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Floquet Transformations for Discrete-time Systems: Equivalence between periodic systems and time-invariant ones

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***Abstract*— This paper considers discrete-time periodic linear systems and their Floquet transformations, by which the periodic systems can be equivalently transformed to time-invariant ones. When the system matrix is nonsingular, it is very easy to derive the discrete-time Floquet transformation, however, when the system matrix is singular, the derivation is essentially different from the continuous-time case. A necessary and sufficient condition for Floquet transformation to exist has been already known and this paper aims to give a simpler proof, by which all the similarity classes of Floquet transformations could be made clear. And also an extension of Floquet transformations to non-autonomous systems is shown.**

I. INTRODUCTION

Recently there has been considerable interest in the study of modeling and control for multi-cylinder engine systems [2][8][9][11], where four strokes, i.e., intake, compression, expansion, and exhaust, repeat sequentially in every cylinder. In order to establish model-based control designs for the automotive engine, a model representation of V6 Spark Ignition SICE benchmark engine is proposed in [7], where by introducing a concept of "role" state variables, the original discrete-time periodic nonlinear state space mode can be equivalently transformed to a time-invariant one.

A lot of model-based control method for continuous-time periodic systems have been proposed, e.g., [10] [13] and references therein. Among them, the theory of Floquet [1][3][14] is very interesting, that is, given a continuous-time periodic linear autonomous system $\dot{x}(t) = A(t)x(t)$ with $A(t+T) = A(t)$, there exists a nonsingular matrix $P(t)$ with $P(t+T) = P(t)$ and $\dot{P}(t)$ being bounded such that the state transformation $\xi(t) = P(t)x(t)$ causes a linear time-invariant system $\dot{\xi}(t) = \tilde{A}\xi(t)$.

The introduction of role state variables in [7] is just a Floquet transformation for the discrete-time periodic *nonlinear* system, which has been derived easily because of the characteristics of automotive engine.

Notice that every continuous-time periodic linear system can be equivalently transformed to a time-invariant one, however, that is not the case with discrete-time periodic systems. In a discrete-time periodic linear system $x_{k+1} = A_k x_k$ with some A_k's being singular, the arguments must be essentially different from the continuous-time case. A discrete-time version of the theory of Floquet has already been considered in [12], which gives a necessary and sufficient condition for Floquet transformations to exist.

The main aim of this paper is to give a novel proof for the result of [12] by showing a self-contained way of constructing Floquet transformations.

Section II formulates the problems to be considered in this paper. In Section III, a necessary and sufficient condition for a square matrix to have N-th roots [4][6] is summarized and also a sufficient condition for an *real* N-th root to exist is given. Section IV is a main part of this paper, where a novel proof for the result of [12] is given. The result for the case of non-autonomous systems is derived in Section V. The conclusion remarks are stated in Section VI.

In this paper, $\mathbf{Z}, \mathbf{R}$ and $\mathbf{C}$ denote sets of all integers, all real numbers and all complex numbers, respectively. Given $N(\geq 2) \in \mathbf{Z}$ and for any $k \in \mathbf{Z}$, $\mathrm{mod}(k/N)$ denotes "k modulo N" and $\mathrm{floor}(k/N)$ "the greatest integer less than or equal to k/N", i.e., $k = \mathrm{floor}(k/N) \times N + \mathrm{mod}(k/N)$. $J_m(0)$ denotes an $m \times m$ Jordan block with eigenvalue 0, i.e.,

$$J_m(0) := \begin{bmatrix} 0_{(m-1)\times 1} & I_{m-1} \\ 0 & 0_{1\times(m-1)} \end{bmatrix}$$

and for any square matrices J_1 and J_2,

$$J_1 \oplus J_2 := \begin{bmatrix} J_1 & 0 \\ 0 & J_2 \end{bmatrix}.$$

II. PROBLEM STATEMENT

Consider a discrete-time periodic system with a period N

$$\begin{aligned} x_{k+1} &= A_k x_k + B_k u_k \\ y_k &= C_k x_k \end{aligned} \quad (1)$$

where $k \in \mathbf{Z}$ is time, $x_k \in \mathbf{R}^n$ is state, $u_k \in \mathbf{R}^m$ input, and $y_k \in \mathbf{R}^p$ output. The matrices A_k, B_k, C_k are real with appropriate sizes, N is a positive integer greater than or equal to 2, and it is assumed that

$$A_k = A_{\mathrm{mod}(k,N)},\ B_k = B_{\mathrm{mod}(k,N)},\ C_k = C_{\mathrm{mod}(k,N)}. \quad (2)$$

Suppose that there exist matrices A, B, C with appropriate sizes and also there exist *nonsingular* matrices $P_k \in \mathbf{R}^{n\times n}, Q_k \in \mathbf{R}^{m\times m}, R_k \in \mathbf{R}^{p\times p}$ such that for any $k \in \mathbf{Z}$,

$$P_{k+1}A_k = AP_k, \qquad P_k = P_{\mathrm{mod}(k/N)} \quad (3)$$

$$P_{k+1}B_k = BQ_k, \qquad Q_k = Q_{\mathrm{mod}(k/N)} \quad (4)$$

$$R_k C_k = CP_k, \qquad R_k = R_{\mathrm{mod}(k/N)}. \quad (5)$$

Then the following equivalent transformations

$$\xi_k = P_k x_k,\ \tau_k = Q_k u_k,\ \eta_k = R_k y_k, \quad (6)$$

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can transform the system (1) to a time-invariant system

$$\begin{array}{rcl} \xi_{k+1} & = & A\xi_k + B\tau_k \\ \eta_k & = & C\xi_k. \end{array} \tag{7}$$

This paper aims to make clear a necessary and sufficient condition for the system (1) to have A, P_k, Q_k, R_k in (3)-(5). In this paper, P_k's and A in (3) are called *Floquet transformation*. A necessary and sufficient condition for the system (1) to have Floquet transformation, which is a condition on rank of A_ks' products, has been derived in [12], where the proof of the sufficiency part is given by showing how to construct P_k's and A from A_k's; 1) extract a structure of nilpotent part of each A_k by using rank data of A_ks' products, 2) update A_k's to be upper-triangular by using "periodic Schur decomposition", 3) construct P_k's and a nonsingular part of A.

