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Local $\mathcal{L}_2$ gain of Hopf Bifurcation Stabilization

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***Abstract*— Local $\mathcal{L}_2$ gain analysis of a class of stabilizing controllers for nonlinear systems with Hopf bifurcations is studied. In particular, a family of Lyapunov functions is first constructed for the corresponding critical system, and simplified sufficient conditions to compute the $\mathcal{L}_2$ gain are derived by solving the Hamilton-Jacobi-Bellman (HJB) inequality. Local robust analysis can then be conducted through computing the local $\mathcal{L}_2$ gain achieved by the stabilizing controllers at the critical situation. The theoretical results obtained in this paper provide useful guidance for selecting a robust controller from a given class of stabilizing controllers under Hopf bifurcation. As an example, application to a modified Van der Pol oscillator is presented.**

I. Introduction

Hopf bifurcation has been found or synthesized in various systems [2], [3], [5], [11], [12]. For the past two decades, stabilizing control of Hopf bifurcated systems has drawn a lot of attention from the control community [1], [7], [8], [10], [14]. Regarding these progresses, an interesting question raised further is: to what extent, can these stabilizing control designs be deemed as robust? In [4], local robust analysis methods are presented based on the projection method [9] for a class of stationary and Hopf bifurcation stabilizing controllers [7]. The results show that the local admissible uncertainty set can be characterized using the coefficients in the Taylor series expansion. It is pointed out that, since the characterizing conditions depend explicitly on the stabilizing control gain, one can numerically compute and compare the 'size' of the uncertainty set that can be tolerated by each controller, which, in other words, provides an effective approach to conduct the so-called 'robust design'. The reason that we use the 'robust analysis' approach to conduct the 'robust design' is due to the fact that, for a nonlinear system, one usually does not know how to characterize all stabilizing controllers but just a subset of them.

In [15], $\mathcal{L}_2$ gain method is applied to characterize the robustness of stationary bifurcation stabilization. It has been shown that the advantage provided by the $\mathcal{L}_2$ approach is that the robustness can be explicitly measured by the $\mathcal{L}_2$ gain values achieved by different controllers. It is known [11], [13] that the key issue for a successful $\mathcal{L}_2$ gain analysis and design is whether one could find a Lyapunov function to solve the famous Hamilton-Jacobi-Bellman equation or inequality. Therefore, in principle, if one could obtain a class of stabilizing controllers and a feasible Lyapunov function candidate, then the $\mathcal{L}_2$ gain, which is an indication of robustness in the norm bound form, could be calculated out for every stabilizing controller and hence the controller achieving the smallest $\mathcal{L}_2$ gain would be deemed as the most robust one in that given class. For bifurcation systems, since a parameter is involved, at the first glimpse, one would think that a Lyapunov function to facilitate the $\mathcal{L}_2$ gain analysis should be also parameterized which may cause extra analytic complexity. However, when one takes a closer look at the stabilized bifurcation system [7], even if there still exists equilibrium point bifurcation, the system behavior will not change significantly around the critical position. Observation of this fact motivates us to conduct local $\mathcal{L}_2$ gain analysis for bifurcation stabilizing controllers when applied to the corresponding critical system and to predict that the $\mathcal{L}_2$ performance of the stabilizing controller at the critical situation would reasonably indicate similarity with that at bifurcated situation.

In this paper, we shall present the detailed results on $\mathcal{L}_2$ gain method to characterize the local robustness of Hopf bifurcation stabilization, which complements the results of [15]. The theoretical results provide useful guidance for selecting a robust controller from a given class of Hopf bifurcation stabilizing controllers.

