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Generalized α-fair resource allocation in wireless networks

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Abstract— **The classical water filling problem is concerned with optimally assigning powers over n independent channels so as to maximize the total transmitted throughput. If each channel is associated with another mobile then it is natural to consider also the problem of fair assignment and to study tradeoffs between fairness and optimality. The object which is allocated is the transmission power, and we are interested in assigning it so as to obtain fairness between either one of three resulting performance measures: the signal to noise ratio, a shifted version of it, or the Shannon capacity. We suggest the generalized α-fairness concept. We obtain explicit solutions for and insight on the fair assignment corresponding to the various performance measures. For the case of a large number of users we consider a variational formulation of the problem. The variational formulation allows us to design distributed resource allocation algorithms.**

I. INTRODUCTION

Fairness concepts have been playing a central role in networking. In the ATM standards [13], the maxmin fairness and its weighted versions appear as the way to allocate throughput to connections using the ABR (Available Bit Rate) best effort service. The proportional fairness has been introduced in [6], [7]. Later it was implemented in wireless communications (e.g. in the Qualcomm High Data Rate (HDR) scheduler) as a way to allocate throughputs (through time slots); it has also been shown to correspond to the way that some versions of the TCP Internet Protocol share bottleneck capacities [10]. A unifying mathematical formulation to fair throughput assignment (which we call the "α-fairness") has been proposed in [11]; the "degree" of fairness is expressed by a parameter α defined on the whole half line $[0,\infty)$; it controls the tradeoff between efficiency (total throughput maximization) on one hand, and fairness, on the other. In particular, the case $\alpha \to \infty$ corresponds to the maxmin fairness (that can be considered to be the most fair allocation), the case $\alpha = 2$ corresponds to the delay minimization, the case $\alpha \to 1$ corresponds to the proportional fair assignment and the case $\alpha = 0$ corresponds to the throughput maximization (that can be considered to be the most efficient).

The above notions have been defined in the telecommunication context for splitting a given available capacity between connections. Fairness can, however, be defined with respect to a utility function f of the resource that is allocated (e.g. throughput). Moreover, the set of possible allocations may be more complex than those corresponding to the split of a fixed quantity. As an example, if we split the throughput of a CDMA link, then the sum of throughputs may itself be a function of the allocation. Indeed, when more than one mobile shares a radio link then the interference adds to the noise at each terminal thus decreasing the SNR (Signal to Noise Ratio) and hence the throughput. This more general context of fair resource allocation has been introduced in the game theory literature already in [12] as the Nash Bargaining concept (of which the proportional fairness is a special case). It was extended to the context of α-fairness in [14].

We investigate and compare in this paper generalized α-fairness resource allocations in downlink cellular networks, related to the α-fairness applied to the SNR, to the shifted SNR and to the throughput. All three can be viewed as utilities (of the power assignments) that we wish to assign fairly. We study and compare the properties of the various fairness criteria. We show that in all three cases the fairness improves monotonously as α goes to infinity. The generalized α-fairness applied to SNR and the shifted SNR admit explicit solutions. The α-fair sharing of the shifted SNR incorporates as particular cases: the SNR maximization ($\alpha = 0$), the Shannon capacity maximization ($\alpha = 1$) and the max-min fairness ($\alpha \to 0$).

The paper is organized in two parts. In the first part we formulate the generalized α-fair resource allocations as convex optimization problems. It turns out that many of these optimization problems have explicit solutions. In the second part, using the law of large numbers, we approximate the generalized α-fair resource allocation problems for the case of many users by variational problems. The variational formulation allows us to design distributed power control algorithms. The example of the Rayleigh fading is analyzed in details.

II. DISCRETE MODEL

We consider the following allocation problem. There is a single decision maker (the Base Station) that decides how to allocate the power between n different users. We further allow there to be a weight of π_i related to the resource of user i. There is a gain parameter h_i related to the channel gain to mobile i.