It is easy to see from (3) that

$$\Phi_{N-1,0} = \left(P_0^{-1} A P_0\right)^N \tag{8}$$

where $\Phi_{N-1,0}$ is defined as

$$\Phi_{N-1,0} := A_{N-1} \cdots A_1 A_0, \tag{9}$$

which is a state transition matrix from x_0 to x_N in the system (1). Note that (8) means $\Phi_{N-1,0}$ has an N-th root $P_0^{-1}AP_0$.

III. N-TH ROOTS OF MATRIX

In this section, we will summarize some properties on N-th roots of a square matrix. Associated with a matrix $\Phi \in \mathbf{R}^{n\times n}$ and a positive integer $N(\geq 2)$, a matrix $A \in \mathbf{C}^{n\times n}$ satisfying

$$A^N = \Phi \tag{10}$$

is called an N-th root of Φ [4] [6].

It is well known [3][4] that every nonsingular Φ has N-th roots, one of which is calculated by using a Lagrange-Hermite interpolation polynomial. However, when Φ is singular, it sometimes happens [4][6] that Φ has no N-th root.

The next theorem gives a necessary and sufficient condition for Φ to have N-th roots, which corresponds to an extension of the case of $N = 2$ in [6].

Theorem 1: Suppose $P^{-1}\Phi P = \Lambda_\Phi \oplus N_\Phi$ where P is nonsingular, $\Lambda_\Phi \in \mathbf{R}^{\bar{n}\times\bar{n}}$ nonsingular, and $N_\Phi \in \mathbf{R}^{(n-\bar{n})\times(n-\bar{n})}$ nilpotent. Then Φ has N-th roots if and only if N_Φ has the Jordan form $J_1 \oplus \cdots \oplus J_M \oplus 0_{r\times r}$ where for $i = 1, \cdots, M$, $m_i > N$ and

$$J_i = \underbrace{J_{p_i+1}(0) \oplus \cdots \oplus J_{p_i+1}(0)}_{r_i} \oplus \underbrace{J_{p_i}(0) \oplus \cdots \oplus J_{p_i}(0)}_{N-r_i} \in \mathbf{R}^{m_i\times m_i} \tag{11}$$

with $p_i = \text{floor}(m_i/N)$ and $r_i = \text{mod}(m_i/N)$. □

Note that if Φ is nonsingular, then $\bar{n} = n$ in Theorem 1, i.e., there exists no N_Φ, therefore Theorem 1 includes the fact that every nonsingular Φ has N-th roots.

It is straightforward to prove Theorem 1 by using the following two lemmas, which are themselves easy to prove.

Lemma 1: Suppose that $2 \leq N < m_i$, associated with $J_{m_i}(0)$, it follows that

$$J_{m_i}(0)^N = \begin{bmatrix} 0_{(m_i-N)\times N} & I_{m_i-N} \\ 0_{N\times N} & 0_{N\times(m_i-N)} \end{bmatrix} \in \mathbf{R}^{m_i\times m_i}$$

has the Jordan form J_i given in (11). □

Lemma 2: Suppose $N \geq 2$. A nilpotent $N_\Phi \in \mathbf{R}^{n\times n}$ has Nth-roots if and only if N_Φ has the Jordan form structure $J_1 \oplus \cdots \oplus J_M \oplus 0_{r\times r}$ with J_i's shown in (11). Furthermore, an N-th root A of N_Φ has the Jordan form $J_A = J_{m_1}(0) \oplus \cdots \oplus J_{m_M}(0) \oplus J_{M+1}$ where $J_{M+1} \in R^{r\times r}$ is any nilpotent matrix with every Jordan block's size $\leq N$. □

Example 1: A nilpotent $N_\Phi = J_2(0) \oplus J_1(0)$ has a square ($N = 2$) root A, i.e., $A^2 = N_\Phi$. In fact, it is easy to see that N_Φ is just a Jordan form satisfying (11) with $M = 1$, $r = 0$, $m_1 = 3$, $p_1 = 1$, $r_1 = 1$, $N - r_1 = 1$ and also A has the Jordan form $J_3(0)$.

On the other hand, a nilpotent $N_\Phi = J_3(0) \oplus J_1(0) \oplus J_1(0)$ has no square root because it does not have the same structure as in Theorem 1. □

Notice that every real nilpotent matrix has a *real* N-th root as shown in Lemma 2. A necessary and sufficient condition for a real matrix to have a *real* square (i.e., $N = 2$) root is given in [6]. The following theorem shows a sufficient condition for Φ to have a real N-th root.

Theorem 2: Suppose that $\Phi \in \mathbf{R}^{n\times n}$ has an N-th root and let $P^{-1}\Phi P = \Lambda_\Phi \oplus N_\Phi$ where P is nonsingular, Λ_Φ nonsingular, and N_Φ nilpotent. If Λ_Φ is *diagonalizable* and also either 1) or 2) holds, then Φ has a *real* N-th root.

1) N is odd.
2) N is even and every negative real eigenvalue of Φ has even multiplicity. □

[Proof] Without loss of generality, it is assumed that $\Lambda_\Phi = \Lambda_p \oplus \Lambda_n \oplus \Lambda_c$ where $\Lambda_p = r_1 I_{k_1} \oplus \cdots \oplus r_k I_{k_k}$ with $r_i > 0$ ($i = 1, \cdots, k$), $\Lambda_n = s_1 I_{\ell_1} \oplus \cdots \oplus s_\ell I_{\ell_\ell}$ with $s_i < 0$ ($i = 1, \cdots, \ell$), and $\Lambda_c = \Omega_1 \oplus \cdots \oplus \Omega_m$ with

$$\Omega_i = t_i \begin{bmatrix} \cos(\theta_i) & \sin(\theta_i) \\ -\sin(\theta_i) & \cos(\theta_i) \end{bmatrix}$$

$t_i > 0$ and $\theta_i \in (-\pi, \pi]$ ($i = 1, \cdots, m$).