II. Problem Formulation

The system under consideration is the following nonlinear control system F with bifurcation parameter γ, subject to uncertainty Δ as a smooth mapping as shown in Fig. 1:

$$\begin{aligned} \dot{x} &= f(\gamma, x) + g(x)u + bw \\ z &= cx \end{aligned} \tag{1}$$

where γ has a critical value 0; $x \in R^n$; b and c are real valued vectors; $z \in R^q$ is the desired output signal for performance evaluation; u is a scalar feedback control; $w \in R$ is a scalar disturbance signal; $f(\cdot,\cdot) : R \times R^n \to R^n$ and $g(\cdot) : R^n \to R^n$ are all smooth functions with $f(\gamma, 0) = 0$. In this paper, the system (1) is assumed to satisfy the Hypothesis H:

(i) $L(\gamma) = \left.\dfrac{df(\gamma,x)}{dx}\right|_{x=0}$ possesses a pair of complex eigenvalues $\lambda(\gamma) = \alpha(\gamma)+j\beta(\gamma)$, $\bar{\lambda}(\gamma) = \alpha(\gamma)-j\beta(\gamma)$ (called 'critical modes') with $\alpha(0) = 0,\quad \beta(0) = \omega_c \neq 0,\quad \alpha'(0) \neq 0$ while all other eigenvalues are stable in a neighborhood of $\gamma = 0$.

(ii) This pair of critical modes are not observable, nor linearly controllable by u in the sense that, for any row vector ℓ and its conjugate $\bar{\ell}$ that satisfy $\ell \neq 0$, $\ell L(0) = j\omega_c$ and $\bar{\ell}L(0) = -j\omega_c$, we have $\ell g(0) = 0$, $\bar{\ell}g(0) = 0$.

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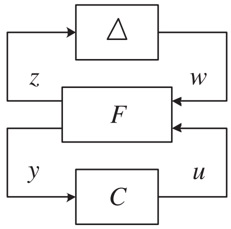


Fig. 1. Nonlinear Control System in Robust Consideration

(iii) This pair of critical modes are not linearly affectable by w in the sense that, for all $\ell \neq 0$ with $\ell L(0) = j\omega_c$ and $\bar{\ell} L(0) = -j\omega_c$, we have $\ell b(0) = 0$ and $\bar{\ell} b(0) = 0$.

The corresponding critical system is then taken as $\gamma = 0$:

$$\begin{array}{l} \dot{x} = f(0, x) + g(x)u + bw = f(x) + g(x)u + bw \\ z = cx \end{array} \qquad (2)$$

Definition 2.1: Assume $u = 0$ and $w = 0$. The nonlinear system in (1) is said to have local bifurcation stability if the origin is locally asymptotically stable in a sufficiently small neighborhood around the origin and a small neighborhood of $\gamma = 0$. □

In [11], the $\mathcal{L}_2$ gain of an input-output mapping and the applications to robust nonlinear control have been discussed in details. By Small Gain Theorem [11], the $\mathcal{L}_2$ gain can be actually interpreted as the measurement of control robustness when the uncertainty signal w is in norm-bounded form. In particular, the following theorem can be applied to find the upper bound of the $\mathcal{L}_2$ gain of a nonlinear mapping (system).

Theorem 2.2: [11] Consider the time-invariant nonlinear mapping (system) from w to z as defined in

$$\begin{array}{l} \dot{x} = f^*(x) + b(x)w, \quad x(0) = x_0 \\ z = h(x) \end{array} \qquad (3)$$

where $f^*(x)$ is locally Lipschitz, and $b(x), h(x)$ are continuous over R^n with compatible dimensions. Both $f^*(x)$ and $h(x)$ vanish at the origin. Let $D \subset R^n$ be a subset in R^n that contains the origin. Suppose that there are an $\eta > 0$ and a continuously differentiable positive semi-definite function $V(x)$ satisfying the inequality

$$HJB := H(V, f^*, h, b, \eta) = \left.\frac{\partial V}{\partial x}\right|_{f^*(x)}$$

$$+\frac{1}{2\eta^2}\frac{\partial V}{\partial x} b(x) b^T(x) (\frac{\partial V}{\partial x})^T + \frac{1}{2} h^T(x) h(x) \leq 0 \qquad (4)$$

for all $x \in D$, and that the origin is a stable equilibrium point of $\dot{x} = f^*(x)$. Then the system is small-signal finite-gain $\mathcal{L}_2$ stable and its $\mathcal{L}_2$ gain, measuring $\frac{\|z\|_2}{\|w\|_2}$, is less than or equal to η. □