Possible interpretations:

(i) The Base Station (BS) transmits to the mobiles simultaneously using independent channels, e.g. different directional antennas or frequency bands (e.g. as in OFDM, where one should assign different power levels

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for different sub-carriers [16]). In that case we may take $\pi_i = 1/n$ for all i.

(ii) The BS transmits to the mobiles using a general periodic polling order. The BS spends a fraction π_i of a cycle to transmit to mobile i. There is a hard constraint on the total energy available during the cycle. The problem is how to assign the available transmission energy per cycle among the mobiles.

The strategy of the decision maker is $x = (x_1, \ldots, x_n)$ such that $x_i \geq 0$ for $i \in [1,n]$ and $\sum_{i=1}^{n} \pi_i x_i = \bar{x}$, where $\bar{x} > 0$ and $\pi_i \geq 0$ for $i \in [1,n]$. The element x_i corresponds to a power level assigned to the i-th user.

As a payoff to the decision maker we take the *generalized α-fairness* utility function ($\alpha \in [0,\infty)$):

$$v(x) = \frac{1}{1-\alpha} \sum_{i=1}^{n} \pi_i \left(f_i(x_i)\right)^{1-\alpha}, \text{ for } \alpha \neq 1, \tag{1}$$

and

$$v(x) := \sum_{i=1}^{n} \pi_i \log(f_i(x_i)), \text{ for } \alpha = 1, \tag{2}$$

where f_i is concave and increasing function in $[0,\infty)$.

Note that even though the optimal solution of (1) converges to the optimal solution of (2) when $\alpha \to 1$, the objective function is discontinuous at $\alpha = 1$. If one wants to deal with an objective function continuous in α, we suggest to make the following small modification:

$$v(x) := \frac{1}{1-\alpha} \sum_{i=1}^{n} \pi_i \left(\left(f_i(x_i)\right)^{1-\alpha} - 1 \right).$$

There is the following interpretation to the above payoff: the decision maker wants to share fairly (in the sense of α-fairness) some function of the resource x. For example, in the context of the downlink power allocation problem in wireless networks, we could wish to share fairly the utility of a throughput instead of sharing fairly the available power. Is the theory that applies to sharing fairly x applicable to sharing $f(x)$?

In [6], [7] and in [11], a fixed amount C of resource is shared. The sum of shares does not depend on the way the resource is shared. In contrast, unless f is linear, the sum $\sum_{n=1}^{N} f_n(x_n)$ will no more be constant.

If f is concave, then the set of $f(x)$ obtained over all x such that the sum of its components is C (or is less than or equal to C) is a convex set. But, more generally, if x belongs to a convex set $\mathbf{X}$ then the set $\mathbf{F} := \{f(x) : x \in \mathbf{X}\}$ is a convex set. Thus we can view the fair allocation of the utility $f(x)$ over $\mathbf{X}$ as the fair allocation of f over $\mathbf{F}$, if f is concave. We note that if f is strictly concave and $\mathbf{X}$ is convex then for each α, there is a unique α-fair assignment.

In the present work we shall consider the following three assignments for $f_i(x)$:

$$\log(1 + h_i x/N_i^0), \quad h_i x/N_i^0, \quad 1 + h_i x/N_i^0,$$

where h_i is the fading coefficient for the sub-carrier i and N_i^0 is the level of the background noise in the sub-carrier i. Usually it is supposed that the background noise in all the sub-carriers is the same, so in that case $N_i^0 = N^0$.

The first choice $f_i(x) = \log(1 + h_i x/N_i^0)$ corresponds to the α-fair assignment of throughputs in a downlink CDMA system. The second choice $f_i(x) = h_i x/N_i^0$ corresponds to the α-fair assignment of SNRs. In the case $f_i(x) = 1 + h_i x/N_i^0$ we assign the shifted SNR according to the α-fairness. The latter choice allows us to treat the SNR maximization, the throughput maximization and the maxmin fairness as particular cases of a unified spectrum of optimization problems.

First we study each criterion separately and then we discuss the relation among them.

