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On the characterisation of stabilisability by means of time-delayed feedback control♠

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Abstract— **In this paper we consider the stabilisability of nonlinear dynamical systems via time-delayed state and output feedback control. Based on an eigenvalue optimisation approach in combination with a continuation argument and a result characterising properties of the optimum eigenvalue configurations, we obtain explicit expressions for stabilisability boundaries for two-dimensional systems. As time-delayed feedback can be used to approximate derivative feedback, we also give a comparison of the stabilisability via both types of feedback.**

I. Introduction

In recent years a growing interest in stability of time-delay systems is emerging in the control literature. The reasons for this are manifold and the interest is motivated by possible applications in networked control systems, chemistry and biology. For a review on recent results and techniques in stability analysis of time-delay systems we refer to e.g. [9],[15].

An important extension of the stability problem for time-delay systems is the problem whether it is possible to achieve stabilisation of a system by means of a so-called time-delayed feedback. The idea of using time-delayed feedback was originally proposed by Pyragas in [13] in the context of stabilisation of periodic solutions of chaotic systems, see also e.g. [8],[12]. Later, the use of time-delayed feedback was extended to the problem of stabilisation of equilibrium points, see e.g. [14].

Recently, [6],[17] (see also [4]) considered the characterisation of stabilisability of unstable equilibria of (mainly two-dimensional) dynamical systems by means of time-delayed feedback. Some interesting analytical results regarding this problem, supplemented with numerical simulations, were given. It is to be noted, however, that both references mainly considered diagonal state feedbacks.

The present paper takes inspiration from [6],[17], and our goal is to provide a complete classification of all possible linear time-delayed state and output feedbacks that may be used to stabilise an unstable equilibrium of a two-dimensional dynamical system. Even though we will be dealing with linear systems, the stabilisability problem gives rise to the analysis of the solutions of a parametrised nonlinear eigenvalue problem. For this analysis we employ methods and tools from bifurcation analysis. In particular, our approach is based on the construction of determining systems and the continuation of their solutions.

In particular, we consider a general nonlinear control system of the form

$$\dot{x} = f(x,u), \quad \eta = h(x) \tag{1}$$

with state $x \in \mathbb{R}^2$, input $u \in \mathbb{R}$ and output $y \in \mathbb{R}$. We assume that the origin is a hyperbolic unstable steady state, i.e., $f(0,0) = 0$. As in e.g. [6],[17], we then consider the linearisation of (1) about $(x,u) = (0,0)$, which is given by

$$\dot{z} = Az + Bu, \quad y = \begin{pmatrix} c_2 & c_1 \end{pmatrix} z \tag{2}$$

where $A = D_x f(0,0)$, $B = D_u f(0,0)$, $\begin{pmatrix} c_2 & c_1 \end{pmatrix} = D_x h(0,0)$. Assuming that the linearised system is controllable, it may then be assumed without loss of generality that A and B have the following forms: $A = \begin{pmatrix} 0 & 1 \\ -a_2 & -a_1 \end{pmatrix}$, $B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. In this paper we will study the question under which conditions on the system parameters a_1, a_2, c_1, c_2 there either exists a *time-delayed state feedback* of the form

$$u(t) = -K(z(t) - z(t-\tau)), \quad K = \begin{pmatrix} K_2 & K_1 \end{pmatrix} \tag{3}$$

or a *time-delayed output feedback* of the form

$$u(t) = -k(y(t) - y(t-\tau)). \tag{4}$$

such that the origin is an exponentially stable equilibrium point of the closed loop system (2,3) or (2,4) respectively.

Note that due to the so-called odd number limitation (see e.g. [8],[12]) the system (3) can never be stabilised by means of a time-delayed feedback of the form (3) or (4) if the matrix A has an odd number of positive real eigenvalues. Therefore, we can throughout restrict ourselves to the case that $a_2 \geq 0$.

The aim of this paper is to give a *complete* answer to the question whether and how the linearised system (2) can be stabilised by means of a time-delayed state feedback (3) or a time-delayed output feedback (4). Our approach uses a methodology from [10], which exploits an eigenvalue optimisation approach in combination with a continuation argument and a result characterising properties of the optimum eigenvalue configurations. The latter argument allows one to "guess" analytical expressions for the boundaries of stabilisability regions that can then be verified by means of numerical computations on a very coarse grid.