If every negative real eigenvalue of Φ has even multiplicity, it follows that $\bar{\ell}_i := \ell_i/2$ ($i = 1, \cdots, \ell$) is integer, which means that Λ_n can be expressed as $\Lambda_n = S_1 \oplus \cdots \oplus S_\ell$ with

$$S_i = \underbrace{\Gamma_i \oplus \cdots \oplus \Gamma_i}_{\bar{\ell}_i},\ \Gamma_i = |s_i| \begin{bmatrix} \cos\pi & \sin\pi \\ -\sin\pi & \cos\pi \end{bmatrix}.$$

From the above observations, it is straightforward to prove that Φ has a real N-th root. (Q.E.D.)

IV. DISCRETE-TIME FLOQUET TRANSFORMATION

This section considers Floquet transformation (3), which is rewritten as

$$\begin{array}{rcl} P_{k+1}A_k & = & AP_k \\ P_k & = & P_{\text{mod}(k/N)} \end{array} \begin{array}{r} (12) \\ (13) \end{array}$$

where P_k's are nonsingular.

In order to make notations simpler, we will assume that $\Phi_{N-1,0}$ in (9) has a *real* N-th root hereafter. Notice that even if $\Phi_{N-1,0}$ has only complex N-th roots, all the results in this section still hold by representing them in $\mathbf{C}^n$ instead of $\mathbf{R}^n$.

Theorem 3: Given $A_k \in \mathbf{R}^{n\times n}$ in (2), there exists nonsingular $P_k \in \mathbf{R}^{n\times n}$ and a matrix $A \in \mathbf{R}^{n\times n}$ satisfying (12) and (13) if and only if it holds that

$$\begin{array}{c}\mathrm{rank}A_{k-1}A_{k-2}\cdots A_{h+1}A_h = \mathrm{rank}A^{k-h} \\ \text{for } 0 \le h \le N-1, \text{ and } 1 \le k-h \le n\end{array} \quad (14)$$

where A is any matrix similar to one of N-th roots of $\Phi_{N-1,0}$. □

The condition given in [12] is as follows.

$$\begin{array}{c}\mathrm{rank}A_{j+k-1}\cdots A_{j+1}A_j = \sigma_k \text{ independent of } j \\ \text{for } 0 \le j \le N-1, \text{ and } 1 \le k \le n\end{array} \quad (15)$$

We claim that (14) and (15) are equivalent. In fact, it is trivial that (14) implies (15). Contrarily, (15) implies (14) because of the observation that a matrix A in (14) can be constructed by using the data $\{\sigma_k \mid 1 \le k \le n\}$ in (15) and eigenspace decomposition of the nonsingular part of $\Phi_{N-1,0}$.

Now we will prove Theorem 3 independently of [12].

The necessity part of Theorem 3 is trivial because of (12), (13) and (8).

The sufficiency part of the theorem will be proven by showing how to construct P_k's in (12) and (13) in a way of using a matrix A in (14) explicitly, which are Lemmas 3, 4 and 5 below.

Before expressing those lemmas, without loss of generality, we assume that the matrix A in (14) is given by

$$A = \Lambda_A \oplus N_A \quad (16)$$

where $\Lambda_A \in \mathbf{R}^{\bar{n}\times\bar{n}}$ is nonsingular and $N_A \in \mathbf{R}^{(n-\bar{n})\times(n-\bar{n})}$ is nilpotent with a structure as follows.

$$N_A = J_1 \oplus J_2 \oplus \cdots \oplus J_\ell \oplus 0_{(n-\bar{n}-m)\times(n-\bar{n}-m)} \quad (17)$$

where for $i = 1, \cdots, \ell$,

$$J_i = \underbrace{J_{m_i}(0) \oplus J_{m_i}(0) \cdots \oplus J_{m_i}(0)}_{r_i} \in \mathbf{R}^{r_i m_i \times r_i m_i} \quad (18)$$

with $m_1 > m_2 > \cdots > m_\ell \ge 2$ and $m := \sum_{i=1}^{\ell} r_i m_i$.

Note that there exists the following relation between the data $\{\bar{n}, m_i, r_i \mid i = 1, \cdots, \ell\}$ in (16)-(18) and $\{\sigma_k \mid k = 1, \cdots, n\}$ in (15).

$$\sigma_k = \bar{n} + \sum_{i=1}^{\ell} r_i \max\{m_i - k, 0\} \quad (19)$$

Therefore, we can see that $\sigma_k = \bar{n}$ for any $k \ge m_1$, $\sigma_k = \bar{n} + r_1(m_1 - k)$ for $m_2 \le k < m_1$, $\sigma_k = \bar{n} + r_1(m_1 - k) + r_2(m_2 - k)$ for $m_3 \le k < m_2$, and so on.

Lemma 3: Suppose that A is given as (16). Then the following facts 1) and 2) are equivalent.

1) There exist nonsingular matrices $P_k \in \mathbf{R}^{n\times n}$ ($k \in \mathbf{Z}$) satisfying (12) and (13).
2) There exist $P_{k,1} \in \mathbf{R}^{\bar{n}\times n}$ and $P_{k,2} \in \mathbf{R}^{(n-\bar{n})\times n}$ ($k \in \mathbf{Z}$) such that

$$P_{k+1,1}A_k = \Lambda_A P_{k,1}, \quad P_{k,1} = P_{\mathrm{mod}(k/N),1} \quad (20)$$
$$P_{k+1,2}A_k = N_A P_{k,2}, \quad P_{k,2} = P_{\mathrm{mod}(k/N),2} \quad (21)$$

where $P_{k,1}$'s and $P_{k,2}$'s are respectively of full row rank.