A. General results

First we note that the first and second assignments correspond to the maximization of the following objective function

$$v(x) = \frac{1}{1-\alpha} \sum_{i=1}^{n} \pi_i U^{1-\alpha}(x_i h_i / N_i),$$

subject to

$$\sum_{i=1}^{n} \pi_i x_i = \bar{x}. \tag{3}$$

and the third assignment corresponds to a shifted version

$$v(x) = \frac{1}{1-\alpha} \sum_{i=1}^{n} \pi_i (U^{1-\alpha}(x_i h_i / N_i) - 1),$$

This change by shift influences only the payoff but not the optimal strategy. For our three cases, we have $U(\tau) = \log(1+\tau), \tau, 1+\tau$. So, $U(\tau)$ is either linear in τ; or $U(\tau)$ is strictly increasing and positive in $(0,\infty)$, $U'(\tau)$ is strictly decreasing, $U(0)$=0, $U(+\infty) = +\infty$ and $U'(0+) < +\infty$.

Define the Lagrangian

$$L_\omega(x) := \frac{1}{1-\alpha} \sum_{i=1}^{n} \pi_i U(x_i h_i / N_i)^{1-\alpha} + \omega(\bar{x} - \sum_{i=1}^{n} \pi_i x_i).$$

Since the optimization problem is convex, the α-fair assignment is obtained by taking the derivative of the Lagrangian L_ω. Thus, the optimal strategy has the form $x_i(\omega) = T(\omega, h_i/N_i^0)$:

(a) if U is nonlinear and $\alpha > 0$ then $T(\omega,\xi)$ with $\xi > 0$ is the positive root of the equation $F(x,\xi) = \omega$ where $F(x,\xi) = \xi U'(x\xi)/U^\alpha(x\xi)$ which exists and unique since $F(0+,\xi) = +\infty$, $F(+\infty,\xi) = 0$ and $F(\cdot,\xi)$ is strictly decreasing. So,

$$T(\omega,\xi) = \frac{1}{\xi}(U'/U^\alpha)^{(-1)}(\omega/\xi). \tag{4}$$

The case $\alpha = 0$ is a particular one since $F(0+,\xi) < \infty$. In this case, we have

$$T(\omega,\xi) = \frac{1}{\xi}\left[(U')^{(-1)}(\omega/\xi)\right]_+.$$

(b) if U is linear then for $U(0) > 0$, $T(\omega,\xi)$ has the form:

$$T(\omega,\xi) = \frac{1}{\xi}\left[(U'/U^\alpha)^{(-1)}(\omega/\xi)\right]_+. \tag{5}$$

If $U(0) = 0$ then $T(\omega,\xi)$ is given by (4).

The Lagrangian multiplier ω can be found as the unique positive root of the following equation

$$H(\omega) := \sum_{i=1}^{n} \pi_i T(\omega, h_i/N_i^0) = \bar{x}.$$

B. Alpha-fair sharing of throughput

The α-fair sharing of the throughput corresponds to the maximization of the following objective function

$$v(x) = \frac{1}{1-\alpha}\sum_{i=1}^{n} \pi_i \log^{1-\alpha}(1 + h_i x_i/N_i^0),$$

subject to (3). Thus, the optimal strategy has the form $x_i(\omega) = T(\omega, h_i/N_i^0)$ where $T(\omega,\xi)$ with $\xi > 0$ is the unique positive root of the equation:

$$F(x,\xi) := \frac{\xi}{1+x\xi}\frac{1}{\log^{\alpha}(1+x\xi)} = \omega,$$

where ω can be found as the unique positive root of the following equation

$$H(\omega) := \sum_{i=1}^{n} \pi_i T(\omega, h_i/N_i^0) = \bar{x}.$$

The case $\alpha = 0$ corresponds to the water filling problem. Namely, the optimal strategy takes the form $x_i(\omega) = [1/\omega - N_i^0/h_i]_+$ and the optimal $\omega = \omega_*$ is defined as the unique root of the equation

$$\sum_{i=1}^{n} \pi_i [1/\omega - N_i^0/h_i]_+ = \bar{x}.$$

C. Alpha-fair sharing of SNR

The SNR α-fair sharing corresponds to the maximization of the following objective function

$$v(x) = \frac{1}{1-\alpha}\sum_{i=1}^{n} \pi_i (h_i x_i/N_i^0)^{1-\alpha},$$

subject to (3). Thus, the optimal strategy is given as follows:

$$x_i = \bar{x}\frac{(h_i/N_i^0)^{1/\alpha-1}}{\sum_{j=1}^{n} \pi_j (h_j/N_j^0)^{1/\alpha-1}}.$$

