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Finite-Time and Practical Stability of a Class of Stochastic Dynamical Systems

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***Abstract*— In practice, one is not only interested in *qualitative characterizations* provided by Lyapunov and Lagrange stability, but also in *quantitative information* concerning system behavior, including estimates of trajectory bounds over finite and infinite time intervals. This type of information has been ascertained in a systematic manner using the notions of *finite-time stability* and *practical stability*. In the present paper we generalize some of the existing finite-time stability and practical stability results for deterministic dynamical systems determined by ordinary differential equations to dynamical systems determined by an important class of stochastic differential equations. We consider two types of stability concepts: finite-time and practical stability in the mean and in the mean square. We demonstrate the applicability of our results by means of several examples.**

1. Introduction

The various concepts of Lyapunov and Lagrange stability are concerned with the *qualitative behavior* of the motions of a dynamical system and usually do not involve quantitative information (e.g., specific estimates of trajectory bounds). Thus, Lyapunov stability results ascertain whether or not an equilibrium is stable, asymptotically stable, unstable, and so forth, and Lagrange stability results ascertain whether or not the motions of a dynamical system are bounded in a certain sense (e.g., [1], [2]). In contrast to these concepts, various notions of stability have been proposed involving specific quantitative information. In this context, a dynamical system is viewed as being stable if whenever a given motion of a dynamical system starts in a *prespecified set*, then the motion will remain in another *prespecified set* of the state space, either over a *specified finite time interval* (*finite-time stability*) or *for all time* (*practical stability*) (e.g., [3]–[5]). The motivation for considering these types of stability concepts is the ability to deduce specific trajectory behavior (e.g., trajectory bounds) for the motions of a dynamical system from the boundaries of the prespecified sets.

As in the case of Lyapunov and Lagrange stability results, the results reported in [3]–[5] concerning finite-time and practical stability are formulated in terms of auxiliary functions (Lyapunov-like functions). As such, these results are characterized by similar attributes and liabilities as the usual Lyapunov and Lagrange stability results: they are very general and powerful; however, there are no general rules on how to determine appropriate Lyapunov-like functions. It needs to be pointed out that in general, the Lyapunov functions of the classical Lyapunov and Lagrange stability results differ significantly from the Lyapunov-like functions of the finite-time and practical stability results.

Although most of the existing results for finite-time and practical stability are concerned with continuous dynamical systems determined by ordinary differential equations (e.g., [3]–[5]), results for discontinuous dynamical systems and switched systems have also been established ([6]–[8]). In addition, practical moment stability of a class of stochastic delay differential equations was addressed in [9] and almost sure practical stability of systems described by Ito differential equation was considered in [10]. The results in [9] and [10] make use of comparison results involving differential inequalities. In the present paper we establish sufficient conditions for practical and finite-time stability in the mean and in the mean square for a class of stochastic dynamical systems determined by linear, time-varying Ito differential equations. Our results, which differ significantly from the results given in [9] and [10], do not make use of the comparison theory, and are in the spirit of the results established in [3]–[5]. We demonstrate the applicability of our results by means of a couple of examples.

2. Notation

Let $\in$ denote the set membership, $\mathbb{R}^n$ a real n-space, $|\cdot|$ the Euclidean norm defined on $\mathbb{R}^n$ and $\mathbb{R}^+ = [0, \infty)$. If A and B are sets, then $A \subset B$, $A \cup B$ and $A \times B$ denote, respectively, that A is a subset of B, the union of A and B and the Cartesian product of A and B. Let $B(a) = \{x \in \mathbb{R}^n\colon |x| < a\}$, $\overline{B(a)} = \{x \in \mathbb{R}^n\colon x \leq a\}$ and $[B(a) - B(b)] = \{x \in \mathbb{R}^n\colon x \in B(a) \text{ and } x \notin B(b)\}$. Let $\partial B(a)$ denote the boundary of $B(a)$. Let $f \in C[\mathbb{R}^+, \mathbb{R}]$ signify that f is a continuous function of $\mathbb{R}^+$ into $\mathbb{R}$ and $f \in C^1[\mathbb{R}^+, \mathbb{R}]$ that f is a continuously differentiable function of $\mathbb{R}^+$ into $\mathbb{R}$. If $V \subset \mathbb{R}^n$ and $W \subset \mathbb{R}^m$, then $f \in C[V, W]$ and $f \in C^1[V, W]$ are defined similarly in the obvious way. For an arbitrary matrix D, we define $|D| = [\lambda_M(D^T D)]^{1/2}$ where λ_M denotes the largest eigenvalue of $D^T D$.