This theorem guarantees that, for any disturbance signal $\|w\|_2 < \frac{1}{\eta}\|z\|_2 + \delta_1$ for some $\delta_1 > 0$, the perturbed nonlinear system will remain stable and, besides, $\|z\|_2 \leq \eta\|w\|_2 + \delta_2$ for some $\delta_2 > 0$. We have the following definition for the $\mathcal{L}_2$ gain of a stabilizing bifurcation control.

Definition 2.3: A feedback control $u(x)$ is called local stabilizing bifurcation control if, for $w = 0$, the closed-loop nonlinear system in (1) $\dot{x} = f(\gamma, x) + g(x)u(x)$ has local bifurcation stability. The local $\mathcal{L}_2$ gain achieved by $u(x)$ with respect to a norm-bounded uncertainty signal w is defined as the local $\mathcal{L}_2$ gain achieved by $u(x)$ when applied to the critical system (2). □

So the main problem addressed in this paper can be stated as follows: **Suppose that u is a class of stabilizing bifurcation controllers for the nonlinear system (1) with $w = 0$. What is the $\mathcal{L}_2$ gain achieved by u with respect to a norm-bounded uncertainty signal w?**

III. LOCAL $\mathcal{L}_2$ GAIN ANALYSIS

The solution to the main problem will lead to the finding of the most robust controller, which has the least $\mathcal{L}_2$ gain in a given class of stabilizing control u. The general procedures to find such a controller can be summarized as follows [15]: 1) Synthesize a class of local stabilizing bifurcation controllers u for the system (1) with $w = 0$; 2) Construct Lyapunov function $V(x)$ for the corresponding critical system; 3) Apply the HJB inequality to find an upper bound of the local $\mathcal{L}_2$ gains η for all the stabilizing bifurcation controllers u; 4) Find the smallest upper bound of the local $\mathcal{L}_2$ gain, say η_0 and its corresponding controller u_0.

Then the u_0 will be the 'most robust' controller among the given class achieving the best performance. For presentation simplicity, only local state feedback stabilizing bifurcation controllers $u = Kx$ [1], [7] will be considered to illustrate the ideas in this paper. It is noted that the results can be generalized (not very trivially) to the other control law u given that its Taylor series expansion exists around the origin. For Lyapunov functions, the result in [6] is applied, where algorithms have been developed to construct a family of Lyapunov functions for nonlinear systems in critical cases. We next introduce a proposition that will be useful to show the local definiteness of the time derivative of Lyapunov function.

Proposition 3.1: [6] For the real variables s and t, the scalar bivariate function

$$\delta(s,t) = c_{20}s^2 + c_{04}t^4 + c_{12}st^2 + c_{21}s^2t + c_{30}s^3$$

$$+c_{13}st^3 + c_{22}s^2t^2 + c_{31}s^3t + c_{40}s^4 + O(|(s,t)|^5)$$

is locally negative definite at a small neighborhood $(s,t) = (0,0)$ if the coefficients $c_{20} < 0$, $c_{04} < 0$ and $|c_{12}| < 2\sqrt{c_{20}c_{04}}$, where $O(|(s,t)|^5)$ stands for the fifth and higher order terms of $|(s,t)|$. □

Consider the critical system (2) with a feedback controller $u = Kx$, where the Hypothesis H applies. The Taylor expansion of system (2) can be expressed as

$$\begin{aligned} \dot{x} &= f^*(x) + bw \\ &= L_0^* x + Q_0^*[x,x] + C_0^*[x,x,x] + \cdots + bw + \cdots \end{aligned}$$

where L_0^*, $Q_0^*[x,x]$, and $C_0^*[x,x,x]$ are vector valued linear, quadratic, and cubic terms of $f^*(x)$, respectively; and $f^*(x) = f(x) + g(x)Kx$. Denote ℓ^* and r^* as a left and a right eigenvector of L_0^* associated with the eigenvalue $j\omega_c$ with $\bar{\ell}^*$ and $\bar{r}^*$ being the conjugate such that $\ell^* r^* = 1$.

