates a closed-loop trajectory $\mathbf{x}(t)$ such that there exists a positive definite and radially unbounded function V such that for each pair of time-instants $t_1\in\mathbb{R}^+$ and $t_2\in\mathbb{R}^+$, with $t_2\geqslant t_1$,

$$V(\mathbf{x}(t_2))\leqslant V(\mathbf{x}(t_1))\,. \tag{4}$$

It is easy to see that in these hypotheses the equilibrium in $\mathbf{x}=\mathbf{0}$ is Lyapunov stable.

A more interesting result is that, with some additional hypotheses global asymptotic stability can be achieved, as stated in the following theorem.

Theorem 2.1: Suppose that there exists a positive integer M such that[4] $V(t)\in\mathcal{C}^{M+1}$ and for all $m\in\{1,2,\ldots,M+1\}$ there exists L_m such that

$$\left|\frac{d^mV(\mathbf{x}(t))}{dt^m}\right|<L_m\,,\quad\forall\; t>0\,,t\neq\tau_k\,. \tag{5}$$

Moreover, suppose that it is possible to associate to $\mathcal{Q}^*$ a set of N continuous negative semidefinite functions from $\mathbb{R}^n$ to $\mathbb{R}$, $\eta_1,\ldots,\eta_N$, such that

$$\forall\;\mathbf{x}\neq\mathbf{0}\,,\quad\exists\; q\in\mathcal{Q}^*\text{ such that }\eta_q(\mathbf{x})<0\,. \tag{6}$$

Suppose also that $\mathcal{Q}^*$ can be partitioned[5] into $M+1$ (disjoint) subsets, $\mathcal{Q}_0^*,\mathcal{Q}_2^*,\ldots,\mathcal{Q}_M^*$, such that, defining, for $i=1,\ldots,M$,

$$\mathcal{X}_i\triangleq\{\mathbf{x}\in\mathbb{R}^n\text{ such that }\mathbf{x}\neq\mathbf{0}\text{ and }\eta_q(\mathbf{x})=0\,,\forall\;q\in\mathcal{Q}_{i-1}^*\}\,,$$

and, for a given set $\mathbf{X}\subset\mathbb{R}^n$,

$$\mathcal{Q}_{\mathbf{X}}^*\triangleq\{q\in\mathcal{Q}^*\text{ such that }\eta_q(\mathbf{x})=0\,,\quad\forall\;\mathbf{x}\in\mathbf{X}\}\,,$$

the following three conditions hold[6].

(C1) For all $\mathbf{x}(\tau_k)\in\mathbb{R}^n$ and for all $p\in\mathcal{Q}_0^*$, $\lim_{t\to\tau_k^+}\left[\dot{V}(t)\Big|_{q=p}\right]=\eta_p(\mathbf{x}(\tau_k))$.

(C2) For all $j=1,\ldots,M$, for all $\mathbf{x}(\tau_k)\in\bigcap_{i=1}^j\mathcal{X}_i$ and for all $p\in\mathcal{Q}_j^*$, $\lim_{t\to\tau_k^+}\left[\frac{d^{j+1}V(t)}{dt^{j+1}}\Big|_{q=p}\right]=\eta_p(\mathbf{x}(\tau_k))$.

(C3) σ_q^* is such that for all $\mathbf{X}\subset\mathbb{R}^n\backslash\{\mathbf{0}\}$ there exists $\varepsilon_{\mathbf{X}}$ such that if[7] $\inf_{\bar{\mathbf{x}}\in\mathbf{X}}\|\mathbf{x}(\tau_k)-\bar{\mathbf{x}}\|<\varepsilon_{\mathbf{X}}$ then there exists $\bar{k}>k$ such that $q_{\bar{k}}\notin\mathcal{Q}_{\mathbf{X}}^*$. Then the equilibrium in $\mathbf{x}=\mathbf{0}$ is globally asymptotically stable. □

Proof. Note that Equation (4) implies that V has a finite non-negative limit for $t\to\infty$: $\lim_{t\to\infty}V(t)\triangleq V_\infty\geqslant 0$. Hence, by conditions (5), Lemma A.2 yields

$$\lim_{k\to\infty}\left[\lim_{t\to\tau_k^+}\frac{d^mV(t)}{dt^m}\right]=0\,,\quad\forall\;m\in\{1,2,\ldots,M\}\,. \tag{7}$$

Suppose that $V_\infty>0$ and define the set $\mathcal{Y}\triangleq\{\mathbf{z}\in\mathbb{R}^n:V(\mathbf{z})=V_\infty\}$. Now, each η_q is, by hypothesis, negative semidefinite; hence, keeping in mind (6), since $\{\mathbf{x}(\tau_k)\}_{k\in\mathbb{Z}^+}$ is a sequence which tends to $\mathcal{Y}$ and by the continuity of each η_q, the only possibility to fulfill C1 (and C2) and (7) is that discrete state of the FSA, from a given switching-time instant h onwards, takes values in $\mathcal{Q}_{\mathcal{Y}}^*$. This means that there exists h such that, for all $k>h$, $q_k\in\mathcal{Q}_{\mathcal{Y}}^*$, i.e.

$$\exists\;h\text{ such that }\forall\;\mathbf{z}\in\mathcal{Y}\,,\forall\;k>h\,,\quad\eta_{q_k}(\mathbf{z})=0\,. \tag{8}$$