Let $(\Omega, \mathfrak{A}, P)$ denote a probability space with probability measure P defined on the σ-algebra $\mathfrak{A}$ of ω-sets ($\omega \in \Omega$) in the sample space Ω. Any $\mathfrak{A}$-measurable function on Ω is called a random variable. A sequence of random variables indexed by $t \in T = \mathbb{R}^+$, $\{X_t(\omega) \in \mathbb{R}^n, t \in T\}$ is called a

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continuous (time) parameter stochastic process. Henceforth, the ω dependence is suppressed and we usually assume that $X_0(\omega) = x$ is known. We often write X_t in places of $\{X_t, t \in T\}$ for such processes. For $A \in \mathfrak{A}$, $P(A)$ denotes the probability of event A and $P(A|B)$ denotes the conditional probability of A under the condition that $B \in \mathfrak{A}$. We let E denote the expectation operator and for a Markov process $\{X_t, t \in T\}$, we let $E_{x,s}X_t$ denote the expected value of X_t at $t \in T$ if it is known that $X_s = x$. If $s = 0$, the notation $E_x X_t$ is used and if x and s are understood from context, the notation EX_t is employed.

3. Stochastic Dynamical Systems

We consider dynamical systems determined by linear, homogeneous equations of the form

$$dX_t = A(t)X_t dt + \sum_{i=1}^{m} B_i(t)X_t dW_t^i, \qquad (1)$$
$$X_{t_0} = c, \qquad t_0 \le t < \tau < \infty,$$

where $A \in C[\mathbb{R}^+, \mathbb{R}^{n\times n}], B_i \in C[\mathbb{R}^+, \mathbb{R}^{n\times m}], W_t$ is an $\mathbb{R}^m$-valued Wiener process and c is a random variable independent of $W_t - W_{t_0}$ for $t \ge t_0$.

It has been shown that (1) has on $[t_0, \tau]$ a unique $\mathbb{R}^n$-valued solution X_t, continuous with probability one, which satisfies the initial condition $X_{t_0} = c$, i.e., if X_t and Y_t are continuous solutions of (1) with the same initial value c, then

$$P\left(\sup_{t_0\le t\le\tau} |X_t - Y_t| > 0\right) = 0.$$

For a proof of the above assertions, refer, e.g., to [11], [12]. We denote the unique global solution of (1) by $X_t = X_t(c)$ on $[t_0, \infty)$.

4. Stability Concepts

We will consider several different notions of stability for system (1).

Definition 1 System (1) is *stable in the mean with respect to* $\{B(\alpha), B(\beta), t_0, T, |\cdot|\}$, $\alpha < \beta$, if $EX_{t_0} = c \in B(\alpha)$ implies that $EX_t \in B(\beta)$ for all $t \in [t_0, t_0 + T)$. If $T < \infty$, system (1) is said to be *finite-time stable in the mean* and if $T = \infty$, system (1) is said to be *practically stable in the mean with respect to* $\{B(\alpha), B(\beta), t_0, |\cdot|\}$. ■

Definition 2 System (1) is *uniformly stable in the mean with respect to* $\{B(\alpha), B(\beta), T, |\cdot|\}$, $\alpha < \beta$, if for all $t_i \in [t_0, t_0 + T)$, $EX_{t_i} \in B(\alpha)$ implies that $EX_t \in B(\beta)$ for all $t \in [t_i, t_0 + T)$. If $T < \infty$, system (1) is said to be *uniformly finite-time stable in the mean* and if $T = \infty$, system (1) is said to be *uniformly practically stable in the mean with respect to* $\{B(\alpha), B(\beta), |\cdot|\}$. ■