For $\alpha \to 1$ we get $x_i = \bar{x}$. It is interesting to note that in the case of the proportional fair assignment of SNR the optimal solution does not depend on the ratios h_i/N_i^0. The α-fair assignment gives strictly positive power for all $\alpha > 0$. For $\alpha = 0$, only mobiles with maximum h_i/N_i^0 receive positive assignment. For $\alpha \to \infty$ we obtain

$$x_i = \bar{x}\frac{N_i^0/h_i}{\sum_{j=1}^{n}(\pi_j N_j^0/h_j)},$$

so that for all i the same value of SNR is obtained:

$$SNR_i = \frac{\bar{x}}{\sum_{j=1}^{n}(\pi_j N_j^0/h_j)}.$$

We note that the ratio

$$\frac{SNR_i}{SNR_j} = a^{1/\alpha} \text{ with } a := \frac{h_i/N_i^0}{h_j/N_j^0},$$

is monotonuously decreasing to one if $a > 1$ and it is monotonuously increasing to one if $a < 1$. Therefore, we can say that there is monotone improvement of fairness when α increases to infinity.

D. Alpha-fair sharing of shifted SNR

In the case of the shifted SNR α-fair sharing, we maximize the following objective function

$$v(x) = \frac{1}{1-\alpha}\sum_{i=1}^{n} \pi_i \left(\left(1 + \frac{h_i x_i}{N_i^0}\right)^{1-\alpha} - 1\right)$$

subject to (3). In two important particular subcases the objective corresponds to the SNR maximization, $\alpha = 0$, and the throughput (Shannon capacity) maximization, $\alpha = 1$. Namely, we have

$$v(x) = \sum_{i=1}^{n} \pi_i \ln\left(1 + \frac{h_i x_i}{N_i^0}\right) \text{ for } \alpha = 1$$

and

$$v(x) = \sum_{i=1}^{n} \pi_i \frac{h_i x_i}{N_i^0} \text{ for } \alpha = 0.$$

In addition to the above observation note that the definition of $v(\cdot)$ is closely related to the definition of Tsallis entropy [15]. The Tsallis entropy is defined for $\alpha > 0$ as follows

$$H(x) = \frac{1}{1-\alpha}\left(\sum_{i=1}^{n} x_i^{1-\alpha} - 1\right)$$

and $H(x)$ reduces to Shannon entropy for $\alpha \to 1$.

When $\alpha > 0$, the optimal solution has the form $x(\omega^*) = (x_1(\omega^*), \ldots, x_n(\omega^*))$ where

$$x_i(\omega) = \frac{N_i^0}{h_i}\left[\left(\frac{h_i}{\omega N_i^0}\right)^{1/\alpha} - 1\right]_+ \text{ for } i \in [1,n]$$

and ω^* is the unique root of the equation

$$H(\omega) := \sum_{i=1}^{n} \pi_i x_i(\omega) = \bar{x}.$$

When $\alpha = 0$, the utility function is linear in x, and then the problem has the solution as in the case of the SNR α-fair sharing with $\alpha = 0$. Namely, only mobiles with maximal h_i/N_i^0 receive positive assignment. It is worthy to note that the solution is not unique if there are a few users with the same maximal ratios h_i/N_i^0.