□

[Proof] 1) ⇒ 2): Divide every P_k into two matrices $P_{k,1} \in \mathbf{R}^{\bar{n}\times n}$ and $P_{k,2} \in \mathbf{R}^{(n-\bar{n})\times n}$ where $P_{k,1}$ consists of the first $\bar{n}$ rows of P_k, then it is trivial that 1) implies 2).

2) ⇒ 1): By using $P_{k,1}$ in (20) and $P_{k,2}$ in (21), make a matrix P_k as

$$P_k = \begin{bmatrix} P_{k,1} \\ P_{k,2} \end{bmatrix} \in \mathbf{R}^{n\times n},$$

then it is trivial that P_k satisfies (12) and (13).

What remains to prove is that P_k made above is nonsingular. Suppose that there exists a vector $v = (v_1, v_2) \in \mathbf{R}^{1\times n}$ with $v_1 \in \mathbf{R}^{1\times\bar{n}}$ and $v_2 \in \mathbf{R}^{1\times(n-\bar{n})}$ such that

$$vP_k = v_1 P_{k,1} + v_2 P_{k,2} = 0. \quad (22)$$

Then it is straightforward to verify from (20) and (21) that for any $h, k \in \mathbf{Z}$ with $h < k$,

$$0 = vP_k A_{k-1} \cdots A_h = v_1 \Lambda_A^{k-h} P_{h,1} + v_2 N_A^{k-h} P_{h,2}.$$

Noticing that N_A is nilpotent, in the case that $k - h \ge m_1$, the above equation becomes $0 = v_1 \Lambda_A^{k-h} P_{h,1}$, which implies $v_1 = 0$ because $\Lambda_A^{k-h} P_{h,1}$ is of full row rank. Thus (22) gives $v_2 P_{k,2} = 0$ and we get $v_2 = 0$ because $P_{k,2}$ is of full row rank. Consequently, (22) implies $v = 0$, so it has been proven that P_k is nonsingular. (Q.E.D.)

Lemma 4: Suppose that A is similar to an N-th root of $\Phi_{N-1,0}$ with A given as (16). Then there exist full row rank matrices $P_{k,1} \in \mathbf{R}^{\bar{n}\times n}$ ($k \in \mathbf{Z}$) satisfying (20). □

[Proof] Notice that there exists a nonsingular matrix $Q \in \mathbf{R}^{n\times n}$ such that

$$(Q^{-1}AQ)^N = \Phi_{N-1,0},$$

which means

$$Q\Phi_{N-1,0} = A^N Q = (\Lambda_A^N \oplus N_A^N)Q. \quad (23)$$

Define a matrix $P_{0,1} \in \mathbf{R}^{\bar{n}\times n}$ as the first $\bar{n}$ rows of Q. Note that $P_{0,1}$ is of full row rank. Then it is easy from (23) to derive

$$P_{0,1}\Phi_{N-1,0} = \Lambda_A^N P_{0,1}$$

that is,

$$P_{0,1} = \Lambda_A^{-N} P_{0,1} A_{N-1} \cdots A_1 A_0, \quad (24)$$

which means that not only $P_{0,1}A_{N-1}\cdots A_1 A_0$ but also $P_{0,1}A_{N-1}\cdots A_k$ for any $k \in \{1, 2, \cdots, N-1\}$ are of full row rank.

Now define $P_{k,1} \in \mathbf{R}^{\bar{n}\times n}$ for $k = 1, 2, \cdots, N-1$ by

$$P_{k,1} := \Lambda_A^{-(N-k)} P_{0,1} A_{N-1} \cdots A_k, \tag{25}$$

then $P_{k,1}$ is of full row rank and satisfies (20). In fact,

$$\begin{aligned} P_{k+1,1}A_k &= \Lambda_A^{-\{N-(k+1)\}} P_{0,1} A_{N-1} \cdots A_{k+1} A_k \\ &= \Lambda_A \Lambda_A^{-(N-k)} P_{0,1} A_{N-1} \cdots A_{k+1} A_k \\ &= \Lambda_A P_{k,1}. \end{aligned}$$

Set $P_{k,1} := P_{\mathrm{mod}(k,N),1}$ for the other $k \in \mathbf{Z}$, then all the $P_{k,1}$'s are of full row rank and satisfy (20). (Q.E.D.)

Lemma 5: Suppose that A is similar to an N-th root of $\Phi_{N-1,0}$ with A given as (16). Then if (14) holds, there exist full row rank matrices $P_{k,2} \in \mathbf{R}^{(n-\bar{n})\times n}$ ($k \in \mathbf{Z}$) satisfying (21). □

Lemma 5 can be proven by Lemmas 6, 7, and 8 below.

Lemma 6: If (14) holds, there exist $p_{k,j}^{(i)} \in \mathbf{R}^{1\times n}$ ($k = 0, 1, \cdots, N-1$; $i = 1, \cdots, \ell$; $j = 1, \cdots, r_i$) such that

$$\begin{array}{l} p_{k,j}^{(i)} A_{k-1} A_{k-2} \cdots A_{k-m_i+1} \neq 0 \\ p_{k,j}^{(i)} A_{k-1} A_{k-2} \cdots A_{k-m_i+1} A_{k-m_i} = 0 \end{array} \tag{26}$$

and also, for any $k = 0, \cdots, N-1$, a set of vectors

$$\left\{ p_{k+m_i-1,j}^{(i)} A_{k+m_i-2} \cdots A_k \;\middle|\; i = 1, \cdots, \ell;\; j = 1, \cdots, r_i \right\} \tag{27}$$

is linearly independent. □

[Proof] When N_A has the structure of (17) and (18), by using (14), it is easy to see from (19) that

$$\begin{array}{l} \mathrm{rank} A_{k-1} A_{k-2} \cdots A_{k-m_i+1} = \mathrm{rank} A^{m_i - 1} \\ \quad = \bar{n} + \mathrm{rank} N_A^{m_i-1} = \bar{n} + \sum_{h=1}^{i} r_h (m_h - m_i + 1) \\ \mathrm{rank} A_{k-1} A_{k-2} \cdots A_{k-m_i+1} A_{k-m_i} = \mathrm{rank} A^{m_i} \\ \quad = \bar{n} + \mathrm{rank} N_A^{m_i} = \bar{n} + \sum_{h=1}^{i-1} r_h (m_h - m_i), \end{array}$$

which means, for any $k = 0, 1, \cdots, N-1$ and $i = 1, 2, \cdots, \ell$,

$$\begin{array}{l} \sum_{h=1}^{i} r_h = \mathrm{rank} A_{k-1} A_{k-2} \cdots A_{k-m_i+1} \\ \qquad - \mathrm{rank} A_{k-1} A_{k-2} \cdots A_{k-m_i+1} A_{k-m_i}. \end{array}$$