♠ An extended version of this paper ([7]) is to appear in the *SIAM Journal on Applied Dynamical Systems*.

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The paper is organised as follows. In Section II we first study the problem of state feedback stabilisation. After that, the problem of output feedback stabilisation will be discussed in Section III, where we will restrict ourselves to the case that $c_1c_2 \leq 0$. In both Sections II and III we will first treat the case of a fixed value of the time-delay τ. After this, the conclusions drawn for this case will be used to derive stabilisability conditions in case the time-delay can also be used as a controller parameter. Since time-delayed feedback could be used to approximate derivative feedback, we also include a discussion of the relationships between stabilisability via both types of feedback at the end of Section III. Finally, in Section IV conclusions will be drawn.

II. State feedback stabilisation

A. Stabilisation through eigenvalue optimisation

The characteristic equation of the closed loop system (2,3) is given by

$$0 = p(s;\ a_1, a_2, \tau, K_1, K_2) := \det(sI - A - (1 - \mathrm{e}^{-s\tau})BK) = s^2 + (a_1 + K_1(1 - \mathrm{e}^{-s\tau}))\, s + (a_2 + K_2(1 - \mathrm{e}^{-s\tau}))\,. \tag{5}$$

Given a_1, a_2 and τ, we define the *spectral abscissa function* $F : \mathbb{R}^2 \to \mathbb{R}$ by $F(K;\ a_1, a_2, \tau) := \sup\{\mathrm{Re}(s) \mid p(s;\ a_1, a_2, \tau, K_1, K_2) = 0\}$. The zeroes of (5) whose real parts are equal to $F(K;\ a_1, a_2, \tau)$ are called the *active* eigenvalues. We further define $c(a_1, a_2, \tau)$ as

$$c(a_1, a_2, \tau) := \min_K F(K;\ a_1, a_2, \tau). \tag{6}$$

This then gives that the stabilisation problem is solvable for given (a_1, a_2, τ) if and only if $c(a_1, a_2, \tau) < 0$, and the boundary of the stabilisability region is determined by values of a_1, a_2 and τ for which $c(a_1, a_2, \tau) = 0$.

From the above we see that the stabilisation problem under consideration can be viewed as an optimisation problem. Furthermore, assertions about stabilisability can be made by minimising the spectral abscissa function $F(K;\ a_1, a_2, \tau)$.

As shown in [16] the spectra abscissa function is not everywhere differentiable, even not everywhere Lipschitz continuous. Moreover, discontinuities of its derivatives typically occur in the minima, which prohibits the use of most standard optimisation methods. However, as the spectral abscissa function is continuous and differentiable almost everywhere, so-called bundle gradient methods like the *gradient sampling algorithms* of [3] are applicable. These methods only rely on the evaluation of the objective function and its gradient in points where the objective function is smooth. In [2], [3] the gradient sampling algorithm, which directly generalises the steepest descent method to piecewise-smooth objective functions, was successfully applied to the design of stabilising fixed-order controllers and, recently, it was included in HIFOO [1], a software package for fixed-order controller design and $\mathcal{H}_\infty$-optimisation. In [16] the algorithm was applied to the stabilisation of time-delay systems, which required a combination with the routines of the package DDE-BIFTOOL [5] for the computation of the eigenvalues determining the stability of these systems. For this paper, we have modified the methods from [16] so that stability by means of time-delayed feedback can be studied.

B. Characterization of stabilisable systems

It is sufficient to compute the stabilisability region in the (a_1, a_2) plane for one *arbitrary* nonzero delay value, because of the following scaling property:

$$p(s;\ a_1, a_2, \tau, K_1, K_2) = 0 \iff p(\hat{s};\ a_1\tau, a_2\tau^2, 1, K_1\tau, K_2\tau^2) = 0,\ \hat{s} = s\tau, \tag{7}$$

which implies the relation

$$c(a_1, a_2, \tau) = 0 \iff c(a_1\tau, a_2\tau^2, 1) = 0.$$

For a given value of τ, an exhaustive approach to compute the stabilisability region in the (a_1, a_2) plane, would consist of applying the gradient sampling algorithm, discussed in Subsection II-A, for a large number of values of (a_1, a_2) on a fine grid, to check whether the system is stabilisable. The stabilisability boundary then separates the regions where the system is stabilisable and where it is not. However, a more efficient calculation is possible by taking into account specific properties of the optimisation problem of $F(K;\ a_1, a_2, \tau)$, as set out in the following proposition, a proof of which can be found in [7].