As such $\bar{\ell}^*$ and $\bar{r}^*$ are a left and a right eigenvector of L_0^* associated with the eigenvalue $-j\omega_c$. Let

$$\beta_2 = 2\Re e\{2\ell^* Q_0^*[r^*, \varsigma] + \ell^* Q_0^*[\bar{r^*}, \tau] + \frac{3}{4}\ell^* C_0^*[r^*, r^*, \bar{r^*}]\}, \tag{5}$$

where
$\varsigma = -\frac{1}{2}L_0^{*-1}Q_0^*[r^*, \bar{r^*}], \quad \tau = \frac{1}{2}(2j\omega_c I - L_0^*)^{-1}Q_0^*[r^*, r^*].$

The Lyapunov function $V(x)$ is in the form:

$$V(x) = \alpha(x^T P x + \rho[x, x, x] + \varphi[x, x, x, x]), \quad \alpha > 0,$$

where each term is obtained as follows.

Algorithm 3.2: [6] Construction of Lyapunov functions.

1) Compute ℓ^* and r^* and choose a basis $\{r^{*1}, r^{*2}, ..., r^{*n}\}$ for R^n with $r^{*1} = r^*$, $r^{*2} = \bar{r^*}$, and for which $\{r^{*3}, r^{*4}, ..., r^{*n}\}$ is a basis for stable subspace E^s, which is defined as the eigenvectors (and the generalized eigenvectors, if any) of L_0^* corresponding to the stable eigenvalues of L_0^*. Let $\ell^{*1} = \ell$ and $\ell^{*2} = \bar{\ell}$. Calculate the row vectors $\{\ell^{*3}, \ell^{*4}, ..., \ell^{*n}\}$ of the associated dual basis. Check that $\beta_2 < 0$ where β_2 is defined in Equation (5).
2) Pick any Π satisfying: $i)$ $\Pi r^* = \Pi \bar{r^*} = 0$; $ii)$ $a^T \Pi a > 0$; $iii)$ $a^T(L_0^{*T}\Pi + \Pi L_0^*)a < 0$ for all $a \in E^s$, and $a \neq 0$, and such that $\beta_2 + \Delta_H(\Pi) < 0$, where

$$\Delta_H(\Pi) = Q_0^{*T}[r^*, \bar{r^*}]((L_0^{*-1})^T\Pi + \Pi L_0^{*-1})Q_0^*[r^*, \bar{r^*}]$$
$$-\frac{1}{4}Q_0^{*T}[r^*, r^*]\{\Pi(L_0^* + 2j\omega_c I)^{-1}$$
$$+((L_0^* - 2j\omega_c I)^{-1})^T\Pi\}Q_0^*[\bar{r^*}, \bar{r^*}]$$

3) Set $P = \Pi + \ell^{*T}\bar{\ell}^* + \bar{\ell}^{*T}\ell^*$
4) Set $\rho[x^1, x^2, x^3]$

$$= \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n}\gamma_{ijk}(\ell^{*i}x^1)(\ell^{*j}x^2)(\ell^{*k}x^3) \tag{6}$$

and

$$\gamma_{111} = -\frac{2}{3j\omega_c}\bar{\ell}^* Q_0^*[r^*, r^*] \tag{7}$$

$$\gamma_{112} = -\frac{1}{3j\omega_c}\left\{4\bar{\ell}^* Q_0^*[r^*, \bar{r^*}] + 2\ell^* Q_0^*[r^*, r^*]\right\} \tag{8}$$