Now we investigate the case when $\alpha > 0$. Without loss of generality, we can assume that the users are arranged by the following ratios in the decreasing order:

$$h_1/N_1^0 \le h_2/N_2^0 \le \ldots \le h_n/N_n^0.$$

Intuitively, we expect that the decision maker gives a higher power level to a user with a larger index. We also note that some users for some values of α can be assigned zero power level. In particular, this can happen if the decision maker uses the Shannon capacity ($\alpha = 1$) as the objective function. This provides a further motivation to use the generalized α-fairness criterion with larger values of α. Thus, we are interested to determine which users obtain zero power level for a given value of α. For this purpose developing approach suggested in [1] we provide an explicit solution to the shifted SNR α-fair sharing problem. Namely, we can prove that the solution to the shifted SNR α-fair sharing problem is given by

$$x_i^* = \begin{cases} \dfrac{\bar{x} + \sum_{t=k}^{n} \pi_t \dfrac{N_t^0}{h_t}\left(1 - \left(\dfrac{N_i^0 h_t}{h_i N_t^0}\right)^{1/\alpha}\right)}{\sum_{t=k}^{n} \pi_t \left(\dfrac{N_i^0 h_t}{h_i N_t^0}\right)^{1/\alpha - 1}}, & i \in [k,n], \\ 0, & \text{otherwise}, \end{cases} \quad (6)$$

where k can be found from the conditions

$$\varphi_k < \bar{x} \le \varphi_{k-1}, \quad (7)$$

where

$$\varphi_t = \sum_{i=t}^{n} \pi_i \frac{N_i^0}{h_i}\left[\left(\frac{N_t^0 h_t}{N_i^0 h_i}\right)^{1/\alpha} - 1\right] \text{ for } t \in [1,n], \; \varphi_0 = \infty. \quad (8)$$

We can consider φ_t as a function $\varphi_t(\alpha)$ on α. It is clear that $\varphi_t(\alpha)$ is decreasing on α and

$$\varphi_t(0) = \infty \text{ and } \varphi_t(\infty) = 0.$$

Thus, for each $t \in [1,n]$ there is unique positive α_t such that $\varphi_t(\alpha_t) = \bar{x}$ and $0 = \alpha_1 < \alpha_2 < \dots$. Then, we have the following result establishing when the optimal strategy assigns non-zero power level to k users.

Proposition 1: For α-utility function with $\alpha \in [\alpha_k, \alpha_{k+1})$, $k \in [1, n-1]$ there are k users with non-zero power level. If $\alpha > \alpha_n$ all users obtain non-zero power levels.

The next proposition describes the limiting case when $\alpha \to \infty$.

Proposition 2: If $\alpha \to \infty$ then the optimal strategy tends to $(x_1^*, \dots, x_n^*)$ where

$$x_i^* = \frac{\bar{x} N_i^0}{\sum_{t=1}^{n} \pi_t N_t^0} \text{ for } i \in [1,n].$$

Proposition 2 implies that the SNRs x_i^*/N_i^0 have the same value for each user i. In turn the latter implies that the Shannon capacities of the users are equal. Besides, since the users are arranged in increasing order, the components of the optimal strategy are also arranged in increasing order. Recall that when α tends to zero the elements of the optimal solution are all zero except the elements corresponding to the minimal noise level and for small α the components of the optimal strategy are arranged in decreasing order. Therefore by changing parameter α we can tune the α-utility function to represent a large variety of criteria from complete fairness to pure efficiency maximization (which results in complete unfairness).

E. Numerical example for discrete model

Let us demonstrate the closed form approach by a numerical example. Suppose that $h_i = i$ for $i = 1, ..., n$ and $N_i^0 = 1$. And let $\pi_i = C\kappa^{i-1}$ with $\kappa > 0$ and $C = (\kappa - 1)/(\kappa^n - 1)$. Assume that $\bar{x} = 5$, $n = 5$, $\kappa = 0.7$ and $\alpha = 0.5$. Then, as the first step we calculate φ_t for $t \in [1,5]$. In our case we get (13.298, 1.979, 0.422, 0.075, 0). Then, by (7), $k = 2$. Thus, by (6), the optimal water-filling strategy is $x^* = (0, 0.400, 1.017, 1.551, 2.051)$ with payoff 10.419.