Proposition II.1. *If $a_2 \neq 0$ and $F(K;\ a_1, a_2, \tau)$ is minimal, then there are at least three active eigenvalues (counting multiplicities).*

By Proposition II.1 the possible configurations of the active eigenvalues in the global minimum of $F(K;\ a_1, a_2, \tau)$ can be reduced to the following two types. Type I: the existence of active roots of multiplicity 2 at $c \pm j\omega$, giving rise to four determining equations with four unknowns c, ω, K_1, K_2; Type II: the existence of active roots of multiplicity 1 at $c \pm j\omega_1, c \pm j\omega_2$ ($\omega_1 \neq \omega_2$), giving rise to four determining equations with five unknowns $c, \omega_1, \omega_2, K_1, K_2$. Theoretically Type II is possible, but is less generic than Type I. Indeed, for Type I the mathematical relations allow a direct computation of the minimal value $c(a_1, a_2, \tau)$ of $F(K;\ a_1, a_2, \tau)$ and the corresponding control parameters K_1 and K_2, provided that good starting values are available. For Type II, however, there is an extra parameter and therefore the determining equations define curves in their unknowns, through the point corresponding to the optimal parameter values. In such a situation, a suitable small parameter change generically reduces the value of $F(K;\ a_1, a_2, \tau)$, meaning that the situation does not correspond to the global minimum, unless an extra condition is satisfied (e.g. a turning point on the curve, or an additional active eigenvalue). Our numerical experiments indicate that this extra condition is characterised by $\omega_2 = \omega_1$, so that Type II reduces to Type I.

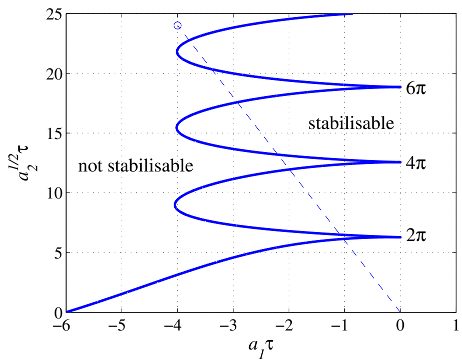

Fig. 1. Stabilisability region for time-delayed state feedback.

In order to compute the stabilisability region in the (a_1, a_2) plane for a particular delay value ($\tau = 1$ was chosen in our simulations), we used the information above as follows. First, we freed the parameters (a_1, a_2) and imposed that $c(a_1, a_2, \tau) = 0$, corresponding to a minimum of F of Type I. In this way we arrived at the four determining equations expressing that there exists $\omega > 0$ such that the real and imaginary parts of p and p' are zero at $j\omega$. These four equations contain the five unknowns $(\omega, K_1, K_2, a_1, a_2)$, which define a curve whose projection on the (a_1, a_2) plane forms the stabilisability boundary in regions where Type I minima occur. Next, we performed a direct minimisation of $F(K;\ a_1, a_2, \tau)$ for values of (a_1, a_2) chosen on a *coarse grid*, in order to exclude Type II minima. Finally, as Type II minima were not observed, the following result was concluded from the determining equations.

Proposition II.2. $c(a_1, a_2, \tau) = 0$ if and only if *there exists an* $\omega \in \mathbb{R}^+$ *such that*

$$a_1\tau = \frac{2\omega(\cos\omega - 1)}{\omega - \sin\omega}, \quad a_2\tau^2 = \frac{\omega^2(\omega - \sin\omega)}{\omega + \sin\omega} \tag{8}$$

According to Proposition II.2, the stabilisability region in the (a_1, a_2) plane is depicted in Figure 1.

