Proceedings of the
47th IEEE Conference on Decision and Control
Cancun, Mexico, Dec. 9-11, 2008

Constrained Linear System under Disturbance Feedback: Convergence with Probability One

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***Abstract*— This paper considers a control parametrization under Model Predictive Control framework for constrained linear discrete time systems with bounded additive disturbances. Like the control parametrization in recent literature, the proposed parametrization uses affine disturbance feedback but includes an additional term. As a result, the parametrization has the same representative ability but has a different closed-loop convergence property. More exactly, the state of the closed-loop system converges to the minimal invariant set with probability one. Deterministic convergence to the same set is also possible if a less intuitive cost function is utilized. Numerical experiments are provided that validate the results.**

I. Introduction

This paper considers the system:

$$x_{t+1} = Ax_t + Bu_t + w_t, \quad (1)$$

$$(x_t, u_t) \in Y,\ w_t \in W,\ \forall\ t \geq 0 \quad (2)$$

where $x_t \in R^n$, $u_t \in R^m$ and $w_t \in W \subset R^n$ are the state, control and disturbance acting on the system at time t, respectively. The set Y represents the joint constraint on x and u of the system. The study of such a system under the Model Predictive Control (MPC) framework has been an active area of research in the past few years [1], [2], [3], [4], [5]. One important issue is the choice of control parametrization within the control horizon. Several choices have been proposed in the literature [2], [3], [5], [6], [7], [8], [9] and a popular choice is $u_t = Kx_t + c_t$ [2] where K is a fixed feedback gain and c_t is the new variable. However, such a choice is known to be conservative and its use will result in a relatively small domain of attraction.

In an effort to reduce conservatism, control parametrization based on affine function of disturbances have been proposed [6], [8], [9], [10]. Löfberg [6] proposes the control parametrization of

$$u_i^L = \sum_{j=1, j\leq i}^{i} M_i^j w_{i-j} + v_i, \quad i = 0, \cdots, N-1 \quad (3)$$

where M_i^j and v_i are the optimization variables and N is the length of the horizon used in MPC. Goulart et.al. [8] show that parametrization (3) is equivalent to that of time-varying affine state feedback in terms of set of states that are reachable within the horizon. They also showed that, under mild assumptions, the origin of the closed-loop system is input-to-state stable (ISS) under the MPC control law derived using time-varying affine state feedback law. Recently, Wang et.al [9], [11] propose an extended disturbance feedback parametrization

$$u_i^W = K_f x_i + c_i + \sum_{j=1}^{N-1} C_i^j w_{i-j}, i = 0, \cdots, N-1 \quad (4)$$

where K_f is a fixed feedback gain such that $\Phi := A + BK_f$ is strictly stable. They show that parametrization (4) under the MPC framework has the same domain of attraction as that using (3) but has a stronger stability result in that the state of closed-loop system converges to the minimal disturbance invariance set, F_∞ [12], of the system $x_{t+1} = \Phi x_t + w_t$. Unlike (3), it is possible that $i < j$ for w_{i-j} in (4). When this happens, w_{i-j} refers to past realized disturbances. This also means that the resulting MPC control law derived from (4) is a dynamic compensator, requiring the values of x_t and $w_{t-1}, \cdots, w_{t-N+1}$ for its evaluation at time t. On the other hand, the parametrization proposed in this paper results in a state feedback MPC control law, requiring only the knowledge of x_t for its evaluation. Correspondingly, a weaker convergence result is obtained : the closed-loop system state converges to F_∞ *with probability one*. Additionally, deterministic convergence to the same set is also possible if a less intuitive cost is used.

The rest of this paper is organized as follows. This section ends with notations used, assumptions needed and a brief review of standard results. Section II gives the proposed control parametrization and the finite horizon (FH) optimization problem including the choice of the cost function. The result of probabilistic convergence of the closed-loop system state is given in section III. Section IV shows a formulation that strengthens the result under weaker assumptions. Numerical examples and discussions are the contents of section V. The last section concludes the paper.

The following notations are used. $\mathbb{Z}_k$ denotes the integer set $\{0, 1, \cdots, k\}$ and $\mathbb{Z}_k^+$ denotes $\{1, \cdots, k\}$; given matrices $A \in R^{n\times m}$ and $B \in R^{p\times q}$: $A \otimes B$ is the Kronecker product of A and B; $\mathrm{vec}(A) = \begin{bmatrix} A_1^T & \cdots & A_m^T \end{bmatrix}^T \in R^{nm}$ is the stacked vector of columns of A and $\|A\| := \sqrt{\lambda_{\max}(A^T A)}$ is the induced norm of matrix A. $A \succ (\succeq) 0$ means that square matrix A is positive definite (semi-definite). For any

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$A \succ 0$, $\|x\|_A^2 = x^T Ax$. $\mathbf{1}_r$ is a r-vector with all elements being 1 and I_n is the $n \times n$ identity matrix. For any set $X, Y \subset R^n$, $X + Y := \{x + y : x \in X, y \in Y\}$ is the Minkowski sum of X and Y.