Definition 3 System (1) is *unstable in the mean with respect to* $\{B(\alpha), B(\beta), t_0, T, |\cdot|\}$, $\alpha < \beta$, if there exist an X_{t_0} and a $t_1 > t_0$ such that $X_{t_0} = c \in B(\alpha)$ and $EX_{t_1}(c) \in \partial B(\beta)$. If $T < \infty$, system (1) is said to be *finite-time unstable in the mean* and if $T = \infty$, system (1) is said to be *practically unstable in the mean with respect to* $\{B(\alpha), B(\beta), t_0, |\cdot|\}$. ■

Definition 4 System (1) is *stable in the mean square with respect to* $\{\alpha, \beta, t_0, T, |\cdot|\}$, $\alpha < \beta$, if $E|X_{t_0}|^2 < \alpha$ implies that $E|X_t|^2 < \beta$ for all $t \in [t_0, t_0 + T)$. If $T < \infty$, system (1) is said to be *finite-time stable in the mean square* and if $T = \infty$, system (1) is said to be *practically stable in the mean square with respect to* $\{\alpha, \beta, t_0, |\cdot|\}$. ■

Definition 5 System (1) is *uniformly stable in the mean square with respect to* $\{\alpha, \beta, t_0, T, |\cdot|\}$, $\alpha < \beta$, if for all $t_i \in [t_0, t_0 + T)$, $E|X_{t_i}|^2 < \alpha$ implies that $E|X_t|^2 < \beta$ for all $t \in [t_i, t_0 + T)$. If $T < \infty$, system (1) is said to be *finite-time stable in the mean square* and if $T = \infty$, system (1) is said to be *uniformly practically stable in the mean square with respect to* $\{\alpha, \beta, |\cdot|\}$. ■

Definition 6 System (1) is *unstable in the mean square with respect to* $\{\alpha, \beta, t_0, T, |\cdot|\}$, $\alpha < \beta$, if there exist an $X_{t_0} = c$ and a $t_1 \in (t_0, t_0 + T)$ such that $E|X_{t_0}|^2 < \alpha$ and $E|X_{t_1}|^2 = \beta$. If $T < \infty$, system (1) is said to be *finite-time unstable in the mean square* and if $T = \infty$, system (1) is said to be *practically unstable in the mean square with respect to* $\{\alpha, \beta, t_0, |\cdot|\}$. ■

When in (1), $B_i(t) \equiv 0$ for all $i = 1, \cdots, m$ and $t \ge t_0 \ge 0$, we have the corresponding deterministic system determined by ordinary differential equations of the form

$$\dot{x} = A(t)x, \qquad x(t_0) = x_0, \quad t \ge t_0, \qquad (2)$$

where $x(t, x_0, t_0)$ is the unique solution of (2) that exists for all $t \ge t_0$. For such systems, the stability definitions given above reduce in the obvious way to the following concepts considered in the literature [3]–[5].

Definition 7 System (2) is *stable with respect to* $\{B(\alpha), B(\beta), t_0, T, |\cdot|\}$, $\alpha < \beta$, if $x(t_0) = x_0 \in B(\alpha)$ implies that $x(t, x_0, t_0) \in B(\beta)$ for all $t \in [t_0, t_0 + T)$. If $T < \infty$, system (2) is said to be *finite-time stable* and if $T = \infty$, system (2) is said to be *practically stable with respect to* $\{B(\alpha), B(\beta), t_0, |\cdot|\}$. ■

The notions of *uniformly stable with respect to* $\{B(\alpha), B(\beta), T, |\cdot|\}$, $\alpha < \beta$, and *unstable with respect to* $\{B(\alpha), B(\beta), t_0, T, |\cdot|\}$ are defined similarly for the cases of *finite-time stability* ($T < \infty$) and *practical stability* ($T = \infty$).