On the other hand, if $\{\mathbf{x}(\tau_k)\}_{k\in\mathbb{Z}^+}$ tends to $\mathcal{Y}$, then $\lim_{h\to\infty}[\inf_{\bar{\mathbf{x}}\in\mathcal{Y}}\|\mathbf{x}(\tau_h)-\bar{\mathbf{x}}\|]=0$. Thus we can apply C3 with $\mathbf{X}=\mathcal{Y}$ finding that there exists $k>h$ such that $q_k\notin\mathcal{Q}_{\mathcal{Y}}^*$, namely that for all $\mathbf{z}\in\mathcal{Y}$ there exists $k>h$ such that $\eta_{q_k}(\mathbf{z})<0$, which is in contradiction with Equation (8). Then it must be $V_\infty=0$ which implies asymptotic stability. ■

A. The switching algorithm

A switching algorithm leading the state $\mathbf{x}$ to $\mathbf{0}$ can now be sketched as follows.

Step 0. The initial state is $(\mathbf{x}_0,q_0)$. The initial time is set to 0. The initial value of k is 0.

Step 1. If there exist a control law $\mathbf{w}_i^*(\mathbf{x}(t))$ such that $\lim_{t\to\tau_k^+}\dot{V}(t)<0$ then $\mathbf{w}(t)=\mathbf{w}_i^*(t)$ for $t\in[\tau_k,\tau_{k+1})$ otherwise go to step 2.

⋮

Step m. If there exist a control law $\mathbf{w}_i^*(\mathbf{x}(t))$ such that $\lim_{t\to\tau_k^+}\frac{\mathrm{d}^mV}{\mathrm{d}t^m}(t)<0$ then $\mathbf{w}(t)=\mathbf{w}_i^*(t)$ for $t\in[\tau_k,\tau_{k+1})$ otherwise go to step $m+1$.

⋮

Step M+1. Go back to step 1 with $k\leftarrow k+1$.

[3]For a comparison, see [20], where an interesting result based on the notion of smooth Lyapunov functions is discussed.

[4]Given a positive integer M we denote by $\mathcal{C}^M$ the set of the functions continuously differentiable up to the order M.

[5]We remind that a *partition* of a set A is a collection of subsets A_i of A such that $\bigcup_i A_i=A$ and, for $i\neq j$, $A_i\cap A_j=\emptyset$.

[6]For a function $F(\mathbf{s}(t))$ we denote by $F(t)|_{q=i}$ the value of $F(\mathbf{s}(t))$ when $\mathbf{s}(t)$ varies according to the dynamics associated to the discrete state $q=i$. The notation $F(t)|_{q\in A}$, for $A\subset\mathbb{N}$, has an analogous meaning.

[7]Condition (C3) introduces a constraint that σ must fulfill to guarantee that the FSA does not get stuck in a particular discrete state $\bar{q}$ while the continuous trajectory $\mathbf{x}(t)$ tends to a state $\bar{\mathbf{x}}\neq\mathbf{0}$ such that $\eta_{\bar{q}}(\bar{\mathbf{x}})=0$.

Note that the switching algorithm, differently from the state-based or logic-based switching rules presented, for instance, in [21], [22], [23], is time-based.

III. Application of the method to a class of non-holonomic systems

The previous result can be exploited to control a class of non-holonomic systems. For $p \in \mathbb{N}$, consider the system constituted by two linear integrators together with, for each $m = 1, \ldots, p$, a basis of the subspace of m-dimensional non-integrable forms, namely

$$\begin{array}{ll} \text{level } 0 & \dot{x} = u\,, \quad \dot{y} = v\,, \\ \text{level } 1 & \dot{z}_1 = xv\,, \\ \quad\vdots & \qquad\vdots \\ \text{level } p & \dot{\mathbf{z}}_p = \left(x^{p-1}, x^{p-2}y, \ldots, y^{p-1}\right)^\top xv\,, \end{array} \tag{9}$$

where u and v are the input variables. The above equations describe a class of systems, as p varies, each of which has an overall order equal to $n = 2 + \sum_{i=1}^{p} i = p(p+1)/2 + 2$.

A. The switching control scheme

We first describe the control structure, then we prove that equilibrium of the closed-loop is Lyapunov stable and that the hypotheses of Theorem 2.1 are also fulfilled. Let $\mathcal{Q}^* = \{0, \ldots, n-2\}$, $\mathbf{s} \triangleq (x, y, \mathbf{z}_1^\top, \ldots, \mathbf{z}_p^\top)^\top$ and

$$\sigma^*_\tau(q, \mathbf{s}) = \kappa\,, \quad \forall\, \mathbf{s} \in \mathbb{R}^n\,, \quad \forall\, q \in \mathcal{Q}^*\,, \tag{10}$$

where $\kappa > 0$ is specified in the sequel. Moreover, introduce the following time-instant[8]:

$$T^V_{\min}(q_k, \mathbf{s}(\tau_k)) \triangleq \sup\left\{\hat{t} \mid \hat{t} \geqslant \tau_k\,,\ \dot{V}(\mathbf{s}(t))\Big|_{q=q_k} \leqslant 0, \forall\, t \in [\tau_k, \hat{t})\right\} - \tau_k\,. \tag{11}$$