The system (1)-(2) is assumed to satisfy the following assumptions:

(A1) system (A, B) is stabilizable;

(A2) the set

$$Y := \{(x, u)|\ Y_x x + Y_u u \leq \mathbf{1}_q\} \subset R^{n+m} \quad (5)$$

is compact and contains the origin;

(A3) the disturbance w_t, $t \geq 0$ are independent and identically distributed (i.i.d.) with zero mean and $W \subset R^n$ is convex and compact;

(A4) a constant feedback gain $K_f \in R^{m \times n}$ is given such that $\Phi := A + BK_f$ has a spectral radius $\rho(\Phi) < 1$.

One other technical assumption is also needed and is discussed in section II. Assumption (A1) is standard. The characterization of Y in (A2) is made out of the need for a concrete computational representation. Assumption (A3) is mild and can be satisfied by many disturbance models. Additionally, the zero mean and i.i.d. condition can be relaxed and this will be discussed in details in section IV. Assumption (A4) is easily satisfied under (A1) and is made for convenience. Under (A1)-(A4) and the results in [12], [13] show that, for sufficiently small W, a constraint-admissible maximal disturbance invariant set,

$$X_f := \{x|\ Gx \leq \mathbf{1}_g\}, \quad (6)$$

exists in the sense that $\Phi x + w \in X_f, (x, K_f x) \in Y$ for all $x \in X_f$ and for all $w \in W$. It is also known [12] that the state of the system $x_{t+1} = \Phi x_t + w_t$ converges to the minimal disturbance invariant set, F_∞, given by

$$F_\infty = W + \Phi W + \Phi^2 W + \cdots \quad (7)$$

and that F_∞ is compact.

II. Control Parametrization

MPC formulation solves an N-stage finite horizon (FH) optimization problem. Let x_i and u_i, $i \in \mathbb{Z}_{N-1}$ denote the predicted state and predicted control at the i^{th} stage, respectively, within the horizon. The proposed control parametrization within the FH optimization problem takes the form

$$u_i = K_f x_i + d_i + \sum_{j=1, j \leq i}^{i} D_i^j w_{i-j} \qquad i \in \mathbb{Z}_{N-1} \quad (8)$$

where $d_i \in R^m, D_i^j \in R^{m \times n}, j = \mathbb{Z}_i^+, i \in \mathbb{Z}_{N-1}$ are the variables of the FH problem and K_f is the feedback gain in (A4). Since $i - j \geq 0$, w_{i-j} is the $(i-j)^{th}$ predicted disturbance at each stage i. In this regard, (8) is similar to (3) in that only predicted disturbances are used in the parametrization. However, in terms of the family of functions that can be represented, u_i is equivalent to u_i^L and u_i^W, the respective parameterizations of Löfberg [6] (or Goulart et. al. [8]) and Wang et. al. [9]. To see this, set $C_i^j = 0$ for all $j > i$ in (4) and it follows that u_i is a special case of u_i^W. To show the converse, let

$$\begin{cases} d_i = c_i + \sum_{j=i+1}^{N-1} C_i^j w_{i-j}, & i \in \mathbb{Z}_{N-1} \\ D_i^j = C_i^j & j \leq i, i \in \mathbb{Z}_{N-1} \end{cases} \quad (9)$$

for any c_i, C_i^j that defines u_i^W. This establishes the equivalence of u_i and u_i^W.

Remark 1: The equivalence of u_i^L and u_i^W, in terms of family of functions that can be represented, has already been established in [9]. With the above result, the representabilities of u_i, u_i^L and u_i^W are all equivalent.

Let the design variables within the control horizon N in (8) be collected in

$$\begin{aligned} \mathbf{D} &:= (D_1^1, D_2^2, D_2^1, \cdots, D_{N-1}^{N-1}, \cdots, D_{N-1}^1), \\ \mathbf{d} &:= (d_0, d_1, \cdots, d_{N-1}) \end{aligned}$$

then the FH optimization problem of (8), referred hereafter as $\mathcal{P}_N(x_t)$, is

$$\min_{\mathbf{d}, \mathbf{D}} \quad J(\mathbf{d}, \mathbf{D}) \quad (10)$$

$$\text{s.t.} \quad x_0 = x_t \quad (11)$$

$$x_{i+1} = Ax_i + Bu_i + w_i, \quad i \in \mathbb{Z}_{N-1} \quad (12)$$

$$u_i = K_f x_i + d_i + \sum_{j=1}^{i} D_i^j w_{i-j}, \quad i \in \mathbb{Z}_{N-1} \quad (13)$$

$$(x_i, u_i) \in Y, \quad \forall w_i \in W, \quad i \in \mathbb{Z}_{N-1} \quad (14)$$

$$x_N \in X_f, \qquad \forall w_i \in W, \quad i \in \mathbb{Z}_{N-1} \quad (15)$$

The above is a standard FH optimization problem for MPC with horizon N with X_f being the maximal disturbance invariant set of (6). The cost function $J(\mathbf{d}, \mathbf{D})$ takes the form

$$J(\mathbf{d}, \mathbf{D}) := \sum_{i=0}^{N-1} \left[\|d_i\|_\Psi^2 + \sum_{j=1}^{i} \|\text{vec}(D_i^j)\|_\Lambda^2 \right] \quad (16)$$

for any choice of Ψ and Λ that satisfy

(A5) $\Psi \succ 0$ and $\Lambda \succeq \Sigma_w \otimes \Psi$ where Σ_w is the covariance matrix of w_t.