Therefore, there exists a set of vectors $\{w_{k,j}^{(i)} \in \mathbf{R}^{1\times n} \mid j = 1, 2, \cdots, (r_1 + \cdots + r_i)\}$ which is linearly independent and satisfy

$$\begin{array}{l} w_{k,j}^{(i)} A_{k-1} A_{k-2} \cdots A_{k-m_i+1} \neq 0 \\ w_{k,j}^{(i)} A_{k-1} A_{k-2} \cdots A_{k-m_i+1} A_{k-m_i} = 0. \end{array} \tag{28}$$

This holds for any $i \in \{1, 2, \cdots, \ell\}$.

From the above definition of $w_{k,j}^{(i)}$'s, it is straightforward to see that we can chose the vectors $w_{k,j}^{(i)}$'s and $w_{k,j}^{(i+1)}$'s such that they have the following relations

$$w_{k-(m_i-m_{i+1}),j}^{(i+1)} = w_{k,j-r_{i+1}}^{(i)} A_{k-1} A_{k-2} \cdots A_{k-(m_i-m_{i+1})}$$

which is equivalent to

$$w_{k,j}^{(i+1)} = w_{k+m_i-m_{i+1},j-r_{i+1}}^{(i)} A_{k+m_i-m_{i+1}-1} \cdots A_{k+1} A_k \tag{29}$$

where $j \in \{1, 2, \cdots, (r_1 + \cdots + r_{i+1})\} / \{1, 2, \cdots, r_{i+1}\}$.

Define $p_{k,j}^{(i)} := w_{k,j}^{(i)}$ for $i = 1, \cdots, \ell$ and $j = 1, \cdots, r_i$, it is easy to see that (26) holds.

Now in order to prove that the set of vectors (27) is linearly independent, we will show that the set of vectors (27) is equal to the set

$$\left\{ w_{k+m_\ell-1,j}^{(\ell)} A_{k+m_\ell-2} \cdots A_k \;\middle|\; j = 1, \cdots, r_1 + \cdots + r_\ell \right\}.$$

which is linearly independent because of its definition.

In fact, it holds that for $i = \ell$ and $j = 1, \cdots, r_\ell$,

$$p_{k+m_\ell-1,j}^{(\ell)} A_{k+m_\ell-2} \cdots A_k = w_{k+m_\ell-1,j}^{(\ell)} A_{k+m_\ell-2} \cdots A_k$$

and also it is easy to verify by using (29) that for $i = 1, 2, \cdots, \ell-1$ and $j = 1, \cdots, r_i$,

$$\begin{array}{l} p_{k+m_i-1,j}^{(i)} A_{k+m_i-2} \cdots A_k = w_{k+m_i-1,j}^{(i)} A_{k+m_i-2} \cdots A_k \\ \quad = w_{k+m_\ell-1,j+r_\ell+\cdots+r_{i+1}}^{(\ell)} A_{k+m_\ell-2} \cdots A_k. \end{array}$$

(Q.E.D.)

Lemma 7: Suppose that (14) holds. By using the vector $p_{k,j}^{(i)} \in \mathbf{R}^{1\times n}$ obtained in Lemma 6, define matrices $P_{k,21} \in \mathbf{R}^{m\times n}$ ($k = 0, 1, \cdots, N-1$) as follows and set $P_{k,21} = P_{\mathrm{mod}(k/N),21}$ for the other $k \in \mathbf{Z}$.

$$P_{k,21} = \begin{bmatrix} P_k^{(1)} \\ P_k^{(2)} \\ \vdots \\ P_k^{(\ell)} \end{bmatrix}, \; P_k^{(i)} = \begin{bmatrix} P_{k,1}^{(i)} \\ P_{k,2}^{(i)} \\ \vdots \\ P_{k,r_i}^{(i)} \end{bmatrix} \in \mathbf{R}^{r_i m_i \times n} \tag{30}$$

and

$$P_{k,j}^{(i)} = \begin{bmatrix} p_{k,j}^{(i)} \\ p_{k+1,j}^{(i)} A_k \\ p_{k+2,j}^{(i)} A_{k+1} A_k \\ \vdots \\ p_{k+m_i-1,j}^{(i)} A_{k+m_i-2} \cdots A_k \end{bmatrix} \in \mathbf{R}^{m_i \times n} \tag{31}$$

Then $P_{k,21}$'s are of full row rank and it holds that

$$P_{k+1,21} A_k = (J_1 \oplus \cdots \oplus J_\ell) P_{k,21} \tag{32}$$

□

[Proof] From (31) and (26), it is easy to derive

$$P_{1,j}^{(i)} A_0 = \begin{bmatrix} p_{1,j}^{(i)} \\ p_{2,j}^{(i)} A_1 \\ p_{3,j}^{(i)} A_2 A_1 \\ \vdots \\ p_{m_i-1,j}^{(i)} A_{m_i-2} \cdots A_2 A_1 \\ p_{m_i,j}^{(i)} A_{m_i-1} \cdots A_2 A_1 \end{bmatrix} A_0$$

$$= \begin{bmatrix} p_{1,j}^{(i)} A_0 \\ p_{2,j}^{(I)} A_1 A_0 \\ p_{3,j}^{(i)} A_2 A_1 A_0 \\ \vdots \\ p_{m_i-1,j}^{(i)} A_{m_i-2} \cdots A_2 A_1 A_0 \\ 0 \end{bmatrix} = J_{m_i}(0) P_{0,j}^{(1)}.$$

That holds for $i = 1, \cdots, \ell$, so we obtain $P_{1,21}A_0 = (J_1 \oplus \cdots \oplus J_\ell)P_{0,21}$, which is (32) with $k = 0$. Similarly it can be proven that (32) holds for the other $k \in \mathbf{Z}$.