For a given $q \in \mathcal{Q}^*$, let $\mathcal{A}(q, \tau_k)$ denote the set of *admissible* switchings at τ_k, i.e. the set of all the discrete states j of the FSA, $j > q$, for which the quantity (11) is greater than κ:

$$\mathcal{A}(q, \tau_k) \triangleq \{j \in \mathcal{Q}^*,\ j > q\ :\ T_{\min}(j, \mathbf{s}(\tau_k)) > \kappa\}\,, \tag{12}$$

and, if $\mathcal{A}(q, \tau_k) \neq \emptyset$, let $l(q) \triangleq \min \mathcal{A}(q, \tau_k)$. Finally,

$$\sigma^*_q(q, \mathbf{s}) = \begin{cases} l(q)\,, & \text{if } \mathcal{A}(q, \tau_k) \neq \emptyset\,, \\ 0 & \text{otherwise}\,. \end{cases} \tag{13}$$

To completely determine the switching control scheme, we need to associate to each discrete state q of the FSA a control law $w^*_q(\mathbf{s}(\tau)) \triangleq (u^*_q(\mathbf{s}(\tau)), v^*_q(\mathbf{s}(\tau)))^\top$. We begin by setting[9] $\gamma = p + 2 + p \bmod 2$ and, for $m = 1, \ldots, p$, $\gamma_m = \min\{d \in \mathbb{N} : d \text{ is even and } d \geqslant (p+2)/(m+1)\}$. Moreover we define the $m \times m$ matrix $\mathbf{P}_m$ as follows[10]:

$$\mathbf{P}_1 = 1\,, \quad \mathbf{e}_i \mathbf{P}_m \mathbf{e}_j = \begin{cases} 1 & \text{if } i = j\,, \\ 2 & \text{if } j = 1 \text{ and } i \neq 1\,, \\ 0 & \text{otherwise.} \end{cases}$$

[8] Here we suppose that the set defined in the bracketed expression is not empty and admits a supremum. It will be clear in the sequel that this is always the case; however, we let $T^V_{\min}(q_k, \mathbf{s}(\tau_k)) \triangleq 0$ when definition (11) is meaningless.

[9] For two integers a and b we denote by $a \bmod b$ the remainder of the division of a by b.

[10] We denote by $\mathbf{e}_i$ the i-th vector of the canonical basis.

Let $\rho_m(i, j) \triangleq \mathbf{e}_i \mathbf{P}_m \mathbf{e}_j$ and $\mathbf{p}_m(i) \triangleq \mathbf{e}_i \mathbf{P}_m$. We introduce the following quantities:

$$\begin{aligned} S_{m,j}(\mathbf{s}) &\triangleq (\mathbf{p}_m(j)\mathbf{z}_m)^{\gamma_m - 1} \sum_{l=1}^{m} \rho_m(j, l) x^{m-l+1} y^{l-1}\,, \\ A(\mathbf{s}) &\triangleq x^{\gamma - 1}\,, \qquad B(\mathbf{s}) \triangleq y^{\gamma-1} + \sum_{m=1}^{p}\sum_{j=1}^{m} S_{m,j}(\mathbf{s})\,, \\ D_m(\mathbf{s}) &\triangleq \sqrt{1 + \sum_{j=m}^{p}\sum_{i=1}^{j} (\mathbf{p}_j(i)\mathbf{z}_j)^{2(\gamma_j - 1)}}\,, \\ \mathcal{Q}^*_m &\triangleq \{m(m-1)/2 + 1, \ldots, m(m-1)/2 + m\}\,, \\ \mathcal{S}_0 &\triangleq \{\mathbf{s} \in \mathbb{R}^n \mid x = 0\,, y = 0\}\,, \\ \mathcal{S}_m &\triangleq \{\mathbf{s} \in \mathbb{R}^n \mid x = y = \|\mathbf{z}_1\| = \ldots = \|\mathbf{z}_m\| = 0\}\,, \end{aligned} \tag{14}$$

Finally, for a particular value of q we denote with $m(q)$ the value of m such that $q \in \mathcal{Q}^*_{m(q)}$ and we define[11]

$$\begin{aligned} L(q) &\triangleq q - m(q)(m(q)-1)/2\,, \\ \delta(\mathbf{s}, q) &\triangleq -\mathrm{sg}\left(\mathbf{p}_{m(q)}(L(q))\mathbf{z}_{m(q)}\right)\,, \\ H(\mathbf{s}, h, q) &\triangleq \left(\mathbf{p}_{m(q)}(h)\mathbf{z}_{m(q)}\right)^{\gamma_m - 1} \times \\ &\qquad \times \sum_{j=1}^{m(q)} \rho_{m(q)}(h, j)\delta(\mathbf{s}, q)^j\,, \\ \varphi_1(\mathbf{s}, q) &\triangleq -\mathrm{sat}(H(\mathbf{s}, L(q), q))/D_m(\mathbf{s})\,, \\ \varphi_2(\mathbf{s}, q) &\triangleq \delta(\mathbf{s}, q)\varphi_1(\mathbf{s}, q)\,, \\ \psi(\mathbf{s}, q) &\triangleq \mathrm{sg}\left(A(\mathbf{s})\varphi_1(\mathbf{s}, q) + B(\mathbf{s})\varphi_2(\mathbf{s}, q)\right)\,, \\ c_i(q, k) &\triangleq \varphi_i(\mathbf{s}(\tau_k), q)\psi(\mathbf{s}(\tau_k), q)\,, \quad i = 1, 2\,. \end{aligned}$$