5. Stability Results

It can be shown (see, e.g., [11]) that the first moment $m_t = EX_t$ for system (1) is the unique solution of the ordinary differential equation

$$\dot{m}_t = A(t)m_t, \qquad m_{t_0} = c_0, \qquad (3)$$

And the second moment $P(t) = EX_tX_t^T$ is the unique non-negative definite symmetric solution of the ordinary differential equation

$$\dot{P}(t) = A(t)P(t) + P(t)A(t)^T + \sum_{i=1}^{m} B_i(t)P(t)B_i(t)^T, \quad (4)$$

$$P(t_0) = Ec_0c_0^T.$$

The mean square $E|X_t|^2$ is the trace of $P(t)$, i.e.,

$$E|X_t|^2 = \text{tr } P(t) = \sum_{j=1}^{n} P_{jj}(t).$$

Therefore, the mean stability in the sense of Definitions 1, 2 and 3 for system (1) can be ascertained by applying corresponding finite-time stability results and practically stability results (see Definition 7) for system (2). We will subsequently address such results. Furthermore, the mean square stability in the sense of Definitions 4, 5 and 6 can be determined similarly. To this end, we note that since

$$P_{ij}(t) = EX_t^iX_t^j = EX_t^jX_t^i = P_{ji}(t), \quad (5)$$

the matrix $P(t)$ is symmetric and (4) represents a system of $l = n(n+1)/2$ linear equations. If we group the l elements $P_{ij}(t)$, $i \geq j$, in such a way as to form a vector q, then (4) can be rewritten in the form

$$\dot{q} = Q(t)q, \qquad q(t_0) = Ec_0, \quad (6)$$

Where $Q \in C[\mathbb{R}^+, \mathbb{R}^{l\times l}]$ can be determined in the obvious way. Accordingly, the mean square stability in the sense of Definitions 4, 5 and 6 for system (1) can be ascertained by applying corresponding finite-time stability results and practical stability results for system (2) (see Definition 2), which we address next.

In the following results, we make use of a Lyapunov-like functions $v \in C^1[\mathbb{R}^+ \times \mathbb{R}^n, \mathbb{R}]$. The notation $\dot{v}_{(2)}(x,t)$ denotes the derivative of v with respect to t evaluated along the solutions of (2), and is given by

$$\dot{v}_{(2)}(t,x) = \nabla v(x,t)^T A(t)x + \frac{\partial v}{\partial t}(t,x), \quad (7)$$

where

$$\nabla v(t,x)^T = \left[\frac{\partial v}{\partial x_1}(t,x), \cdots, \frac{\partial v}{\partial x_n}(t,x)\right].$$

Theorem 1 Assume that there exist a function $v \in C^1[\mathbb{R}^+ \times \mathbb{R}^n, \mathbb{R}]$ and an integrable function $\varphi\colon \mathbb{R}^+ \to \mathbb{R}$ such that

(i) $\dot{v}_{(2)}(t,x) \leq \varphi(t)$ for all $t \in J = [t_0, t_0 + T)$ and $x \in B(\beta)$ and

(ii) $\displaystyle\int_{t_0}^{t} \varphi(s)ds < \inf_{x\in\partial B(\beta)} v(t,x) - \sup_{x\in\overline{B(\alpha)}} v(t_0,x).$

Then system (2) is *stable with respect to* $\{B(\alpha), B(\beta), t_0, T, |\cdot|\}$, $\alpha < \beta$. If $T < \infty$, system (2) is *finite-time stable*. If $T = \infty$, system (2) is *practically stable*.

Proof. The proof of this result is given in [3]–[5]. We repeat it here for purposes of completeness.

The proof is by contradiction. Assume that $x(t_0) \in B(\alpha)$ and that there exists a $t_1 \in (t_0, t_0, +T)$, the first time such that $x(t_1, x(t_0), t_0) \in \partial B(\beta)$. Now

$$\begin{aligned} & v(t_1, x(t_1, x(t_0), t_0)) \\ &= v(t_0, x(t_0)) + \int_{t_0}^{t_1} \dot{v}_{(2)}(t, x(t, x(t_0), t_0))dt \\ &\leq \sup_{x\in\overline{B(\alpha)}} v(t_0,x) + \int_{t_0}^{t_1} \varphi(s)ds \\ &< \sup_{x\in\overline{B(\alpha)}} v(t_0,x) + \inf_{x\in\partial B(\beta)} v(t_1,x) - \sup_{x\in\overline{B(\alpha)}} v(t_0,x) \\ &= \inf_{x\in\partial B(\alpha)} v(t_1,x) \end{aligned}$$

The above inequality implies that $x(t_1, x(t_0), t_0) \notin \partial B(\beta)$, a contradiction to the original assumption. Therefore, there does not exist a $t_1 \in J$ as asserted. ■