Now we will prove that $P_{k,21}$ defined by (30) and (31) is of full row rank.

First, we will prove that each submatrix $P_k^{(i)}$ of $P_{k,21}$ is of full row rank.

Suppose $vP_k^{(i)} = 0$, denoting $v = [v_1 \ v_2 \ \cdots \ v_{r_i}]$ with every $v_j \in \mathbf{R}^{1\times m_i}$, that is

$$vP_k^{(i)} = v_1 P_{k,1}^{(i)} + v_2 P_{k,2}^{(i)} + \cdots + v_{r_i} P_{k,r_i}^{(i)} = 0. \quad (33)$$

Then postmultiply $A_{k-1}\cdots A_{k-m_i+1}$ and it is easy to show from (26) that all the rows of $P_{k,j}^{(i)}A_{k-1}\cdots A_{k-m_i+1}$ except the first row are zeros, therefore we get

$$\sum_{j=1}^{r_i} \alpha_j p_{k,j}^{(i)} A_{k-1}\cdots A_{k-m_i+1} = 0$$

where $\alpha_j \in \mathbf{R}$ is the first element of v_j. Recall that the vectors $p_{k,j}^{(i)}A_{k-1}\cdots A_{k-m_i+1}$'s are linearly independent, which was shown in Lemma 6, thus it follows that all the α_j's are zeros. This implies that (33) becomes

$$vP_k^{(i)} = \tilde{v}_1 P_{k,1}^{(i)} + \tilde{v}_2 P_{k,2}^{(i)} + \cdots + \tilde{v}_{r_i} P_{k,r_i}^{(i)} = 0 \quad (34)$$

where $\tilde{v}_j = [0 \ v_{j1}], v_{j1} \in \mathbf{R}^{1\times(m_i-1)}$.

Postmultiply (34) by $A_{k-1}\cdots A_{k-m_i+2}$ and use the same logic as shown above, we can show the first element of v_{j1} is zero.

Continue the above procedure, finally we obtain the fact that $vP_k^{(i)} = 0$ implies $v = 0$, that is, $P_k^{(i)}$ is of full row rank.

Now, we will prove that $P_{k,21} \in \mathbf{R}^{m\times n}$ is of full row rank.

Suppose $sP_{k,21} = 0$, denoting $s = [s_1 \ s_2 \ \cdots \ s_\ell]$ with $s_i \in \mathbf{R}^{1\times r_i m_i}$, that is

$$sP_{k,21} = s_1 P_k^{(1)} + s_2 P_k^{(2)} + \cdots + s_\ell P_k^{(\ell)} = 0. \quad (35)$$

Multiply this equation by $A_{k-1}\cdots A_{k-m_1+1}$ to get

$$sP_{k,21}A_{k-1}\cdots A_{k-m_1+1} = 0. \quad (36)$$

Notice $P_k^{(i)}A_{k-1}\cdots A_{k-m_1+1} = 0$ for $i = 2, \cdots, \ell$ and then (36) goes to $s_1 P_k^{(1)} A_{k-1}\cdots A_{k-m_1+1} = 0$, which means $s_1 = 0$ from the same logic above. Therefore (35) becomes

$$sP_{k,21} = s_2 P_k^{(2)} + \cdots + s_\ell P_k^{(\ell)} = 0,$$

multiply this by $A_{k-1}\cdots A_{k-m_2+1}$, then by using the same logic, it follows $s_2 = 0$. This process finally gives $s_i = 0$ for $i = 1, \cdots, \ell$, that is, $s = 0$ which means that $P_{k,21}$ is of full row rank. (Q.E.D.)

Lemma 8: Suppose that (14) holds. Then there exist $P_{k,22} \in \mathbf{R}^{(n-\bar{n}-m)\times n}$ ($k \in \mathbf{Z}$) such that

$$P_{k+1,22}A_k = 0, \quad P_{k,22} = P_{\mathrm{mod}(k/N),22} \quad (37)$$

and each matrix $P_{k,2} \in \mathbf{R}^{(n-\bar{n})\times n}$ defined by

$$P_{k,2} := \begin{bmatrix} P_{k,21} \\ P_{k,22} \end{bmatrix} \in \mathbf{R}^{(n-\bar{n})\times n} \quad (38)$$

is of full row rank where $P_{k,21}$ are given in Lemma 7. □

[Proof] Consider a subspace $\mathcal{W}_k \subset \mathbf{R}^n$ defined by

$$\mathcal{W}_k = \left\{ w \in \mathbf{R}^{1\times n} \middle| \, wA_{k-1} = 0 \right\}.$$

Note that (14) includes $\mathrm{rank}A_{k-1} = \mathrm{rank}A$, and also recall N_A's structure (17) and (18). Then it is easy to see $\dim\mathcal{W}_k = n - \bar{n} - m + (r_1 + \cdots + r_\ell)$.

Denote the last row vector of $P_{k,j}^{(i)}$ ($i = 1, \cdots, \ell$; $j = 1, \cdots, r_i$) belongs to$\mathcal{W}_k$, which means that we have already obtained $r_1 + \cdots + r_\ell$ linearly independent vectors in $\mathcal{W}_k$.

Therefore we can find another $(n - \bar{n} - m)$ linearly independent vectors in $\mathcal{W}_k$, of which $P_{k,22} \in \mathbf{R}^{(n-\bar{n}-m)\times n}$ can be composed. In fact, it is easy to see (37). And also it is easy to verify that $P_{k,2}$ given in (38) is of full row rank by using the same logic as used in the proof of Lemma 7. (Q.E.D.)