Then, the control law takes the form

$$w^*_0(\mathbf{s}) = \begin{bmatrix} -x \\ -\left(y^{\gamma-1} - \sum_{m=1}^{p}\sum_{h=1}^{m} S_{m,h}(\mathbf{s})\right) \end{bmatrix}, \tag{15}$$

$$w^*_q(\mathbf{s}) = \begin{bmatrix} c_1(q, k) \\ c_2(q, k) \end{bmatrix}, \quad q \in \{1, \ldots, n-2\}\,. \tag{16}$$

Note that $c_1(q, k)$ and $c_2(q, k)$ are to be updated every time the finite state machine switches to one of the states $1, \ldots, n-2$ according to the specific value $\mathbf{s}(\tau)$ that the continuous state takes at the switching time–instant τ.

B. Stability analysis

Lyapunov stability is proven by showing that the function

$$V(\mathbf{s}) \triangleq \frac{x^\gamma}{\gamma} + \frac{y^\gamma}{\gamma} + \sum_{m=1}^{p} \frac{1}{\gamma_m} \sum_{h=1}^{m} [\mathbf{p}_m(h)\mathbf{z}_m]^{\gamma_m} \tag{17}$$

fulfills Equation (4).

Lemma 3.1: Consider the function in (17). If $q = 0$ and the control law (15)-(16) is applied to the system (9) then $T^V_{\min}(0, \mathbf{s}) = +\infty$, $\forall\, \mathbf{s} \in \mathbb{R}^n$. □

[11] We define the *sign* and *saturation* functions as:

$$\mathrm{sg}(x) \triangleq \begin{cases} 1 & \text{if } x \geqslant 0\,, \\ -1 & \text{otherwise}\,, \end{cases} \qquad \mathrm{sat}(x) \triangleq \begin{cases} \mathrm{sg}\,(x) & \text{if } |x| > 1\,, \\ x & \text{if } |x| \leqslant 1\,, \end{cases}$$

Proof: The first time derivative of V is

$$\dot{V}(t) = x^{\gamma-1}u_0^* + y^{\gamma-1}v_0^* + \sum_{i=1}^{p}\sum_{h=1}^{i} S_{i,h}(\mathbf{s})v_0^* \,. \qquad (18)$$

By collecting the factor v_0^* and substituting the control laws (15) and (16), Equation (18) becomes

$$\dot{V}(t) = -x^{\gamma} - \left[y^{\gamma-1} + \sum_{i=1}^{p}\sum_{h=1}^{i} S_{i,h}(\mathbf{s})\right]^2 \qquad (19)$$

and the claim follows by recalling that γ is even. ■

As a consequence of Lemma 3.1, when $q = 0$ V is always non increasing which implies Lyapunov stability.

To prove global asymptotic stability we show that the hypotheses of Theorem 2.1 are fulfilled. First, note that the time-derivatives of any order of V can be expressed as polynomials in t the coefficients of which depend on $\mathbf{s}(t)$, c_1 and c_2; therefore (5) is fulfilled since $\mathbf{s}(t)$, c_1 and c_2 are bounded for all t (by Lyapunov stability and by construction). To prove that there exists κ such that $T_{\min}[q_k, \mathbf{s}(\tau_k)] > \kappa$ for all $k \in \mathbb{Z}^+$, we need some preliminary results.

Lemma 3.2: If $\mathbf{s}(\tau_k) \in \mathcal{S}_{m-1}\backslash\mathcal{S}_m$ there exists $q^\star \in \mathcal{Q}_m$ such that $T_{\min}[q^\star, \mathbf{s}(\tau)] > 0$, for the function (17). □

Proof: The general solution of system (9) for constant inputs $u = c_1$ and $v = c_2$ and the initial condition $\mathbf{s}(\tau) \in \mathcal{S}_{m-1}\backslash\mathcal{S}_m$ is:

$$\begin{cases} x(t) = c_1 r\,, \quad y(t) = c_2 r\,, & \\ \mathbf{z}_i(t) = \mathbf{a}_i r^{i+1}\,, & \text{for } i = 1,\ldots,m-1\,, \\ \mathbf{z}_i(t) = \mathbf{z}_i(\tau) + \mathbf{a}_i r^{i+1}\,, & \text{for } i = m,\ldots,p\,. \end{cases}$$

where $r = t - \tau$ and

$$\mathbf{a}_i = \frac{1}{i+1}\left(c_1^i c_2, \ldots, c_1 c_2^i\right) \qquad (20)$$