We conclude the section with some observations.

From Figure 1 we can deduce stabilisability information for a system with fixed parameters (a_1, a_2) as a function of the delay τ. When the delay changes, the normalised plant parameters $(a_1\tau,\ \tau\sqrt{a_2})$ move on a (half) straight line, as indicated in the figure. Note that several disjunct delay intervals may exist for which the system is stabilisable. If the delay can also be used as a controller parameter then the system is stabilisable if and only if $a_2 > 0$, i.e. it does not satisfy the odd number condition. Furthermore, stability can always be achieved for sufficiently small values of the delay.

III. Stabilisation by means of time-delayed output feedback

In this section we consider the system (2) and consider the question whether or not it can be stabilised with a time-delayed output feedback of the form (4) with $c_1c_2 \leq 0$. The cases $c_1c_2 > 0$ and $c_1 = 0, c_2 \neq 0$ are omitted because they can be treated in a completely analogous way, and unlike the other cases, the results are not essential for the analysis of Section III.C.

Note that now the characteristic equation of the closed loop system is given by

$$0 = p(s;\ a_1, a_2, c_1, c_2, \tau, k) =:$$
$$s^2 + (a_1 + c_1k(1 - \mathrm{e}^{-s\tau}))\, s + (a_2 + c_2k(1 - \mathrm{e}^{-s\tau}))\,. \tag{9}$$

Analogously to (7) we now have the following scaling property:

$$p(s; a_1, a_2, c_1, c_2, \tau, k) = 0 \iff$$
$$p(\hat{s}; a_1\tau, a_2\tau^2, 1, \tfrac{c_2}{c_1}\tau, 1, c_1k\tau) = 0,\ \hat{s} = s\tau. \tag{10}$$

As a consequence, it is again sufficient to compute the stabilisability region for one arbitrary nonzero delay value.

In the following subsection we will first consider the case that $c_2 = 0,\ c_1 = -1$. After this, we will consider the cases that $c_1c_2 < 0$.

As there now is only one free parameter (viz. the control parameter k), it may be shown in a similar way as in Proposition II.1 that if $F(k;\ a_1, a_2, \tau)$ is minimal, there are at least *two* active eigenvalues.

A. The case that $c_2 = 0,\ c_1 = 1$

In this case, the stabilisability region is the region to the right of the thick black and red curves in Figure 2. In this figure, the red curve is characterised by the existence of $\omega \in \mathbb{R}^+$, $\hat{k} \in \mathbb{R}$ such that

$$p(j\omega;\ a_1, a_2, \hat{k}) = 0,\ \frac{d\mathrm{Re}(\lambda_k)}{dk}(\hat{k}) = 0 \tag{11}$$

where λ_k satisfies $p(\lambda_k;\ a_1, a_2, k) = 0,\ \lambda_{\hat{k}} = j\omega$. This represents the case that an unstable eigenvalue in the right-half plane becomes critically stable, but moves back into the right-half plane when k is changed. It may be shown that this gives rise to the following equalities describing the red curve:

$$a_1\tau = \frac{2\omega(\cos\omega - 1)}{\omega - \sin\omega}, \quad a_2\tau^2 = \frac{\omega^2(\omega + \sin\omega)}{\omega - \sin\omega}. \tag{12}$$

The black curves are characterised by the existence of $\omega_1, \omega_2 \in \mathbb{R}^+$ and k such that one pair of roots of $p_k(s)$ crosses to the right-half plane through $j\omega_1$, while another pair of roots of $p_k(s)$ crosses to the left-half plane through $j\omega_2$ (or vice versa). It may be shown that this gives rise to the following equalities:

$$a_1\tau = k\tau(1 - \cos\omega_1), \quad a_2\tau^2 = \omega_1^2 - k\tau\omega_1\sin\omega_1,$$
$$k\tau = \tfrac{\omega_2^2 - \omega_1^2}{\omega_2\sin\omega_2 - \omega_1\sin\omega_1}, \quad \cos\omega_1 = \cos\omega_2 \tag{13}$$