Now, we will demonstrate how the optimal strategy depends on α. Namely, let $\alpha = 0, 0.5, ..., 2.5, 3$. Then, the optimal strategies are given in Figure 1. Thus, the optimal strategies with change of α continuously changes from (0,0, 0, 0, 7.510) with payoff equals 25 for $\alpha = 0$ through (0.491, 0.723, 0.756, 0.753, 0.741) with payoff equals 5.546 for $\alpha = 1.4$ to (1.095, 0.547, 0.365, 0.274, 0.219) with zero payoff for $\alpha = \infty$. For the Shannon capacity the optimal strategy is (0.244, 0.744, 0.911, 0.994, 1.044) with payoff 5.952.

Also, we calculate that $\alpha_1 = 0$, $\alpha_2 = 0.312$, $\alpha_3 = 0.519$, $\alpha_4 = 0.684$, $\alpha_5 = 0.821$. Thus, if we prefer that all the sub-carriers were in use we have to set α to a value larger than 0.821.

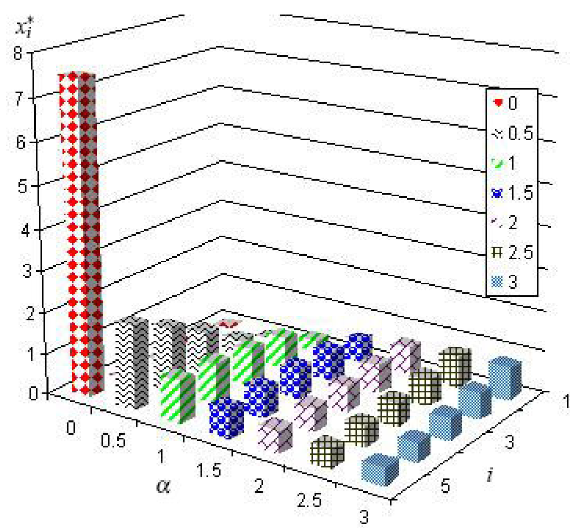

Fig. 1. Dependence of the optimal strategies on α

F. Relation among different alpha-fair allocations

Let us discuss the relation among different α-fair allocations. In the next proposition we discuss the important particular cases: $\alpha = 0$, $\alpha \to 1$, $\alpha = 2$ and $\alpha \to \infty$.

Proposition 3: (i) For $\alpha = 0$, with the function $f_i(x) = \log(1 + h_i x/N_i^0)$, the throughput (Shannon capacity) is maximized. With the the functions $f_i(x) = h_i x/N_i^0$ and $f_i(x) = 1 + h_i x/N_i^0$, SNR is maximized.

(ii) For $\alpha \to 1$, with the function $f_i(x) = \log(1 + h_i x/N_i^0)$, the throughputs are assigned according to the proportional fair paradigm. With the function $f_i(x) = h_i x/N_i^0$, the SNRs are assigned proportionally fair. And with $f_i(x) = 1 + h_i x/N_i^0$, the throughput is maximized.

(iii) For $\alpha = 2$, with the function $f_i(x) = \log(1 + h_i x/N_i^0)$ the delay is minimized.

(iv) For $\alpha \to \infty$, the allocation under all three utilities is the same and corresponds to the maxmin fair assignment of SNRs.

We note that

$$\log(x) < \log(1 + x) < x.$$

Moreover, for the left hand bound becomes tight for large x and the right hand side becomes tight for small x. This suggests that maximizing the sum of the Shannon capacity is equivalent to maximizing the sum of SNR at a regime of low SNRs, and is equivalent to the proportional fair assignment of the SNRs at the regime of large SNRs.

Curiously enough, in the case of the SNR proportional fair sharing the optimal allocation does not depend on the ratios h_i/N_i^0.

Next, for each α-fair allocation we plot Jain's fairness index

$$J = \frac{(\sum_{i=1}^n SNR_i)^2}{n(\sum_{i=1}^n SNR_i^2)}$$

as a function of α (see Figure 2). We calculate the Jain's index with respect to SNRs.