Clearly, all the coordinates of the state evolve in time as polynomials in the variable r. The time derivative of V is

$$\begin{aligned}\dot{V}(r) = (c_1 r)^{\gamma-1}c_1 + (c_2 r)^{\gamma-1}c_2 + \\ + \sum_{i=1}^{m-1}\sum_{j=1}^{i}\left(\mathbf{p}_i(j)\mathbf{a}_i r^{i+1}\right)^{\gamma_i-1} G(i) + \\ + \sum_{i=m}^{p}\sum_{j=1}^{i}\left(\mathbf{p}_i(j)\left(\mathbf{z}_i(\tau) + \mathbf{a}_i r^{i+1}\right)\right)^{\gamma_i-1} G(i)\,, \end{aligned} \qquad (21)$$

where $G(i) = \sum_{h=1}^{i}\rho_i(j,h)c_1^{i-h+1}c_2^h r^i$. Note that (21) is a polynomial in r and that a common factor r^m can be collected. In fact, due to the choices of γ and γ_m, the following conditions hold:

$$\gamma - 1 \geqslant p+1 \geqslant m+1 > m\,,$$

$$(\gamma_i - 1)(i+1) + i = \gamma_i(i+1) - 1 \geqslant p+2-1 > m\,.$$

As a consequence, (21) can be rewritten as

$$\dot{V}(r) = r^m(b_0 + b_1 r + \ldots + b_N r^N)\,, \qquad (22)$$

for some constants $b_0, \ldots, b_N$ depending on c_1 and c_2. Then, the following properties hold. First, the first m order time derivative of V are zero for $r \to 0^+$, that is for $t \to \tau_k^+$:

$$\lim_{t\to\tau_k^+}\dot{V}(t) = \lim_{t\to\tau_k^+}\ddot{V}(t) = \ldots = \lim_{t\to\tau_k^+}\frac{d^m}{dt^m}V(t) = 0\,.$$

Furthermore, the sign of the $(m+1)$-th time-derivative, calculated for $r \to 0^+$, is the sign of b_0; in particular, if we take c_1 and c_2 as in (15) and (16), and b_0 depends on $\mathbf{s}$ and q. Considering only the value of $q \in \mathcal{Q}_m^*$, we have[12] (from (21))

$$\begin{aligned}\lim_{t\to\tau_k^+}\frac{d^{m+1}V(t)}{dt^{m+1}} &= m!\,b_0(\mathbf{s},q) = \\ &= m!\,\varphi_1(\mathbf{s},q)^{m+1}\sum_{j=1}^{m}\left[\left(\mathbf{p}_m(j)\mathbf{z}_m(\tau)\right)^{\gamma_m-1}F(j)\right] = \\ &= m!\,\varphi_1(\mathbf{s},q)^{m+1}\sum_{j=1}^{m}H(\mathbf{s},j,q)\,,\end{aligned}$$

where $F(j) = \sum_{h=1}^{m}\rho_m(j,h)\delta(\mathbf{s}(\tau),q)^h$. We want to prove that $b_0(\mathbf{s},q)$ is always strictly negative. For, two cases have to be taken into account, according to the sign of $\mathbf{p}_m(L(q))\mathbf{z}_m(\tau)$, and consequently the value of $\delta(\mathbf{s}(\tau),q)$. Consider first the case $\mathbf{p}_m(L(q))\mathbf{z}_m(\tau) < 0$. In this case $\delta(\mathbf{s}(\tau),q) = 1$ and $\sum_{h=1}^{m}\rho_m(j,h)\delta(\mathbf{s}(\tau),q)^j = \mathbf{p}_m(j)(1,1,1,\ldots)^\top > 0$, where the last inequality comes from the definition of $\mathbf{P}_m$. Moreover, since $\gamma_m - 1$ is odd, it turns out that $H(\mathbf{s},j,q) < 0$, for all $j = 1,\ldots,m$. This means $\varphi_1(\mathbf{s},q) > 0$ whatever q is, and $\sum_{j=1}^{m}H(\mathbf{s},j,q) < 0$ and, in conclusion, $b_0(\mathbf{s},q) < 0$.

Suppose, now $\mathbf{p}_m(L(q))\mathbf{z}_m(\tau) \geqslant 0$. In this case $\delta(\mathbf{s}(\tau),q) = -1$ and $\sum_{h=1}^{m}\rho_m(j,h)\delta(\mathbf{s}(\tau),q)^j = \mathbf{p}_m(j)(-1,1,-1,1,\ldots)^\top < 0$, where the last inequality comes, again, from the definition of $\mathbf{P}_m$. Then $H(\mathbf{s},j,q) \leqslant 0$, for every $j = 1,\ldots,m$. This means $\varphi_1(\mathbf{s}(\tau),q) \geqslant 0$ whatever q is. Moreover, as $\mathbf{P}_m$ has full rank, at least for one $q^\star \in \mathcal{Q}_m^*$ we have $\mathbf{p}_m(L(q^\star))\mathbf{z}_m(\tau) > 0$ which means that $H(\mathbf{s},L(q^\star),q)$ is strictly negative and, as a consequence, $\sum_{j=1}^{m}H(\mathbf{s},j,q) < 0$. If $H(\mathbf{s},L(q^\star),q)$ is strictly negative, then $\varphi_1(\mathbf{s}(\tau),q^\star)$, is strictly positive. We can conclude that $b_0(\mathbf{s},q^\star) < 0$. So there exists at least one value $q^\star \in \mathcal{Q}_m$ for which $b_0(\mathbf{s},q^\star)$ is negative, which means $T_{\min}[q^\star,\mathbf{s}] > 0$. ■

This means that for every value of $m = 1,\ldots,p$ at least one of the control laws associated to the discrete states in $\mathcal{Q}_m$, the one associated to $q^\star$, is such that $T_{\min}[q^\star,\mathbf{s}(\tau)] > 0$, for $\mathbf{s}(\tau) \in \mathcal{S}_{m-1}\backslash\mathcal{S}_m$. What we want to prove now is that there exists a lowerbound $T_{q^\star} > 0$ such that $T_{\min}[q^\star,\mathbf{s}(\tau)] > T_{q^\star}$, for all $\mathbf{s}(\tau)$. For, we first prove the following lemma.