The technical conditions of (A5) are needed to ensure the convergence property of the closed-loop system and its role will become clear in the proof of Theorem 2. However, some comments on the ease of verification of (A5) is appropriate.

Remark 2: Since $\Lambda - \Sigma_w \otimes \Psi$ has to be positive definite, (A5) can be easily satisfied even when the covariance matrix Σ_w is unknown. For example, let $\Lambda = \alpha^2 I_n \otimes \Psi$ where $\alpha = \max_{w \in W} \|w\|_2$. Then it follows that $\Lambda \succeq \Sigma_w \otimes \Psi$ because $\alpha^2 I_n \succeq ww^T$ for all $w \in W$ which implies that $\alpha^2 I_n \otimes \Psi \succeq \text{E}[ww^T] \otimes \Psi$.

Remark 3: Although Remark 2 implies that (A5) will be satisfied as long as eigenvalues of Λ are large enough, an over-large Λ will degrade the performance of the resulting MPC controller. This will be verified in the numerical examples and discussed further in section V.

Remark 4: Give matrices $Q \succeq 0$, $R \succ 0$ and $P \succ 0$ satisfying algebraic Riccati equation, it is shown [9], [14],

[15] that

$$\mathrm{E}\left[\sum_{i=0}^{N-1}(\|x_i\|_Q^2+\|u_i\|_R^2)+\|x_N\|_P^2\right]= x_0^TPx_0+N\mathrm{trace}(\Sigma_w P)+J(\mathbf{d},\mathbf{D}).$$

if $\Psi = R + B^TPB$, $\Lambda = \Sigma_w \otimes \Psi$ and $K_f = -(R + B^TPB)^{-1}B^TPA$. Hence, cost function (16) can be related to expected value of standard LQ cost.

From (12) and (13), it is obvious that x_i and u_i are affine functions of $w_i, i \in \mathbb{Z}_{N-1}$. Correspondingly, constraints (14) and (15) under assumptions (A2) and expression of X_f in (6) are affine in $w_i \in W, i \in \mathbb{Z}_{N-1}$. Since $w_i,\ i \in \mathbb{Z}_{N-1}$ are predicted disturbances within the horizon and have not been realized at time t, $\mathcal{P}_N(x_t)$ is a quadratic programming problem with linear uncertainties in its constraints. Its numerical solution is obtained from the deterministic equivalence of $\mathcal{P}_N(x_t)$. This process is done using the dual variables of the constraints and is a standard procedure in robust optimization [10]. The exact procedure has been discussed in [8], [9] for the case where W is a polytope and will not be elaborated here. It is also possible to formulate the deterministic equivalence when W is a conic or second-order cone representable set [16], [17].

Let the feasible set of optimization problem $\mathcal{P}_N(x_t)$ be

$$\Pi_N(x_t) := \{(\mathbf{d},\mathbf{D})|\ \mathcal{P}_N(x_t) \text{ is feasbile }\} \tag{17}$$

and the set of admissible initial states be

$$\mathcal{X}_N := \{x|\ \Pi_N(x) \neq \emptyset\}. \tag{18}$$

Remark 5: Consider the FH optimization problem under different control parameterizations, it follows from Remark 1 that the same admissible set $\mathcal{X}_N$ is achieved for the case where (3) or (4) replaces (13).

The rest of the MPC formulation is standard: $\mathcal{P}_N(x_t)$ is solved at each time t to obtain the optimizer $(\mathbf{d}_t^*, \mathbf{D}_t^*) := (\mathbf{d}^*(x_t), \mathbf{D}^*(x_t))$ and the corresponding $u_{0|t}^* := u_0^*(x_t)$ is applied to system (1) resulting in the MPC control law,

$$u_t = u_{0|t}^* = K_f x_t + d_{0|t}^* \tag{19}$$

III. Feasibility and Stability

The feasibility of $\mathcal{P}_N(x_t)$ at different time instants and stability of the closed-loop system under the feedback law (19) are addressed in this section.

Theorem 1: If $\mathcal{P}_N(x_t)$ admits an optimal solution, so does $\mathcal{P}_N(x_{t+1})$ under the feedback law (19) for all possible $w_t \in W$.