Given $\omega_1 > 0$, it is obviously the case that there exist multiple $\omega_2 > 0$ such that the last equality in (13) holds. However, from numerical experiments it turned out that the curves described by (13) are only active for $\omega_1 = \omega \in [0, \pi]$,

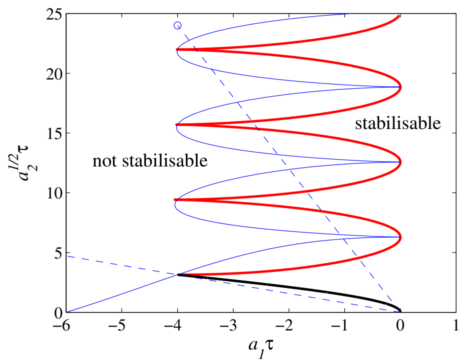

Fig. 2. Stabilisability region for time-delayed output feedback with $c_2 = 0$, $c_1 = 1$. Thick red and black curves: stabilisability boundary. Blue curve: stabilisability boundary for state feedback.

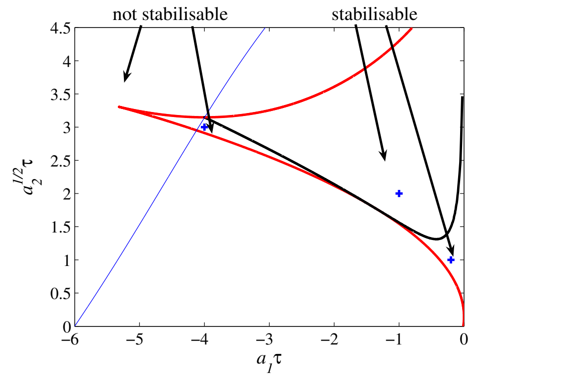

Fig. 3. Stabilisability region for time-delayed output feedback with $\frac{c_2}{c_1}\tau = -0.01$. Blue curve: stabilisability boundary for time-delayed state feedback; red curve: Case 1; black curve: Case 2.

$\omega_2 = 2\pi - \omega$. This gives that in fact the black curve in Figure 2 is given by

$$a_1\tau = \frac{2(\pi-\omega)(\cos\omega - 1)}{\sin\omega}, \quad a_2\tau^2 = \omega(2\pi - \omega) \qquad (0 \le \omega \le \pi) \tag{14}$$

As before, we can deduce stability information for a system with fixed parameters (a_1, a_2) as a function of the delay τ. As illustrated in Figure 2, the normalised plant parameters $(a_1\tau, \tau\sqrt{a_2})$ will again move on a (half) straight line. We see from the figure that again several disjunct delay intervals may exist for which the system is stabilisable. Furthermore, as it may be shown that the black curve has an infinite slope at the origin, we have, unlike the static state feedback case, that the system is never stabilisable for small enough values of τ. Finally, the lower dashed line in Figure 2 illustrates the construction of the stabilisability boundary if k as well as τ are control parameters. Namely, we see that in this case (a_1, a_2) is on the stabilisability boundary if and only if there exists a $\tau > 0$ such that $(a_1\tau, \tau\sqrt{a_2}) = (-4, \pi)$. This then gives that there exist $k \in \mathbb{R}$, $\tau > 0$ such that the closed loop system is stable if and only if $\sqrt{a_2} > -\frac{\pi a_1}{4}$.

B. The case that $c_1c_2 < 0$

Recall that in in this case the characteristic equation of the closed loop system is given by (9). Define $\tilde{c} = \frac{c_2}{c_1}\tau$. As for the case treated in the previous subsection, the stabilisability boundary is formed by points characterised by one of the following two properties.