Lemma 3.3: Let $q^\star \in \mathcal{Q}_m$ be the value of the discrete variable for which the coefficient[13] b_0 in (22) is negative. Then for each $\xi = 1,\ldots,N$ there exists a positive constant a_ξ such that $-b_\xi/b_0 < a_\xi$. □

[12] Note that $\mathbf{s} \in \mathcal{S}_{m-1}$ implies, in particular, $x = 0$ and $y = 0$ which, in turn, implies $A(\mathbf{s}) = B(\mathbf{s}) = 0$ and $\psi(\mathbf{s},q) = 1$.

[13] For sake of clarity we drop the dependency from k, $\mathbf{s}$ and $q\star$.

Proof: From the expression of the first derivative, it turns out that each b_ξ is the sum of a number of term. If we can prove that the assertion holds for each of these terms, then the assertion is easily proven for b_ξ. First note that, with the given definition of saturation, for every $x \in \mathbb{R}$ and every $\beta \geqslant 1$, $|\mathrm{sat}(x)|^\beta \leqslant |x|$. Consider, then, the first coefficient, namely $c_1(q^\star)^\gamma$. Given the expression of $c_1(q^\star)$ and of $b_0(\mathbf{s}, q^\star)$, we have:

$$-\frac{c_1^\gamma}{b_0} \leqslant \frac{|c_1|^\gamma}{|b_0|} = \frac{|\varphi_1|^\gamma}{|\varphi_1|^{m+1}\left|\sum_{j=1}^m H(j)\right|} = \frac{|\varphi_1|^{\gamma-m-1}}{\left|\sum_{j=1}^m H(j)\right|}.$$

Moreover, $H(j) \leqslant 0$ for all $j = 1, \ldots, m$, hence $\left|\sum_{j=1}^m H(j)\right| = \sum_{j=1}^m |H(j)|$ and we can conclude that

$$-\frac{c_1^\gamma}{b_0} \leqslant \frac{|\mathrm{sat}(H(L(q^\star))|^{\gamma-m-1}}{|D_m|^{\gamma-m-1}\sum_{j=1}^m |H(j)|} \leqslant \leqslant \frac{|\mathrm{sat}(H(L(q^\star)))|^{\gamma-m-1}}{|H(L(q^\star))|} \leqslant 1$$

where the last two inequalities hold since $D_m \geqslant 1$ and $\gamma - m - 1 \geqslant 1$. Analogously, it can be proven that $-c_2^\gamma/b_0 \leqslant 1$.

Consider now, the generic term of the first summation in (21) for a fixed $i < m$ and a fixed $j \in \{1, \ldots, i\}$; recalling the expression of $\mathbf{a}_i$, by simple algebra it follows that

$$(\mathbf{p}_i(j)\mathbf{a}_i r^{i+1})^{\gamma_i-1}\left(\sum_{h=1}^i \rho_i(j,h)c_1^{i-h+1}c_2^h r^i\right) = \frac{r^{(\gamma_i-1)(i+1)+i}}{(i+1)^{\gamma_i-1}}\left(\sum_{h=1}^i \rho_i(j,h)c_1^{i-h+1}c_2^h\right)^{\gamma_i}$$

As a consequence, it will be:

$$-\frac{(\mathbf{p}_i(j)\mathbf{a}_i)^{\gamma_i-1}\left(\sum_{h=1}^i \rho_i(j,h)c_1^{i-h+1}c_2^h\right)}{b_0} \leqslant \leqslant \frac{|\varphi_1|^{\gamma_i(i+1)}\left|\sum_{h=1}^i \rho_i(j,h)\delta^j\right|^{\gamma_i}}{|\varphi_1|^{m+1}\left|\sum_{j=1}^m H(j)\right|} \leqslant \leqslant \frac{|\mathrm{sat}(H(L(q^\star)))|^{\gamma_i(i+1)-m-1}\left|\sum_{h=1}^i \rho_i(j,h)\delta^j\right|^{\gamma_i}}{|D_m|^{\gamma_i(i+1)-m-1}|H(L(q^\star))|} \leqslant 3^{\gamma_i}$$

where the last inequality holds since $D_m \geqslant 1$ and $\gamma_i(i+1) - m - 1 \geqslant 1$ and recalling the definition of $\mathbf{P}_m$. Then, for $i < m$, $-b_{(\gamma_i-1)(i+1)+i}/b_0$ is bounded by a positive constant.

Finally, consider the generic term of the second summation in Equation (21), for a fixed $i \geqslant m$ and a fixed $j \in \{1, \ldots, i\}$, namely

$$\left(\mathbf{p}_i(j)\mathbf{z}_i(\tau)+\mathbf{p}_i(j)\mathbf{a}_i r^{i+1}\right)^{\gamma_i-1}\left(\sum_{h=1}^i \rho_i(j,h)c_1^{i-h+1}c_2^h r^i\right).$$