$$\gamma_{i11} = -\frac{1}{3}\left\{4\bar{\ell}^* Q_0^*[r^*, (2j\omega_c I + L_0^*)^{-1}r^{*i}]\right.$$
$$\left. + 2Q_0^{*T}[r^*, r^*]\Pi(L_0^* + 2j\omega_c I)^{-1}r^{*i}\right\} \tag{9}$$

$$\gamma_{i12} = -\frac{2}{3}\left\{\bar{\ell}^* Q_0^*[\bar{r^*}, L_0^{*-1}r^{*i}] + \ell^* Q_0^*[r^*, L_0^{*-1}r^{*i}]\right.$$
$$\left. + Q_0^{*T}[r^*, \bar{r^*}]\Pi L_0^{*-1}r^{*i}\right\} \tag{10}$$

where $i = 3, 4, ..., n$ in Eq. (9) and (10).

5) Set $\varphi[x^1, x^2, x^3, x^4] = \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n}\sum_{p=1}^{n}\{$

$$\varepsilon_{ijkp}(\ell^{*i}x^1)(\ell^{*j}x^2)(\ell^{*k}x^3)(\ell^{*p}x^4)\} \tag{11}$$

where, $\varepsilon_{1111} = \varphi[r^*, r^*, r^*, r^*]$ and $\varepsilon_{1112} = \varphi[r^*, r^*, r^*, \bar{r^*}]$ are selected according to

$$2\bar{\ell}^* C_0^*[r^*, r^*, r^*] + 3\rho[r^*, r^*, Q_0^*[r^*, r^*]]$$
$$+4j\omega_c\varphi[r^*, r^*, r^*, r^*] = 0 \tag{12}$$

and

$$\ell^* C_0^*[r^*, r^*, r^*] + 3\bar{\ell}^* C_0^*[r^*, r^*, \bar{r^*}]$$
$$+3\rho[r^*, r^*, Q_0^*[r^*, \bar{r^*}]] + 3\rho[r^*, \bar{r^*}, Q_0^*[r^*, r^*]]$$
$$+4j\omega_c\varphi[r^*, r^*, r^*, \bar{r^*}] = 0 \tag{13}$$

6) All structural coefficients γ_{ijk} and ε_{ijkp} which have not been specified in the above steps are assigned arbitrarily, modulo the symmetry requirement and the conjugate symmetry requirement: $\bar{\gamma}_{ijk} = \gamma_{[i][j][k]}$ and $\bar{\varepsilon}_{ijkp} = \varepsilon_{[i][j][k][p]}$ with $[i] \equiv 2$ if $i = 1$, $[i] \equiv 1$ if $i = 2$, and $[i] \equiv 0$ otherwise. □

Now we are in the position to present the main theorem in this paper.

Theorem 3.3: Given a class of local stabilizing bifurcation (Hopf) controllers $u = Kx$ for the Hopf bifurcated system (1). Let

$$V(x) = \alpha\left(x^T P x + \rho[x, x, x] + \varphi[x, x, x, x]\right)$$

be the Lyapunov function constructed from Algorithm 3.2 for the system (2). If there is an $\eta > 0$ such that

$$M := \alpha(L_0^{*T}\Pi + \Pi L_0^*) + \frac{2\alpha^2}{\eta^2}\Pi b b^T \Pi + \frac{1}{2}c^T c \leq 0,$$

then there exists a small neighborhood $D \subset R^n$ of the origin such that, $\forall x(0) \in D$, the local $\mathcal{L}_2$ gain achieved by u will be no greater than η as a function of α. Furthermore, the smallest possible η can be obtained through a search conducted on varying α.