Proof: The proof is standard, but the details are given for their relevance to Theorem 2. For clarity, additional subscripts "$|t$" and "$|t+1$" are used to denote the variables at the different times. Let $(\mathbf{d}_t^*, \mathbf{D}_t^*)$ denote the optimal solution of $\mathcal{P}_N(x_t)$. At time $t+1$ when w_t is realized, choose $(\hat{\mathbf{d}}_{t+1}, \hat{\mathbf{D}}_{t+1})$ by letting

$$\hat{d}_{i|t+1} = \begin{cases} d_{i+1|t}^* + (D_{i+1|t}^{i+1})^* w_t & i \in \mathbb{Z}_{N-2} \\ 0 & i = N-1 \end{cases} \tag{20}$$

$$\hat{D}_{i|t+1}^j = \begin{cases} (D_{i+1|t}^j)^* & j \in \mathbb{Z}_i^+,\ i \in \mathbb{Z}_{N-2}^+ \\ 0 & j \in \mathbb{Z}_{N-1}^+,\ i = N-1 \end{cases} \tag{21}$$

and it is feasible to $\mathcal{P}_N(x_{t+1})$ for all possible $w_t \in W$ due to the disturbance invariance of X_f for system (1) under control law $u_t = K_f x_t$. It is clear that $\Pi_N(x)$ is compact for all $x \in \mathcal{X}_N$. Since W is bounded and J is a norm function, $\max_{w_t} J(\hat{\mathbf{d}}_{t+1}, \hat{\mathbf{D}}_{t+1}) < \infty$ and the set $\{(\mathbf{d},\mathbf{D}) \in \Pi_N(x_{t+1}) | J(\mathbf{d},\mathbf{D}) \le \max_{w_t} J(\hat{\mathbf{d}}_{t+1}, \hat{\mathbf{D}}_{t+1})\}$ is compact. Hence, the optimum of $\mathcal{P}_N(x_{t+1})$ exists, following the Weierstrass' theorem. □

The main result of probabilistic convergence of the closed-loop system is stated in the next theorem.

Theorem 2: Suppose $x_0 \in \mathcal{X}_N$ and (A1)-(A5) are satisfied. System (1) under MPC control law (19) has the following properties: (i) $(x_t, u_t) \in Y$ for all $t \ge 0$, (ii) $x_t \to F_\infty(K_f)$ with probability one as $t \to \infty$.

Proof: (i) The stated result follows directly from Theorem 1. (ii) Let $J_t^* := J(\mathbf{d}_t^*, \mathbf{D}_t^*)$ and $\hat{J}_{t+1}(w_t) := J(\hat{\mathbf{d}}_{t+1}(w_t), \hat{\mathbf{D}}_{t+1})$ where $(\hat{\mathbf{d}}_{t+1}(w_t), \hat{\mathbf{D}}_{t+1})$ are given by (20)-(21). Then it follows that

$$\begin{aligned} & J_t^* - \hat{J}_{t+1}(w_t) \\ =\ & \sum_{i=0}^{N-1}(\|d_{i|t}^*\|_\Psi^2 - \|\hat{d}_{i|t+1}\|_\Psi^2) + \sum_{i=1}^{N-1}\|\mathrm{vec}(D_{i|t}^i)^*\|_\Lambda^2 \\ =\ & \|d_{0|t}^*\|_\Psi^2 + \sum_{i=1}^{N-1}(\|d_{i|t}^*\|_\Psi^2 - \|\hat{d}_{i-1|t+1}\|_\Psi^2) \\ & \qquad + \sum_{i=1}^{N-1}\|\mathrm{vec}(D_{i|t}^i)^*\|_\Lambda^2 \\ =\ & \|d_{0|t}^*\|_\Psi^2 + \sum_{i=1}^{N-1}(\|d_{i|t}^*\|_\Psi^2 - \|d_{i|t}^* + (D_{i|t}^i)^* w_t\|_\Psi^2) \\ & \qquad + \sum_{i=1}^{N-1}\|\mathrm{vec}(D_{i|t}^i)^*\|_\Lambda^2 \\ =\ & \|d_{0|t}^*\|_\Psi^2 + g(w_t) \end{aligned} \tag{22}$$

where

$$\begin{aligned} g(w_t) := \sum_{i=1}^{N-1}(&\|\mathrm{vec}(D_{i|t}^i)^*\|_\Lambda^2 - 2(d_{i|t}^*)^T\Psi(D_{i|t}^i)^* w_t \\ &- \|(D_{i|t}^i)^* w_t\|_\Psi^2). \end{aligned} \tag{23}$$

Taking the expectation of (22) over w_t, it follows that

$$\begin{aligned} J_t^* - \|d_{0|t}^*\|_\Psi^2 &= \mathrm{E}_{w_t}\left[\hat{J}_{t+1}(w_t)\right] + \mathrm{E}_{w_t}[g(w_t)] \\ &\ge \mathrm{E}_{w_t}\left[\hat{J}_{t+1}(w_t)\right] \qquad (24) \\ &\ge \mathrm{E}_{w_t}\left[J_{t+1}^*(w_t)\right] = \mathrm{E}_t\left[J_{t+1}^*(w_t)\right]. \qquad (25) \end{aligned}$$

where E_t is the expectation taken over $w_i,\ i \ge t$. Inequality (24) follows from the fact that $\mathrm{E}_{w_t}[g(w_t)] \ge 0$. This is true because by taking the expectation of (23), one gets

$$\mathrm{E}_{w_t}[g(w_t)] = \sum_{i=1}^{N-1}(\|\mathrm{vec}(D_{i|t}^i)^*\|_\Lambda^2 - \|\mathrm{vec}(D_{i|t}^i)^*\|_{\Sigma_w\otimes\Psi}^2$$

$$-2(d^*_{i|t})^T\Psi(D^i_{i|t})^*\mathrm{E}[w_t])$$

where the last term is zero due to (A3) and the rest is non-negative due to (A5).