Theorem 2 Assume that there exist a function $v \in C^1[\mathbb{R}^+ \times \mathbb{R}^n, \mathbb{R}]$ and an integrable function $\varphi\colon \mathbb{R}^+ \to \mathbb{R}$ such that

(i) $\dot{v}_{(2)}(t,x) \leq \varphi(t)$ for all $t \in J = [t_0, t_0 + T)$ and $x \in B(\beta) - B(\alpha)$ and

(ii) $\displaystyle\int_{t_1}^{t_2} \varphi(s)ds < \inf_{x\in\partial B(\beta)} v(t_2,x) - \sup_{x\in B(\alpha)} v(t_1,x)$ for all $t_2 > t_1$, $t_1, t_2 \in J$.

Then system (2) is *uniformly stable with respect to* $\{B(\alpha), B(\beta), T, |\cdot|\}$. ■

The proof of Theorem 2 is similar to the proof of Theorem 1. For details, refer to [3]–[5].

Theorem 3 Assume that there exist a function $v \in C^1[\mathbb{R}^+ \times \mathbb{R}^n, \mathbb{R}]$, an integrable function $\varphi\colon \mathbb{R}^+ \to \mathbb{R}$ and a constant t_1, $t_0 < t_1 < t_0 + T$, such that

(i) $\dot{v}_{(2)}(t,x) \geq \varphi(t)$ for all $x \in B(\beta)$ and $t \in J = [t_0, t_0 + T)$;

(ii) $\displaystyle\int_{t_0}^{t_1} \varphi(t)dt > \sup_{x\in\partial B(\beta)} v(t_1,x) - v(t_0,x_0)$; and

(iii) $v(t_1,x) \leq \displaystyle\sup_{x\in\partial B(\beta)} v(t_1,x)$, $x \in B(\beta)$, $t_1 > t_0$.

Then system (2) is *unstable with respect to* $\{B(\alpha), B(\beta), t_0, T, |\cdot|\}$. ■

The proof of Theorem 3 is similar to the proof of Theorem 1. For details, refer to [3]–[5].

Summarizing, to ascertain the various finite-time stability in the mean and practical stability in the mean, properties of the stochastic dynamical system (1) (in the sense of Definitions 1–3), we apply Theorems 1–3 in the analysis of the deterministic dynamical system (3) (involving the mean for system (1)). To ascertain the various mean square finite-time stability and mean-square practical stability properties of the stochastic dynamical system (1) (in the sense of Definitions 4–6), we apply Theorems 1–3 in the analysis of the deterministic dynamical system (6) (involving the mean square).

6. Examples

We now address the mean and mean square finite-time and practical stability of the stochastic linear homogeneous system (1). To this end, we let $x(t) = m_t$. Then the deterministic systems (2) and (3) have identical form. We will use the x-notation (rather than the m-notation).

Example 1 We choose

$$v(x) = x^T x.$$

Along the solutions of (2) (resp., of (3)), we have

$$\dot{v}_{(2)}(x) = x^T [A(t) + A(t)^T] x.$$

Let $\Lambda(t)$ denote the laregst eigenvalue of $A(t)+A(t)^T$. Then

$$\dot{v}_{(2)}(x) \leq \Lambda(t)|x|^2,$$

for all $x \in \mathbb{R}^n$, $t \in J = [t_0, t_0 + T)$.

(a) Assume that $\Lambda(t) \leq 0$ for all $t \in J$. Then

$$\dot{v}_{(2)}(x) \leq \Lambda(t)|x|^2 \leq \Lambda(t)\alpha^2,$$

for all $x \in B(\beta) - \overline{B(\alpha)}$, $t \in J$.

Let

$$\varphi(t) = \Lambda(t)\alpha^2, \qquad t \in J.$$

To satisfy the hypotheses of Theorem 2, we require that

$$\int_{t_1}^{t_2} \Lambda(t)\alpha^2 dt \leq \beta^2 - \alpha^2, \qquad t_1, t_2 \in J, \ \alpha < \beta. \tag{8}$$

Inequality (8) is satisfied for any $\alpha < \beta$ and $T = \infty$. Therefore, when $\Lambda(t) \leq 0$ for all $t \in [t_0, \infty)$, system (1) is *uniformly practically stable in the mean with respect to* $\{B(\alpha), B(\beta), t_0, |\cdot|\}$ for $\beta > \alpha$.