1) There exist $\omega \in \mathbb{R}^+$, $\hat{k} \in \mathbb{R}$ such that (11) holds. It may be shown that this gives rise to the following equalities describing this curve:

$$\begin{cases} a_1\tau = \dfrac{2\omega(-\tilde{c}^2\cos\omega + \omega^2\cos\omega - \omega^2 - 2\tilde{c}\omega\sin\omega - \tilde{c}^2)}{-2\tilde{c}\omega\cos\omega + \tilde{c}^2\omega + \omega^3 - \omega^2\sin\omega + \tilde{c}^2\sin\omega - 2\tilde{c}\omega} \\ a_2\tau^2 = \dfrac{\omega^2(2\tilde{c}\omega\cos\omega + \omega^3 + \tilde{c}^2\omega + \omega^2\sin\omega - \tilde{c}^2\sin\omega - 2\tilde{c}\omega)}{-2\tilde{c}\omega\cos\omega + \tilde{c}^2\omega + \omega^3 - \omega^2\sin\omega + \tilde{c}^2\sin\omega - 2\tilde{c}\omega} \end{cases} \tag{15}$$

2) There exist $\omega_1, \omega_2 \in \mathbb{R}^+$, $k \in \mathbb{R}$ such that one pair of roots of $p_k(s)$ crosses to the right-half plane through $j\omega_1$, while another pair of roots of $p_k(s)$ crosses to the left-half plane through $j\omega_2$ (or vice versa). It may be shown that this gives rise to the following equalities:

$$\begin{cases} a_1\tau = -\frac{k\tau}{\omega_1}(\omega_1 - \omega_1\cos\omega_1 + \tilde{c}\sin\omega_1) \\ a_2\tau^2 = \omega_1(\omega_1 + k\sin\omega_1) - \tilde{c}k\tau(1-\cos\omega_1) \\ c_1k\tau = \dfrac{\omega_2^2 - \omega_1^2}{\omega_1\sin\omega_1 - \omega_2\sin\omega_2 + \tilde{c}(\cos\omega_1 - \cos\omega_2)} \\ -\cos\omega_1 + \frac{\tilde{c}}{\omega_1}\sin\omega_1 = -\cos\omega_2 + \frac{\tilde{c}}{\omega_2}\sin\omega_2. \end{cases} \tag{16}$$

As in the previous subsection, in principle for each $\omega_1 > 0$ there exist multiple $\omega_2 > 0$ such that the last equality in (16) is satisfied. Again, it turns out that it suffices to only consider the smallest $\omega_2 > \omega_1$ satisfying the equality.

As for intersections of the two types of curves in the (a_1, a_2)-plane, numerical experiments have indicated that at these intersections the values of k associated with both curves are identical. An intersection can then in the first place occur if for ω associated with point (a_1, a_2) on the first curve, we have that (a_1, a_2) is on the second curve with $\omega_1 = \omega_2 = \omega$. It may be shown that this occurs when both curves cross the static state feedback stability boundary given by (8). Secondly, an intersection may occur when we have the situation where cases 1 and 2 hold at the same time with $\omega_1 \neq \omega_2$ and $\omega_1, \omega_2 \neq \omega$.

In Figure 3,10 we have plotted the stabilisability area for $\frac{c_2}{c_1}\tau = -0.01$. In Figure 3, the left-hand intersection of the red and black curves is one of the first kind described above, while the right-hand intersection is one of the second kind described above.

A qualitatively completely different case occurs as $\frac{c_2}{c_1}\tau$ is decreased. In Figures 4,7 this is illustrated for the case that $\frac{c_2}{c_1}\tau = -10$. In this case, the intersections of the red and black curves are all of the first kind described above.

In Figures 5,···,10 some more plots of stabilisability regions are given.

C. Comparison with derivative output feedback

In [11] the stabilisability with control laws involving state derivative feedback is addressed, motivated by po-

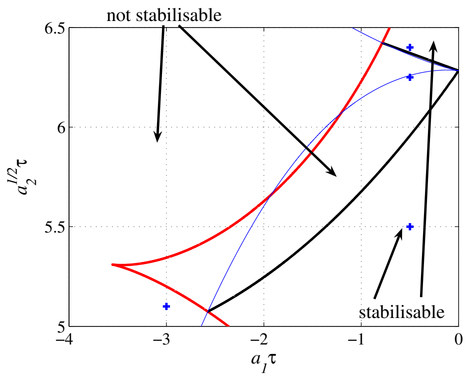

Fig. 4. Stabilisability region for time-delayed output feedback with $\frac{c_2}{c_1}\tau = -10$. Blue curve: stabilisability boundary for time-delayed state feedback; red curve: Case 1; black curve: Case 2.