Inequality (25) follows from the fact that $\hat{J}_{t+1}(w_t) \geq J^*_{t+1}(w_t)$ for every $w_t \in W$ which implies that $\mathrm{E}_{w_t}[\hat{J}_{t+1}(w_t)] \geq \mathrm{E}_{w_t}[J^*_{t+1}(w_t)]$. The last equality of (25) follows from the fact that $J^*_{t+1}(w_t)$ depends on w_t only and not on any $w_i, i > t$.

Repeating the inequality of (25) for increasing t, one gets,

$$J^*_{t+1}(x_{t+1}) - \|d^*_{0|t+1}(x_{t+1})\|^2_\Psi \geq \mathrm{E}_{w_{t+1}}\left[J^*_{t+2}(x_{t+1}, w_{t+1})\right]$$

where the dependence of the various quantities on x_{t+1} are added for clarity. Since x_{t+1} depends on x_t and w_t, the above can be equivalently written as

$$J^*_{t+1}(w_t) - \|d^*_{0|t+1}(w_t)\|^2_\Psi \geq \mathrm{E}_{w_{t+1}}\left[J^*_{t+2}(w_t, w_{t+1})\right]. \quad (26)$$

The above inequality holds true for all possible w_t, hence

$$\begin{aligned}&\mathrm{E}_{w_t}[J^*_{t+1}(w_t)] - \mathrm{E}_{w_t}[\|d^*_{0|t+1}(w_t)\|^2_\Psi]\\ &\geq \mathrm{E}_{w_t}[\mathrm{E}_{w_{t+1}}\left[J^*_{t+2}(w_t, w_{t+1})\right]]\\ &= \mathrm{E}_t[J^*_{t+2}(w_t, w_{t+1})]\end{aligned} \quad (27)$$

or

$$\mathrm{E}_t[J^*_{t+1}(w_t)] - \mathrm{E}_t[\|d^*_{0|t+1}(w_t)\|^2_\Psi] \geq \mathrm{E}_t[J^*_{t+2}(w_t, w_{t+1})] \quad (28)$$

The equality in (27) follows from assumption (A3), particularly,

$$\begin{aligned}&\mathrm{E}_{w_t}[\mathrm{E}_{w_{t+1}}\left[J^*_{t+2}(w_t, w_{t+1})\right]]\\ &= \mathrm{E}_{w_t}\left[\int J^*_{t+2}(w_t, w_{t+1}) f_{w_{t+1}}(w_{t+1})dw_{t+1}\right]\\ &= \int\int J^*_{t+2}(w_t, w_{t+1}) f_{w_{t+1}}(w_{t+1})dw_{t+1} f_{w_t}(w_t)dw_t\\ &= \int\int J^*_{t+2}(w_t, w_{t+1}) f_{w_t,w_{t+1}}(w_t, w_{t+1})dw_{t+1}dw_t\\ &= \mathrm{E}_{w_t,w_{t+1}}[J^*_{t+2}(w_t, w_{t+1})] = \mathrm{E}_t[J^*_{t+2}(w_t, w_{t+1})]\end{aligned}$$

where $f_{w_t}(\cdot)$, $f_{w_{t+1}}(\cdot)$ and $f_{w_t,w_{t+1}}(\cdot,\cdot)$ are density functions of w_t, w_{t+1} and their joint density function, respectively, and $f_{w_t,w_{t+1}}(\cdot,\cdot) = f_{w_t}(\cdot)f_{w_{t+1}}(\cdot)$ from assumption (A3). Summing (25) and (28) leads to

$$J^*_t \geq \|d^*_{0|t}\|^2_\Psi + \mathrm{E}_t[\|d^*_{0|t+1}(w_t)\|^2_\Psi] + \mathrm{E}_t[J^*_{t+2}(w_t, w_{t+1})]$$

Repeating the above procedure infinite times leads to

$$\infty > J^*_t \geq \sum_{i=t}^{\infty} \mathrm{E}_t\left[\|d^*_{0|i}\|^2_\Psi\right]$$

By applying Markov bound (given non-negative random variable R and any $\epsilon \geq 0$, $\mathrm{E}[R] \geq \epsilon\mathrm{Pr}\{R \geq \epsilon\}$), we have

$$\infty > \epsilon\sum_{i=t}^{\infty}\mathrm{Pr}(\|d^*_{0|i}\|^2_\Psi \geq \epsilon) \quad (29)$$

for any arbitrary small $\epsilon > 0$. From the First Borel-Cantelli Lemma [18], this implies that $\lim_{i\to\infty}\mathrm{Pr}(\|d^*_{0|i}\|^2_\Psi \geq \epsilon) = 0$. Hence $d^*_{0|i}$ approaches zero with probability one as t increases. Consequently, the MPC control law (19) converges to $K_f x_t$ with probability one. When this happens, the closed-loop system converges to $x_{t+1} = \Phi x_t + w_t$ and, hence, x_t converges to $F_\infty(K_f)$ with probability one. □