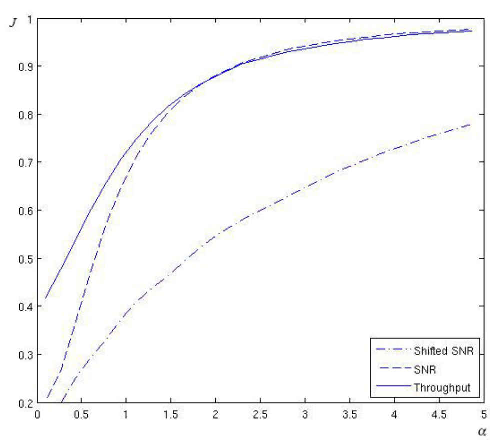

Fig. 2. Jain's fairness index as a function of α.

We can see that the Jain's fairness index improves monotonously in the all three cases of the generalized α-fair resource allocation.

Both SNR α-fair sharing and the shifted SNR α-fair sharing have the same limiting cases. If $\alpha \to 0$, they correspond to the SNR maximization, and if $\alpha \to \infty$ they correspond to maxmin fair sharing. However, as one can see from Figure 2, the shifted SNR α-fair sharing provides a finer tuned scale of the resource allocations.

III. Variational model

Consider the case when $\pi_i = 1/n$ for $i = 1, \ldots, n$, the gain coefficients h_i are i.i.d. random variables distributed with the density function $\sigma(\cdot)$, $N_i^0 = N^0$ (without loss of generality, we take $N^0 = 1$), and the number of users is large. Then, according to the law of large numbers, the objective

$$v(x) = \frac{1}{1-\alpha} \sum_{i=1}^n \pi_i \left((f_i(x_i))^{1-\alpha} - 1 \right),$$

can be replaced by the expectation

$$\frac{1}{1-\alpha} E\{(f_i(x_i))^{1-\alpha} - 1\} =$$
$$\frac{1}{1-\alpha} \int_0^\infty \sigma(h)[(f(hx(h)))^{1-\alpha} - 1]dh.$$

The constraint

$$\sum_{i=1}^n \pi_n x_n = \bar{x}$$

can be respectively replaced by

$$\int_0^\infty \sigma(h) x(h) dh = \bar{x}. \qquad (9)$$

Now the optimal strategy depends on the gain coefficient h. This allows us to design distributed algorithms that do not require the knowledge of the gains for all the users. We also do not need anymore to know the number of mobiles.

A. Alpha-fair sharing of throughput

In the case of the variational version of the throughput α-fair sharing, we maximize the following objective function

$$v(x) = \frac{1}{1-\alpha} \int_0^\infty \sigma(h) \log^{1-\alpha}(1 + hx(h))\, dh,$$

subject to (9).

The problem has the unique optimal solution for positive α and it is of the form $x(h, \omega^*)$ where $x(h, \omega)$ is the unique positive root of the equation:

$$F(x,h) := \frac{h}{1 + xh} \frac{1}{\log^\alpha(1 + xh)} = \omega, \qquad (10)$$

and ω^* is the unique root of the equation

$$\int_0^\infty \sigma(h) x(h, \omega) = \bar{x}.$$

B. Alpha-fair sharing of SNR

In the case of the variational version of SNR α-fair sharing, we maximize the following objective function

$$v(x) = \frac{1}{1-\alpha} \int_0^\infty \sigma(h) \left(hx(h)\right)^{1-\alpha} dh.$$

The problem has the unique optimal solution for positive α given as follows:

$$x(h) = h^{1/\alpha - 1} \frac{\bar{x}}{\int_0^\infty \sigma(t) t^{1/\alpha - 1} dt}. \qquad (11)$$