PROOF :

The time derivative of $V(x)$ evaluated along trajectories of the controlled system (2) under $u = Kx$ (with $w = 0$) can be written in Taylor series as

$$\left.\frac{\partial V}{\partial x}\right|_{f^*(x)} = \alpha\left\{x^T(L_0^{*T}P + PL_0^*)x + 2Q_0^{*T}[x, x]Px\right.$$
$$+3\rho[x, x, L_0^* x] + 2C_0^{*T}[x, x, x]Px$$
$$\left.+3\rho[x, x, Q_0^*[x, x]] + 4\varphi[x, x, x, L_0^* x]...\right\}$$

The HJB (4) of this system, for an $\eta > 0$, can then be obtained as:

$$HJB = H_{(2)} + H_{(3)} + H_{(4)} + O(x^5)$$

where the integer subscripts represent the quadratic, cubic and quartic terms in x, and $O(x^5)$ represents terms of fifth and higher orders of x. $H_{(2)}$ is given by

$$H_{(2)} = x^T\left\{\alpha(L_0^{*T}P + PL_0^*) + \frac{2\alpha^2}{\eta^2}Pbb^T P + \frac{1}{2}c^T c\right\}x$$
$$= x^T\left\{\alpha(L_0^{*T}\Pi + \Pi L_0^*) + \frac{2\alpha^2}{\eta^2}\Pi b b^T \Pi + \frac{1}{2}c^T c\right\}x$$
$$= x^T M x$$

While $M \leq 0$ (or $M < 0$ in E^s) implies $H_{(2)} \leq 0$ (or $H_{(2)} < 0$ in E^s), it does not of course guarantee that

$HJB \leq 0$ in the entire support space. Next we shall show that the HJB is indeed negative definite by following the method similar to that of [6].

Recall that, in Hypothesis H, any vector $x \in R^n$ has a unique representation $x = dr^* + \bar{d}\bar{r^*} + a$, where d is a complex scalar, r^* is the right eigenvector of L_0^* associated with the eigenvalue $j\omega_c$, and $a \in E^s$. From Equations (7) and (8), we readily obtain:

$$\rho[r^*, r^*, r^*] = -\frac{2}{3j\omega_c}\bar{\ell^*}Q_0^*[r^*, r^*] \tag{14}$$

$$\rho[r^*, r^*, \bar{r^*}] = -\frac{1}{3j\omega_c}\left\{4\bar{\ell^*}Q_0^*[r^*, \bar{r^*}] + 2\ell^*Q_0^*[r^*, r^*]\right\} \tag{15}$$

Since the vectors $r^{*k}, k = 3, 4, \cdots, n$, form a basis for E^s, Equation (9) is a solution of

$$\sum_{i=3}^{n}\gamma_{i11}(\ell^{*i}\tilde{a}) = -\frac{1}{3}\left\{4\bar{\ell^*}Q_0^*[r^*, (2j\omega_c I + L_0^*)^{-1}\tilde{a}] + 2Q_0^{*T}[r^*, r^*]\Pi(L_0^* + 2j\omega_c I)^{-1}\tilde{a}\right\} \tag{16}$$

where $\tilde{a} = r^{*k}, k = 3, 4, \cdots, n$. Moreover, knowledge of a vector $\tilde{a} \in E^s$ is tantamount to knowledge of another vector $a = (L_0^* + 2j\omega_c I)^{-1}\tilde{a} \in E^s$. Thus above equation can be further written as

$$\sum_{i=3}^{n}\gamma_{i11}\left(\ell^{*i}(L_0^* + 2j\omega_c I)a\right) = -\frac{1}{3}\left\{4\bar{\ell^*}Q_0^*[r^*, a] + 2Q_0^{*T}[r^*, r^*]\Pi a\right\} \tag{17}$$

In addition, from Equation (6), we have

$$\rho[r^*, r^*, (L_0^* + 2j\omega_c I)a] = \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n}\gamma_{ijk}(\ell^{*i}r^*)(\ell^{*j}r^*)(\ell^{*k}L_0^*a) = \sum_{i=2}^{n}\gamma_{i11}\left(\ell^{*i}(L_0^* + 2j\omega_c I)^*a\right) \tag{18}$$