IV. Deterministic Convergence

While the assumption of W being a convex compact set is reasonable, the assumption of w_t being zero mean and i.i.d. is harder to verify in practice. This section is concerned with the relaxation of assumption (A3) while achieving a stronger convergence result than that of Theorem 2. Consider

(A3a) $w_t \in W$ and W is convex and compact.

and define the cost function

$$V(\mathbf{d}, \mathbf{D}) := \sum_{i=0}^{N-1}\left[\|d_i\|^2_\Psi + \sum_{j=1}^{i}(\gamma_1\|\mathrm{vec}(D^j_i)\|^2 + \gamma_2\|\mathrm{vec}(D^j_i)\|)\right] \quad (30)$$

for some constants γ_1 and γ_2 satisfying

(A5a) $\gamma_1 \geq \alpha^2\|\Psi\|$, $\gamma_2 \geq 2\alpha\beta\|\Psi\|$.

where $\beta := \max_{(x,\mathbf{d},\mathbf{D})\in T_N, i\in\mathbb{Z}_{N-1}}\|d_i\|$, T_N is the set of $(x, \mathbf{d}, \mathbf{D})$ defined by (11)-(15) and $\alpha := \max_{w\in W}\|w\|$. The existence of α and β are guaranteed by compactness of the W and T_N sets.

Theorem 3: Suppose $x_0 \in \mathcal{X}_N$ and (A1-A2), (A3a), (A4) and (A5a) are satisfied and $J(\mathbf{d}, \mathbf{D})$ is replaced by $V(\mathbf{d}, \mathbf{D})$ in $\mathcal{P}_N(x)$, then system (1) under the MPC control law (19) satisfies (i) $(x_t, u_t) \in Y$ for all $t \geq 0$, (ii) $x_t \to F_\infty(K_f)$ as $t \to \infty$.

Proof: (i) The replacement of cost function $J(\mathbf{d}, \mathbf{D})$ by $V(\mathbf{d}, \mathbf{D})$ does not affect the feasibility of problem $\mathcal{P}_N(x)$. This means that part (i) of Theorem 2 remains valid. (ii) Let V^*_t and $\hat{V}_{t+1}$ be defined in the same manner as J^*_t and $\hat{J}_{t+1}$ in the statement of proofs of Theorem 2. Following the same reasoning as in (22), it can be shown that

$$V^*_t - \hat{V}_{t+1}(w_t) = \|d^*_{0|t}\|^2_\Psi + p(w_t) \quad (31)$$

where

$$\begin{aligned}p(w_t) = \sum_{i=1}^{N-1}&(\gamma_1\|\mathrm{vec}(D^i_{i|t})^*\|^2 + \gamma_2\|\mathrm{vec}(D^i_{i|t})^*\|\\ &-2(d^*_{i|t})^T\Psi(D^i_{i|t})^*w_t - \|(D^i_{i|t})^*w_t\|^2_\Psi).\end{aligned} \quad (32)$$

Hence

$$\begin{aligned}p(w_t) &\geq \sum_{i=1}^{N-1}(\gamma_1\|\mathrm{vec}(D^i_{i|t})^*\|^2 + \gamma_2\|\mathrm{vec}(D^i_{i|t})^*\|\\ &\quad - 2\|d^*_{i|t}\|\|\Psi\|\|w_t\|\|(D^i_{i|t})^*\| - \|\Psi\|\|w_t\|^2\|(D^i_{i|t})^*\|^2)\\ &\geq \sum_{i=1}^{N-1}(\gamma_1\|\mathrm{vec}(D^i_{i|t})^*\|^2 + \gamma_2\|\mathrm{vec}(D^i_{i|t})^*\|\\ &\quad - 2\alpha\beta\|\Psi\|\|\mathrm{vec}(D^i_{i|t})^*\| - \alpha^2\|\Psi\|\|\mathrm{vec}(D^i_{i|t})^*\|^2)\\ &= \sum_{i=1}^{N-1}((\gamma_1 - \alpha^2\|\Psi\|)\|\mathrm{vec}(D^i_{i|t})^*\|^2\\ &\quad + (\gamma_2 - 2\alpha\beta\|\Psi\|)\|\mathrm{vec}(D^i_{i|t})^*\|)\\ &\geq 0\end{aligned}$$

where the fact $\|(D^i_{i|t})^*\| \leq \|\mathrm{vec}(D^i_{i|t})^*\|$, i.e. 2-norm of a matrix is less than its Frobenius norm, is used. Hence,

$p(w_t) \geq 0$ under (A5a). As a consequence, equation (31) implies

$$V_t^* - \|d_{0|t}^*\|_\Psi^2 \geq V_{t+1}^* \geq 0 \tag{33}$$

Hence, $\{V_t^*\}$ is a monotonic non-increasing sequence and is bounded from below by zero. This means that $V_\infty := \lim_{t\to\infty} V_t^* \geq 0$ exists. Repeating (33) for t from 0 to ∞ and summing them up, it follows that

$$\infty > V_0^* - V_\infty \geq \sum_{t=0}^{\infty} \|d_{0|t}^*\|_\Psi^2 \tag{34}$$