The first bracketed quantity is, in turn, the sum of different terms of the form

$$b_\zeta(i,\mu) = \binom{\gamma_i-1}{\mu}(\mathbf{p}_i(j)\mathbf{z}_i(\tau))^\mu\left(\mathbf{p}_i(j)\mathbf{a}_i r^{i+1}\right)^{\gamma_i-1-\mu},$$

where $\mu = 0, \ldots, \gamma_i - 1$. For this term the following inequality holds:

$$-\frac{\binom{\gamma_i-1}{\mu}(\mathbf{p}_i(j)\mathbf{z}_i(\tau))^\mu\left(\sum_{h=1}^i \rho_i(j,h)\varphi_1^{i+1}\delta^h\right)^{\gamma_i-\mu}}{(i+1)^{\gamma_i-\mu-1}\varphi_1^{m+1}\sum_{j=1}^m H(j)} \leqslant \leqslant \frac{(\gamma_i-1)!\,|\mathbf{p}_i(j)\mathbf{z}_i(\tau)|^\mu\left|\sum_{h=1}^i \rho_i(j,h)\delta^h\right|^{\gamma_i-\mu}}{\left(\frac{D_m}{\mathrm{sat}(H(L(q^\star)))}\right)^{(i+1)(\gamma_i-\mu)-m-1}\left|\sum_{j=1}^m H(j)\right|}.$$

Now, since we are considering $i \geqslant m$ and $\mu \leqslant \gamma_i - 1$ it is always $(i+1)(\gamma_i-\mu)-m-1 \geqslant 0$; the equality holds only if $i = m$ and $\mu = \gamma_m - 1$ but these values imply that the corresponding exponent of r is m, which means that we are taking into account $b_0(\mathbf{s}, q^\star)$; as a consequence, for all $b_\xi(\mathbf{s}, q^\star)$ with $\xi \neq 0$ it must be either $i > m$ or $\mu < \gamma_i - 1$ which means $(i+1)(\gamma_i-\mu)-m-1 \geqslant 1$; this, in turn, due to the definition of D_m, yields $|\mathbf{p}_i(j)\mathbf{z}_i(\tau)|^\mu \leqslant D_m^{(i+1)(\gamma_i-\mu)-m-1}$. Moreover, as $\left|\sum_{j=1}^m H(j)\right| \geqslant \sum_{j=1}^m |H(j)| \geqslant |H(L(q^\star))|$,

$$|\mathrm{sat}(H(L(q^\star)))|^{(i+1)(\gamma_i-\mu)-m-1} \leqslant \left|\sum_{j=1}^m H(j)\right|.$$

In conclusion,

$$-\frac{b_\zeta(i,\mu)}{b_0} \leqslant (\gamma_i-1)!\left|\sum_{h=1}^i \rho_i(j,h)\delta^h\right|^{\gamma_i-\mu} \leqslant (\gamma_i-1)!\,3^{\gamma_i-\mu}$$

which is a positive constant. Then, for every $\xi = 1, \ldots, N$ the existence of a_ξ is proven. ■

These Lemmas are now used to prove the existence of a lower bound for $T_{\min}^V(q^\star, \mathbf{s}(\tau))$ independent from $\mathbf{s}(\tau)$.

Theorem 3.4: If the control law applied to system (9) is as in (15) and (16) and V is as in (17), then for all $\mathbf{s}(\tau) \in \mathcal{S}_{m-1}\backslash\mathcal{S}_m$, there exist $q^\star \in \mathcal{Q}_m$ and $T(m) > 0$ such that $T_{\min}^V(q^\star, \mathbf{s}(\tau)) > T(m)$. □

Proof: Lemma 3.3 guarantees that there exist positive constants $a_1, \ldots, a_N$ such that $b_0 a_\xi > -b_\xi$. Consider, now, the function $f(r) = -1 + a_1 r + \ldots + a_N r^N$. It is easy to see that $f(0) = -1$ and that $\lim_{r\to+\infty} f(r) = +\infty$; so there exists at least one $r \in (0, +\infty)$ such that $f(r) = 0$. Let $T(m) \triangleq \min\{r \in (0,+\infty) \,|\, f(r) = 0\}$. Obviously, $f(r) < 0$ for all $r \in (0, T(m))$; by using the inequalities $b_0 a_\xi > -b_\xi$ and multiplying by $-b_0 r^m$, one obtains $r^m(b_0 + b_1 r + \ldots + b_N r^N) < 0$, for all $r \in (0, T(m))$, hence the claim. ■

Then the switching control scheme is completely specified by choosing in (10) $\kappa = \alpha \min_m\{T(m)\}$, with $0 < \alpha < 1$.

Finally we show that the functions

$$\eta_0(\mathbf{s}) = -x^\gamma - \left(y^{\gamma-1} + \sum_{i=1}^p \sum_{h=1}^i S_{i,h}(\mathbf{s})\right)^2, \qquad (23)$$

$$\eta_q(\mathbf{s}) = \varphi_1(\mathbf{s}, q)^{m(q)+1}\sum_{j=1}^{m(q)} H(\mathbf{s}, j, q), \quad q \in \mathcal{Q}_{m^\star} \qquad (24)$$

fulfill (6) and the conditions (C1), (C2) and (C3) of Theorem 2.1. By the reasoning made in the proof of Lemma 3.2 it is clear that the η's are negative semidefinite and that condition (6) holds.