The correspondence between the above lemmas and how to construct the submatrices $P_{k,1}$, $P_{k,21}$, and $P_{k,22}$ of P_k in Theorem 3 is expressed as follows.

$$P_k = \begin{bmatrix} P_{k,1} \\ P_{k,21} \\ P_{k,22} \end{bmatrix} \quad \begin{matrix} \longleftarrow & \text{Lemma 4} \\ \longleftarrow & \text{Lemmas 6 and 7} \\ \longleftarrow & \text{Lemma 8} \end{matrix}$$

Note that $P_{k,1}$ is given by finding a basis consisting of eigenvectors associated with all of the nonzero eigenvalues of $\Phi_{N-1,0}$. Also $P_{k,21}$ and $P_{k,22}$ are given by finding a basis consisting of eigenvectors associated with zero eigenvalues of A_k.

Now let us look at a numerical example.

Example 2: Consider $x_{k+1} = A_k x_k$ where $N = 3$ and

$$A_0 = \begin{bmatrix} 0&0&0&0&0 \\ 0&1&0&0&0 \\ 0&0&0&0&2 \\ 0&0&0&1&0 \\ 0&0&0&0&0 \end{bmatrix}, \ A_1 = \begin{bmatrix} 0&0&1&0&0 \\ 0&0&0&0&0 \\ 1&0&0&0&0 \\ 0&1&0&0&0 \\ 0&0&0&0&0 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} 0&0&0&0&0 \\ 0&0&0&0&0 \\ 0&0&1&0&0 \\ 0&1&0&0&0 \\ 4&0&0&0&0 \end{bmatrix}, \ A_k = A_{\mathrm{mod}(k/3)}.$$

It is easy to see that $\Phi_{2,0} := A_2A_1A_0 = 0_{4\times4} \oplus 8$ satisfies the fact 2) of Theorem 1 and so it has cube root similar to $A = 2 \oplus N_A$ where N_A is any nilpotent matrix which is composed of Jordan blocks with their sizes ≤ 3.

Furthermore it is easy to verify that

$$\begin{aligned} &\mathrm{rank}A_0 = \mathrm{rank}A_1 = \mathrm{rank}A_2 = 3 \\ &\mathrm{rank}A_1A_0 = \mathrm{rank}A_2A_1 = \mathrm{rank}A_0A_2 = 2 \\ &\mathrm{rank}A_2A_1A_0 = \mathrm{rank}A_0A_2A_1 = \mathrm{rank}A_1A_0A_2 = 1 \\ &\mathrm{rank}A_{k-1}\cdots A_h = 1 \text{ for } \forall(k-h) \geq 4. \end{aligned}$$

From those observations, if the nilpotent N_A of A is chosen as $N_A = J_3(0) \oplus J_1(0)$, then A_k's and A satisfy (14). Note that $n = 5$, $\bar{n} = 1$, and $\ell = 1, r_1 = 1, m_1 = m = 3$.

Now let us construct $P_{k,1}$. First (24) in Lemma 4 gives

$$P_{0,1} = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \end{bmatrix} \in \mathbf{R}^{\bar{n}\times n}$$

and (25) in Lemma 4 gives

$$\begin{aligned} P_{2,1} &= \Lambda_A^{-1} P_{0,1} A_2 = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 \end{bmatrix} \\ P_{1,1} &= \Lambda_A^{-2} P_{0,1} A_2 A_1 = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \end{bmatrix}. \end{aligned}$$

Next let us construct $P_{k,21}$. It is easy to find $p_{k,1}^{(1)} \in \mathbf{R}^{1\times n}$ $(k = 0, 1, 2)$ in (26), Lemma 6 as follows.

$$p_{0,1}^{(1)} = \begin{bmatrix} \alpha_0 & \beta_0 & a_0 & \gamma_0 & 0 \end{bmatrix},\ p_{1,1}^{(1)} = \begin{bmatrix} \alpha_1 & \beta_1 & 0 & a_1 & \gamma_1 \end{bmatrix}$$
$$p_{2,1}^{(1)} = \begin{bmatrix} 0 & \alpha_2 & \beta_2 & a_2 & \gamma_2 \end{bmatrix}$$

where $a_k (\neq 0), \alpha_k, \beta_k, \gamma_k \in \mathbf{R}$ are free parameters $(k = 0, 1, 2)$. Then we can calculate $P_{k,21}$ in (30), Lemma 7 as follows.

$$P_{0,21} = \begin{bmatrix} \alpha_0 & \beta_0 & a_0 & \gamma_0 & 0 \\ 0 & \beta_1 & 0 & a_1 & 0 \\ 0 & a_2 & 0 & 0 & 0 \end{bmatrix},\ P_{1,21} = \begin{bmatrix} \alpha_1 & \beta_1 & 0 & a_1 & \gamma_1 \\ \beta_2 & a_2 & 0 & 0 & 0 \\ a_0 & 0 & 0 & 0 & 0 \end{bmatrix}$$
$$P_{2,21} = \begin{bmatrix} 0 & \alpha_2 & \beta_2 & a_2 & \gamma_2 \\ 0 & \gamma_0 & a_0 & 0 & 0 \\ 0 & a_1 & 0 & 0 & 0 \end{bmatrix}.$$

$P_{k,22}$'s in (37), Lemma 8 are easily obtained as follows.

$$P_{0,22} = \begin{bmatrix} b_0 & \delta_0 & 0 & 0 & 0 \end{bmatrix},\ P_{1,22} = \begin{bmatrix} \delta_1 & 0 & 0 & 0 & b_1 \end{bmatrix}$$
$$P_{0,22} = \begin{bmatrix} 0 & \delta_2 & 0 & 0 & b_2 \end{bmatrix}$$

where $b_k (\neq 0), \delta_k \in \mathbf{R}$ are free parameters $(k = 0, 1, 2)$.