(b) Assume that $\Lambda(t) \geq 0$ for all $t \in J$. Then

$$\dot{v}_{(2)}(x) \leq \Lambda(t)|x|^2 \leq \Lambda(t)\beta^2,$$

for all $x \in B(\beta)$, $t \in J$.

Let

$$\varphi(t) = \Lambda(t)\beta^2, \qquad t \in J.$$

To satisfy the hypotheses of Theorem 1, we require that

$$\int_{t_0}^{t} \Lambda(s)\beta^2 ds \leq \beta^2 - \alpha^2, \qquad t \in J, \ \alpha < \beta. \tag{9}$$

Therefore, when $\Lambda(t) \geq 0$ for all $t \in [t_0, t_0 + T)$ and if inequality (9) is satisfied for $T = \infty$, then system (1) is *practically stable in the mean with respect to* $\{B(\alpha), B(\beta), t_0, |\cdot|\}$ for $\beta > \alpha$.

(c) If in part (b) we assume that $\Lambda(t) = \Lambda > 0$ for all $t \in J$, then it follows from (9) that system (1) is *finite-time stable in the mean with respect to* $\{B(\alpha), B(\beta), t_0, T, |\cdot|\}$ if

$$\Lambda T \leq 1 - (\alpha/\beta)^2.$$

■

Example 2 We can improve part (a) of Example 1 by utilizing the Lyapunov-like function

$$v(x) = \ln(x^T x).$$

Along the solutions of (2) (resp., (3)), we have

$$\nabla v(x) = \frac{2}{x^T x} x$$

and

$$\begin{aligned} \dot{v}_{(2)}(x) &= \nabla v(x)^T A(t) x \\ &= \frac{2}{x^T x}\left[x^T \frac{[A(t) + A(t)^T]}{2} x\right] \\ &\leq \frac{x^T \Lambda(t) x}{x^T x}, \qquad x \neq 0 \end{aligned}$$

i.e.,

$$\dot{v}_{(2)}(x) \leq \Lambda(t), \qquad x \neq 0, \ t \in \mathbb{R}^+$$

where $\Lambda(t)$ denotes again the largest eigenvalue of the matrix $A(t) + A(t)^T$.

Now let

$$\dot{v}_{(2)}(x) \leq \Lambda(t) = \varphi(t), \qquad t \in \mathbb{R}^+.$$

The hypotheses of Theorem 2 are satisfied if for all $t_1, t_2 \in J = [t_0, t_0 + T)$, $t_2 > t_1$,

$$\begin{aligned} \int_{t_1}^{t_2} \varphi(s) ds &= \int_{t_1}^{t_2} \Lambda(s) ds \\ &< \inf_{|x|=\beta} \ln|x|^2 - \sup_{|x|=\alpha} \ln|x|^2 \\ &= \ln\beta^2 - \ln\alpha^2 \\ &= 2\ln(\beta/\alpha), \end{aligned}$$

i.e.,

$$\int_{t_1}^{t_2} \Lambda(s) ds \leq 2\ln(\beta/\alpha), \qquad \beta > \alpha, \ \ t_1, t_2 \in J. \tag{10}$$

Therefore, by Theorem 3, system (1) is *uniformly practically stable in the mean with respect to* $\{B(\alpha), B(\beta), t_0, |\cdot|\}$ if inequality (10) can equivalently be written as

$$e^{\frac{1}{2}\int_{t_1}^{t_2} \Lambda(s) ds} < \beta/\alpha, \qquad \beta > \alpha, \ \ t_1, t_2 \in J.$$

■

Example 3 The *mean square finite-time stability* and *practical stability* of the stochastic dynamical system (1) can be accomplished similarly as in Examples 1 and 2, applying Theorems 1–3 to the deterministic system (6). ■

7. Concluding Remarks

We established sufficient conditions for finite-time and practical stability in the mean and in the mean square for an important class of linear stochastic differential equations. In arriving at our results, we bring to bear existing finite-time and practical stability results for deterministic dynamical systems. We demonstrate the applicability of our results by means of two specific examples.

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