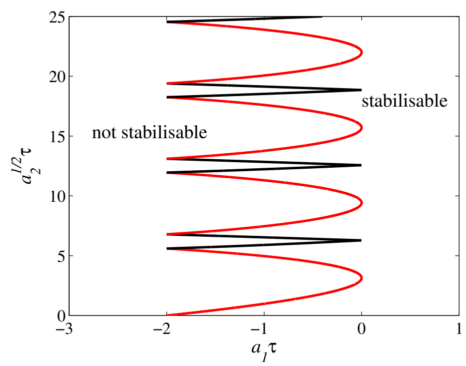

Fig. 5. Stabilisability region for time-delayed output feedback with $c_1c_2 \to -\infty$ (note: this is equivalent to $c_1 = 0, c_2 \neq 0$). Red curve: Case 1; black curve: Case 2.

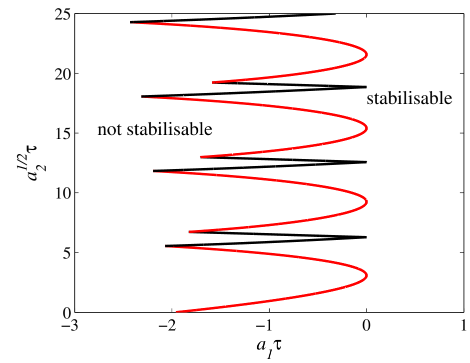

Fig. 6. Stabilisability region for time-delayed output feedback with $\frac{c_2}{c_1}\tau = -100$. Red curve: Case 1; black curve: Case 2.

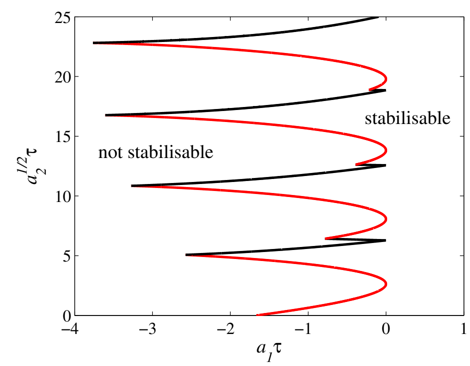

Fig. 7. Stabilisability region for time-delayed output feedback with $\frac{c_2}{c_1}\tau = -10$. Red curve: Case 1; black curve: Case 2.

tential applications in vibration control, where the outputs of accelerometers are used directly for feedback. Given the fact that for small delays time-delayed feedback can be interpreted as an approximation of derivative feedback, following from $\dot{x}(t) \approx \frac{x(t)-x(t-\tau)}{\tau}$, it seems appropriate to make a comparison between the stabilisability regions for time-delayed output feedback (with k and τ as parameters) and the stabilisability regions using the control law $u(t) = k\dot{y}(t)$ where $y(t)$ is defined in (2). We will do this here for the different cases also encountered in this section.

When $c_1c_2 > 0$, stabilisation by means of derivative feedback is always possible, while due to the odd number limitation stabilisation by means of time-delayed feedback is possible if and only if $a_2 > 0$. At first sight the conclusion regarding the case that $a_2 \leq 0$ might look counter-intuitive, because it could be argued that derivative feedback is just time-delayed feedback with a very small time-delay. However, it may be shown that the approximation by time-delayed feedback introduces an unstable half plane pole whose real part moves off to infinity as the delay goes to zero. In fact, it may be shown that for $a_2 \leq 0$ the stabilisation by means of derivative feedback is *fragile* (see [11]), in that small implementation or modelling errors will lead to an unstable closed loop, even though the feedback is stabilisable for the nominal system.

For $c_1c_2 < 0$, the system is stabilisable by means of derivative output feedback if and only if $\left(a_1\left|\frac{c_1}{c_2}\right|+1\right)a_2 \geq 0$. As above, for the part of this region where $a_2 \leq 0$, the stabilisability is fragile. For time-delayed feedback, stabilisation is possible if $a_2 > 0$ and stabilisation by means of derivative feedback is possible, but there is also still a region where $\left(a_1\left|\frac{c_1}{c_2}\right|+1\right) < 0$ and where stabilisation is possible, see Subsection III.B.