Now, define $M = p$, $\mathcal{Q}_0^* = \{\mathbf{0}\}$ and, for $m = 3, ..., M$, $\mathcal{Q}_m^* = \{1\}$ as in Equation (14) and note that $\mathcal{X}_1 = \mathcal{S}_0 \backslash \{\mathbf{0}\}$ and that for all $j = 1, \dots, p$, $\mathcal{X}_j = \{\mathbf{s} \in \mathbb{R}^n \backslash \{\mathbf{0}\} : \mathbf{z}_{j-1} = \mathbf{0}\}$. Hence $\bigcap_{i=1}^{j} \mathcal{X}_i = \mathcal{S}_{j-1}$. The following two cases are then in order. For all $\mathbf{s}(\tau_k) \in \mathbb{R}^n$, $\lim_{t \to \tau_k^+} [\dot{V}(t)|_{q=0}] = \eta_0[\mathbf{s}(\tau_k)]$ (see Equations (19) and (23)), hence C1 holds. For all $\mathbf{s}(\tau_k) \in \bigcap_{i=1}^{j} \mathcal{X}_i$, $\lim_{t \to \tau_k^+} [\frac{d^{j+1}V}{dt^{j+1}}|_{q \in \mathcal{Q}_j^*}] = \eta_q(\mathbf{s}(\tau_k))$ (see the proof of Lemma 3.2), hence C2 holds.

Finally, we need to show that Condition C3 holds. For, note that, by construction, if $\mathbf{X}$ is not a subset of $\bigcup_i \mathcal{X}_i$, then $\mathcal{Q}_{\mathbf{X}} = \emptyset$ and (C3) holds obviously. On the other hand, if $\mathbf{X} \subset \mathcal{X}_1 \backslash \mathcal{X}_2$, then, for all $\mathbf{s} \in \mathbf{X}$, $\eta_1(\mathbf{s}) < 0$ and, by Lemma 3.2 and Theorem 3.4 $T_{\min}(1, \mathbf{s}) > T_D$ for all $\mathbf{s} \in \mathbf{X}$. By continuity of V, there exists $\varepsilon_{\mathbf{X}}$ such that if $\|\mathbf{s}(\tau_k) - \bar{\mathbf{s}}\| < \varepsilon_{\mathbf{X}}$ for some $\bar{\mathbf{s}} \in \mathbf{X}$ and some k, then $T_{\min}(1, \mathbf{s}(\tau_k)) > T_D$. Hence the switching strategy (13) guarantees that $q_k = 1$.

In general, if $\mathbf{X} \subset \bigcup_{i=1}^{j} \mathcal{X}_i \backslash \mathcal{X}_{j+1}$ then for all $\mathbf{s} \in \mathbf{X}$ and for all $q \in \mathcal{Q}_j^*$ $\eta_q(\mathbf{s}) < 0$. Again, by Lemmas 3.2 and Theorem 3.4 $T_{\min}(q^\star, \mathbf{s}) > T_D$ for some $q^\star \in \mathcal{Q}_j^*$ and by continuity of V there exists $\varepsilon_{\mathbf{x}}$ such that if $\|\mathbf{s}(\tau_k) - \bar{\mathbf{s}}\| < \varepsilon_{\mathbf{X}}$ for some $\bar{\mathbf{s}} \in \mathbf{X}$ and some k, then $T_{\min}(q^\star, \mathbf{s}(\tau_k)) > T_D$. Hence the switching strategy (13) guarantees that $q_k = q^\star$. Then condition (C3) holds.

We have then proven the following result.

Theorem 3.5: Consider system (9). Let the control law be determined according to (15) and (16). Let the discrete state q of the FSA be updated according to the switching strategy (13). Then the equilibrium of system (9) is globally asymptotically stable. $\square$

IV. CONCLUDING REMARKS

We have designed a "hybrid" control scheme to asymptotically stabilize the equilibrium of a class of non-holonomic nonlinear systems. The main idea of the proposed tools lies on the fact that when the first-order time-derivative of the candidate Lyapunov function is zero, higher-order derivatives are considered in the design of the switching algorithm. The proposed switching strategy has been proven to globally asymptotically stabilize the zero equilibrium by means of *saturated* control signals.

APPENDIX

The following lemma is an extension of Barbalat's lemma and can easily be proven.

Lemma A.1: Suppose that a function $f : \mathbb{R} \to \mathbb{R}$ is $\mathcal{C}^2$ in $\mathbb{R} \backslash X_d$ where $X_d \triangleq \{x_1, x_2, x_3 \dots\}$ is such that $x_1 < x_2 < x_3 < \dots$ and there exists $\rho > 0$ such that, for all $k \in \mathbb{Z}^+$, $x_{k+1} - x_k \geqslant \rho$. If there exists $\lim_{x \to +\infty} f(x) = l < \infty$ and if there exists L such that $|f''(x)| < L$, for all $x > 0$, then $\lim_{k \to \infty} \left[\lim_{x \to x_k^+} \frac{df}{dx}\right] = 0$. $\square$

More in general, the following result holds.

Lemma A.2: Suppose that a function $f : \mathbb{R} \to \mathbb{R}$ is $\mathcal{C}^M$ in $\mathbb{R} \backslash X_d$. If there exists $\lim_{x \to +\infty} f(x) = l < \infty$ and if for all $m \in \{1, \dots, M\}$ there exists L_m such that $\left|\frac{d^{m+1}f}{dx^{m+1}}\right| < L_m$, for all $x > 0$, then for all $m \in \{1, \dots, M\}$ $\lim_{k \to \infty} \left(\lim_{x \to x_k^+} \frac{d^m f}{dx^m}\right) = 0$. $\square$

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