Finally we can construct P_k's by composing $P_{k,1}$,$P_{k,21}$, and $P_{k,22}$ as follows.

$$P_0 = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ \alpha_0 & \beta_0 & a_0 & \gamma_0 & 0 \\ 0 & \beta_1 & 0 & a_1 & 0 \\ 0 & a_2 & 0 & 0 & 0 \\ b_0 & \delta_0 & 0 & 0 & 0 \end{bmatrix},\ P_1 = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ \alpha_1 & \beta_1 & 0 & a_1 & \gamma_1 \\ \beta_2 & a_2 & 0 & 0 & 0 \\ a_0 & 0 & 0 & 0 & 0 \\ \delta_1 & 0 & 0 & 0 & b_1 \end{bmatrix}$$
$$P_2 = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & \alpha_2 & \beta_2 & a_2 & \gamma_2 \\ 0 & \gamma_0 & a_0 & 0 & 0 \\ 0 & a_1 & 0 & 0 & 0 \\ 0 & \delta_2 & 0 & 0 & b_2 \end{bmatrix},\ P_k = P_{\mathrm{mod}(k/3)}$$

□

V. Non-Autonomous System

In this section, associated with the discrete-time periodic system (1), suppose that Floquet transformation P_k's in (3) with A are already known, then consider a problem of finding Q_k, R_k, B, C in (4) and (5).

The following theorem answers to this problem.

Theorem 4: Given A_k, B_k, C_k satisfying (2), suppose that there exist A and nonsingular P_k satisfying (3). Then there exist B, C and nonsingular Q_k, R_k satisfying (4),(5) if and only if it holds that for any $k \in \mathbf{Z}$,

$$\begin{aligned} \mathrm{Im} P_{k+1} B_k &= \mathrm{Im} P_k B_{k-1} &\quad (39) \\ \mathrm{Ker} C_k P_k^{-1} &= \mathrm{Ker} C_{k-1} P_{k-1}^{-1}. &\quad (40) \end{aligned}$$

□

[Proof] The necessity part is trivial, because (4) and (5) mean $\mathrm{Im} P_{k+1} B_k = \mathrm{Im} B$ and $\mathrm{Ker} C_k P_k^{-1} = \mathrm{Ker} C$, respectively.

The sufficiency part can be proven as follows.

It is easy to see from (39) that there exists an $B \in \mathbf{R}^{n\times m}$ with $\mathrm{Im} P_{k+1} B_k = \mathrm{Im} B$ for any $k \in \mathbf{Z}$. Therefore you can find nonsingular $Q_k \in \mathbf{R}^{m\times m}$ satisfying (4).

Similarly, (40) implies that there exists a $C \in \mathbf{R}^{p\times n}$ with $\mathrm{Ker} C_k P_k^{-1} = \mathrm{Ker} C$ for any $k \in \mathbf{Z}$, and then there exist nonsingular $R_k \in \mathbf{R}^{p\times p}$ satisfying (5). (Q.E.D.)

VI. Conclusions

It is well known as theory of Floquet that every continuous-time periodic linear system can be equivalently transformed in the sense of Liapnov to a time-invariant one. For discrete-time periodic linear systems, the theorem of Floquet transformation has been already shown in the previous research [12].

This paper gave a new proof of discrete-time Floquet transformation, which is suitable for listing up all the different similarity classes of Floquet transformations [5]. One of interesting future researches is to make this list and to find out a real Floquet transformation among them.

Floquet transformations for discrete-time periodic nonlinear systems of automotive engines have been done successfully in [7], The other future research is to expand this result to general nonlinear systems.

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References

[1] R.W. Brockett, "Finite Dimensional Linear System", *John Wiley and Sons,inc.*, p46-48, 1970.
[2] J. Chauvin, G. Corde, N. Petit, and P.Rouchon, "Periodic input estimation for linear periodic systems: Automotive engine applications", *Automatica*, V0l.43, pp971-980, 2007.
[3] C.T. Chen, "Linear System Theory and Design", *Holt, Rinehart and Winston*, p153-154, 1984.
[4] F.R. Gantmacher, "The Theory of Matrices", Vol.1, *Chelsea*, New York, Chapter VIII, 1959.
[5] Y.Hayakawa and T.Jimbo, "On Similarity Classes of Discrete-time Floquet Transformations"(in Japanese), The 37th SICE Symposium on Control Theory, 2008.9, to appear.
[6] G.A. Horn and C.R. Johnson, "Topics in Matrix Analysis", *Cambridge University Press*, Chapter 6, 1991.
[7] T. Jimbo and Y. Hayakawa, "Physical-Model-Based Control of Engine Cold Start via Role State Variables", *17th World Congress of IFAC*(Seoul,Korea), pp1024-1029, 2008
[8] R. Johansson and A. Rantzer, "Nonlinear and Hybrid Systems in Automotive Control", *Springer-Verlag London*, 2003.
[9] R.K. Jurgen, "Electronic Engine Control Technologies" 2ndEdition, *SAE*, 2004.
[10] P. Montagnier and R.J. Spiteri, "A Gramian-Based Controller for Linear Periodic Systems", *IEEE Trans. Automat. Contr.*, Vol.49, No.8, pp1380-1385, 2004.
[11] A. Ohata, "An Adaptive Fuel Injection Control with Internal Model inAutomotive Engines",*IECON'90*, pp78-83, 1990.
[12] P. Van Dooren and J. Sreedhar, "When is a Periodic Discrete-Time System Equivalent to a Time-Invariant One?", *Linear Algebra and its Applications*, 212/213, pp131-151, 1994.
[13] J.J. Yamé and R.Hanus, "On Stabilization and Spectrum Assignment in Periodically Time-Varying Continuous-Time Systems", *IEEE Trans. Automat. Contr.*, Vol.46, No.6, pp979-983, 2001.
[14] J. Zhou, "Floquet Factorization Algorithms in Linear Continuous-Time Periodic Systems", *SICE Annual Conference 2007*, pp2955-2959, 2007.