Since Ψ is positive definite, this implies that $\lim_{t\to\infty} d_{0|t}^* = 0$ and $\lim_{t\to\infty} u_t = K_f x_t$. Therefore, the stated result follows. $\square$

Remark 6: Several choices of the cost function of (30) are possible. For example, the results of Theorem 3 remain true if $\|\mathrm{vec}(D_i^j)\|$ is replaced by $\|D_i^j\|$. This may be more appealing as less conservative bounds on γ_1 and γ_2 can be found to ensure the non-negativity of $p(w_t)$. However, its use will result in a semi-definite programming problem for $\mathcal{P}_N(x)$ and is less desirable computationally. The use of $\|\mathrm{vec}(D_i^j)\|$ results in a second-order cone programming for $\mathcal{P}_N(x)$ and is computationally more amiable.

Remark 7: The computation of β can be simplified to $\beta = \max_{(x,\mathbf{d},\mathbf{D})\in T_N} \|d_0\|$, see Appendix for details. Note that any upper bound of β can be used to guarantee the results of Theorem 3. One such upper bound is $\bar{\beta} := \|\sigma\|$ where $\sigma_i := \max_{(x,\mathbf{d},\mathbf{D})\in T_N} |d_0(i)|$ and $d_0(i)$ is the ith element of d_0.

V. NUMERICAL EXAMPLES AND DISCUSSIONS

The performance of the proposed MPC control law is illustrated on an example having $n = 2$ and $m = 1$. The system parameters and constraints are:

$$A = \begin{bmatrix} 1.1 & 1 \\ 0 & 1.3 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad K_f = [-0.7434\ -1.0922],$$

$$Y = \{(x,u) |\ |u| \leq 1,\ \|x\|_\infty \leq 8\},$$

$$W = \{w |\ w(1) = \hat{w} - 0.2\breve{w}, w(2) = \breve{w}, -0.2 \leq \hat{w}, \breve{w} \leq 0.2\}$$

where $\hat{w}$ and $\breve{w}$ are random variables uniformly distributed over $[-0.2, 0.2]$. Terminal set X_f is the corresponding maximal constraint-admissible disturbance invariant set of (1) under $u_t = K_f x_t$. The weight matrices in the cost function (16) are chosen to be

$$\Psi = 1, \quad \Lambda = \Sigma_w \otimes \Psi = \begin{bmatrix} 0.0139 & -0.0027 \\ -0.0027 & 0.0133 \end{bmatrix}$$

The proposed algorithm is simulated with $N = 8$ and $x_0 = [-4\ \ 2]^T$ over 15 realizations of disturbance sequences and resulting trajectories are shown in Fig. 1 to 4 by solid lines. $\hat{F}_\infty$ in Fig. 1 is a tight outer bound of F_∞ obtained using procedures given in [19].

It is clear from Fig. 1 and 3 that both the state and control constraints are satisfied by all trajectories, in accordance to property (i) of Theorem 2. Figure 4 shows the convergence of $d_t = d_{0|t}^*$ to zero as t increases. Hence, the closed-loop state converges to $\hat{F}_\infty(K_f)$ as shown in Fig. 2 where $dis(x_t, \hat{F}_\infty) = \min_{x\in\hat{F}_\infty} \|x - x_t\|$.