In the case that $c_1 = 0, c_2 \neq 0$, stabilisation by means of derivative output feedback as well as time-delayed output feedback is possible if and only if $a_2 > 0$.

Finally, for $c_1 \neq 0, c_2 = 0$ stabilisation by means of derivative output feedback is possible if and only if $a_1a_2 > 0$, where for $a_2 < 0$ the stabilisability is fragile again. For time-delayed output feedback stabilisation is possible if $a_1, a_2 > 0$ and if $a_1 < 0, a_2 > \frac{\pi^2 a_1^2}{16}$, see Subsection III.A.

IV. Conclusions

In this paper we have explored the limits of the stabilisation approach via linear time-delayed state and output feedback. Hereto we have made a paramaterisation of stabilisable system. This has yielded many insights into the mechanism of time-delayed output feedback which could not have been determined by means of conservative analysis methods that

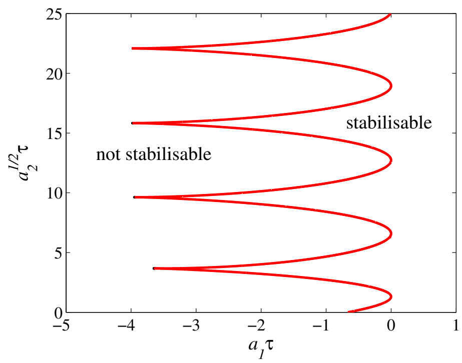

Fig. 8. Stabilisability region for time-delayed output feedback with $\frac{c_2}{c_1}\tau = -1$. Red curve: Case 1; black curve: Case 2.

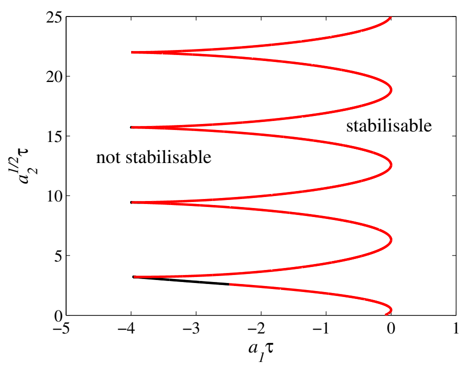

Fig. 9. Stabilisability region for time-delayed output feedback with $\frac{c_2}{c_1}\tau = -0.1$. Red curve: Case 1; black curve: Case 2.

yield sufficient but not necessary stability or stabilisability conditions. However, as has become apparent from the shapes of the stabilisability regions, the parameterisation problem is a very difficult problem even for the second order systems that have been discussed in this paper. We have solved the problem by means of a combination of analytical results (see e.g. Proposition II.1) and recently developed numerical tools like rightmost eigenvalue computation and nonsmooth optimisation. The advantage of our method over a purely numerical approach is that *explicit* expressions of stabilisability boundaries have been obtained based on numerics performed on a *course* grid.

It should be noted that even though our paper was inspired by the references [6],[17], we have not explicitly solved the control problems stated in these references. The main reason for this is that from a purely practical control point of view the problem formulations in our paper are more relevant. The results presented in this paper have direct application in controlling certain systems using delayed output feedback which can be interpreted as either an approximation of derivative feedback, or as a feedback that weighs the difference of an output signal over a time-interval of length τ. However, the methods applied in the present paper can, *mutatis mutandis*, be straightforwardly applied to the control problems in [6],[17].

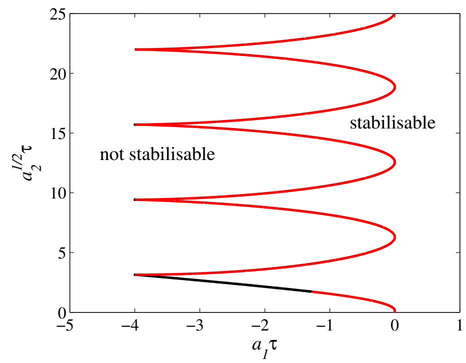

Fig. 10. Stabilisability region for time-delayed output feedback with $\frac{c_2}{c_1}\tau = -0.01$. Red curve: Case 1; black curve: Case 2.

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