In the important particular case when h is distributed according to the Rayleigh's law $\sigma(t) = \kappa \exp(-\kappa t)$ with $\kappa > 0$, we have

$$x(h) = h^{1-1/\alpha} \frac{\bar{x} \kappa^{1-1/\alpha}}{\Gamma(1/\alpha)}.$$

C. Alpha-fair sharing of shifted SNR

In the case of the variational version of α-fair sharing of shifted SNR, we maximize the following objective function

$$v(x) = \frac{1}{1-\alpha}\int_0^\infty \sigma(h)\left((1+hx(h))^{1-\alpha}-1\right)dh,$$

The problem has the unique optimal solution for positive α and it is of the form $x(h,\omega_*)$ where

$$x(h,\omega^*) = \frac{1}{h}\left[\left(\frac{h}{\omega^*}\right)^{1/\alpha}-1\right]_+ \quad \text{for } h\in[0,\infty) \qquad (12)$$

and ω^* is the unique root of the equation

$$H(\omega) := \int_\omega^\infty \frac{\sigma(h)}{h}\left[\left(\frac{h}{\omega}\right)^{1/\alpha}-1\right]dh = \bar{x}. \qquad (13)$$

Let us now establish some properties of the optimal solution. We will show that the optimal strategy has some monotonous property, namely, that for small α it is increasing function, but for large α there is a switching point between increasing and decreasing branches of the optimal strategy and this switching point is very closely related to the base of the natural logarithm.

Proposition 4: The optimal strategy has the following monotonous properties:
(a) if $\alpha \le 1$ then the optimal strategy $x(h,\omega)$ is increasing on h in $[\omega,\infty)$,
(b) $\alpha > 1$ then the optimal strategy $x(h,\omega)$ is increasing on h in $[\omega, z_\alpha\omega)$ and strictly decreasing in $[\omega z_\alpha,\infty)$, where $z_\alpha = (\alpha/(\alpha-1))^\alpha$.

This optimal strategy has the following very nice property connecting limit switching point and the base of the natural logarithm. Namely, $z_\alpha \to e$ as $\alpha\to\infty$.

Proposition 5: For any positive ω one can find an $\alpha(\omega)$ utility function such that the optimal strategy will employ all the sub-carries from the interval (ω,∞) and will not employ any the sub-carries from the interval $[0,\omega]$.

D. Distributed approach for alpha-fair sharing allocations

We note that in the variational model the optimal strategy depends only on the parameter α, the gain coefficient h, the level of the background noise N^0 and the total available power resource $\bar{x}$. This observation allows us to design distributed algorithms for the implementation of the generalized α-fair resource allocation. There is a particular need for distributed algorithms in Ad Hoc, Sensor and Mesh networks, see e.g. [2], [4], [9], [8]. We suggest here two variants of the distributed algorithm. In the first variant, we assume that the nodes are capable of arithmetic operation and the Base Station trusts them so that it conveys them the total available power resource $\bar{x}$ and the parameter α. In this variant the nodes themselves compute the optimal strategy either by solving equation (10) or by using formula (11) or (12) depending on the choice of the objective function. In the second variant, the Base Station computes the optimal strategy as a function of the gain coefficient and distributes this strategy to all nodes. The advantages of the second variant is that the Base Station is not obliged to disclose the value of the parameter α and the total available power resource and the nodes do not need to perform arithmetic operations for the computation of the optimal strategy.

IV. Conclusions

The classical water filling problem is concerned with optimally assigning powers over n independent channels so as to maximize the total transmitted throughput. If each channel is associated with another mobile then it is natural to consider also the problem of fair assignment and study tradeoffs between fairness and optimality. The object which is allocated is the transmission power, and we are interested in assigning it so as to obtain fairness between either one of three resulting performance measures: the signal to noise ratio, a shifted version of it, or the Shannon capacity. We suggest the generalized α-fairness concept. By means of parameter α it is possible to perform a spectrum of fair allocations resolving the tradeoff between fairness and efficiency. Furthermore, we obtain explicit solutions for and insight on the fair assignment corresponding to the various performance measures. For the case of large number of users we suggest variational formulations, which lead to the design of distributed algorithms.

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