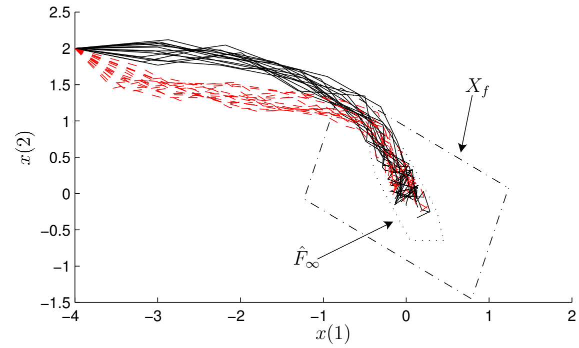

Fig. 1. State trajectories

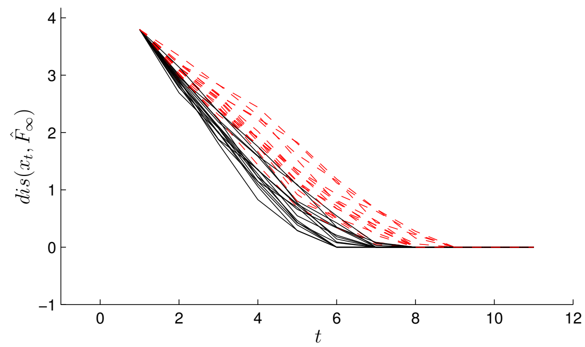

Fig. 2. Distance between states and $\hat{F}_\infty(K_f)$

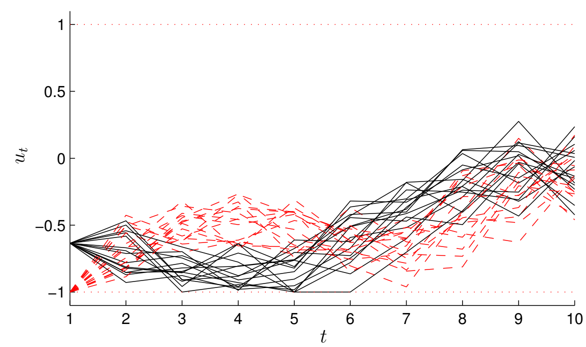

Fig. 3. Control trajectories

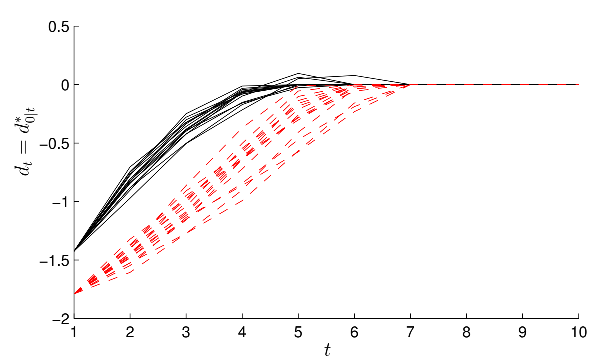

Fig. 4. Values of d_t

Next experiment attempts to shown the influence of the weight matrices on the performance of the MPC controller. Without of loss of generality, only Λ is regulated instead of both Ψ and Λ. In order to make the difference obvious, Λ is multiplied by 10000 and the system is simulated with same initial conditions and disturbance realizations as in the previous experiment. The results are shown in Fig. 1 to 4 by dash lines.

It can be observed that although the constraints are satisfied and the state converges to $\hat{F}_\infty$ set as well, the convergence is much slower this time as shown in Fig. 2 and 4. The reason is that by using a large Λ the time-varying disturbance feedback gains D_i^j becomes the dominating factors of the cost function and they are forced to vanish as fast as possible. As a consequence, the control law (8) is forced to be a fixed disturbance feedback control law as the one in [2]. Hence, the advantage of time-varying disturbance feedback is lost, leading to a degraded performance of the MPC controller. The results also verify the statement in Remark 3.

VI. Conclusions

A control parametrization is proposed for MPC of constrained linear systems with disturbances. This parametrization has the same feasible domain as that achieved by parametrization using affine time-varying state feedback law. Under the resultant controller, the closed-loop system state converges to the minimal robust invariant set F_∞ with probability one and this is achieved by minimizing a norm-like cost function. If a less intuitive cost is minimized, deterministic convergence to the same set is also achievable.

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Appendix

$\beta := \max_{(x,\mathbf{d},\mathbf{D})\in T_N, i\in\mathbb{Z}_{N-1}} \|d_i\| = \max_{(x,\mathbf{d},\mathbf{D})\in T_N} \|d_0\|$ is due to the fact that for any $(x,\mathbf{d},\mathbf{D}) \in T_N$ and integer $i \in \mathbb{Z}^+_{N-1}$, a set of $(\bar{x},\bar{\mathbf{d}},\bar{\mathbf{D}}) \in T_N$ can be found such that $\bar{d}_0 = d_i$. Specifically, given $(x,\mathbf{d},\mathbf{D}) \in T_N$ and let the correspondingly defined state and control sequence be $\{x_0,\ldots,x_N\}$ and $\{u_0,\ldots,u_{N-1}\}$. According to (15) $x_N \in X_f$ for all possible disturbances. Then for any $i \in \mathbb{Z}^+_{N-1}$, $(\bar{x},\bar{\mathbf{d}},\bar{\mathbf{D}})$ can be defined by

$$\bar{x} = \Phi^i x + \sum_{j=0}^{i-1} \Phi^{i-1-j} B d_j,\ \bar{d}_j = \begin{cases} d_{j+i} & j \in \mathbb{Z}_{N-1-i} \\ 0 & N-i \le j \le N-1 \end{cases},$$

$$\bar{D}_j^k = \begin{cases} D_{j+i}^k & j \in \mathbb{Z}^+_{N-1-i} \\ 0 & N-i \le j \le N-1 \end{cases} \ k \in \mathbb{Z}^+_j.$$

where $\bar{x}$ is the nominal state of x_i defined by $(x,\mathbf{d},\mathbf{D})$ and $(\bar{\mathbf{d}},\bar{\mathbf{D}})$ define the control sequence $\{u_i,\ldots,u_{N-1}, K_f x_N,\ldots,K_f x_{N-1+i}\}$. According to (A4) under controller $u_t = K_f x_t$ all the constraints are satisfied and $x_t \in X_f$ for $t \ge N$ since $x_N \in X_f$. Therefore, $(\bar{x},\bar{\mathbf{d}},\bar{\mathbf{D}})$ satisfies (11)-(15), namely $(\bar{x},\bar{\mathbf{d}},\bar{\mathbf{D}}) \in T_N$. As a result, $\max_{(x,\mathbf{d},\mathbf{D})\in T_N} \|d_0\| \ge \max_{(x,\mathbf{d},\mathbf{D})\in T_N} \|d_i\|$, for any $i \in \mathbb{Z}_{N-1}$ and $\beta = \max_{(x,\mathbf{d},\mathbf{D})\in T_N} \|d_0\|$.