Combining Equation (17) and (18), we have

$$\rho[r^*, r^*, (L_0^* + 2j\omega_c I)a] = -\frac{1}{3}\left\{4\bar{\ell^*}Q_0^*[r^*, a] + 2Q_0^{*T}[r^*, r^*]\Pi a\right\} \tag{19}$$

In a similar fashion, we obtain from Equation (10) that

$$\rho[r^*, \bar{r^*}, L_0^*a] = -\frac{2}{3}\left\{\bar{\ell^*}Q_0^*[\bar{r^*}, a] + \ell^*Q_0^*[r^*, a] + Q_0^{*T}[r^*, \bar{r^*}]\Pi a\right\} \tag{20}$$

Since $\ell^*r^* = 1$, the r-component of any vector x is given by $(\ell^*x)r^*$, the $\bar{r^*}$-component is $(\bar{\ell^*}x)\bar{r^*}$, and the E^s-component is $x^s = x - (\ell^*x)r^* - (\bar{\ell^*}x)\bar{r^*}$, then by Equations (14), (15) and (19), we have

$$\rho[r, r, x] = -\frac{2}{3j\omega_c}(\ell^*x)\bar{\ell^*}Q_0^*[r^*, r^*] - \frac{2}{3j\omega_c}(\bar{\ell^*}x)\left\{\ell^*Q_0^*[r^*, r^*] + 2\bar{\ell^*}Q_0^*[r^*, \bar{r^*}]\right\} - \frac{1}{3}\left\{2Q_0^{*T}[r^*, r^*]\Pi(L_0^* + 2j\omega_c I)^{-1}x^s + 4\bar{\ell^*}Q_0^*[r^*, (L_0^* + 2j\omega_c I)^{-1}x^s]\right\}$$

Letting $x = Q_0^*[\bar{r^*}, \bar{r^*}]$, we may rewrite the above equation as

$$\rho[r^*, r^*, Q_0^*[\bar{r^*}, \bar{r^*}]] = \frac{8}{3}\bar{\ell^*}Q_0^*[r^*, \bar{\tau}] + \frac{4}{3}Q_0^{*T}[r^*, r^*]\Pi\bar{\tau} - \frac{2}{9j\omega_c}\left\{(\ell^*Q_0^*[\bar{r^*}, \bar{r^*}])(\bar{\ell^*}Q_0^*[r^*, r^*])\right\} - \frac{2}{3j\omega_c}\left\{(\bar{\ell^*}Q_0^*[\bar{r^*}, \bar{r^*}])(\ell^*Q_0^*[r^*, r^*])\right\}$$

Therefore, we obtain

$$\Re e\left\{\rho[r^*, r^*, Q_0^*[\bar{r^*}, \bar{r^*}]]\right\} = \Re e\left\{\frac{8}{3}\bar{\ell^*}Q_0^*[r^*, \bar{\tau}] + \frac{4}{3}Q_0^{*T}[r^*, r^*]\Pi\bar{\tau}\right\} \tag{21}$$

In a similar way, we have

$$\Re e\left\{\rho[r^*, \bar{r^*}, Q_0^*[r^*, \bar{r^*}]]\right\} = \Re e\left\{\frac{8}{3}\ell^*Q_0^*[r^*, \varsigma] + \frac{4}{3}Q_0^{*T}[r^*, \bar{r^*}]\Pi\varsigma\right\} \tag{22}$$

From Algorithm 3.2 Step 2 we know that $\beta_2 + \Delta_H(\Pi) < 0$. From Equations (21) and (22) we can verify that

$$\Re e\left\{2\ell^*C_0^*[r^*, r^*, \bar{r^*}] + \rho[r^*, r^*, Q_0^*[\bar{r^*}, \bar{r^*}]] + 2\rho[r^*, \bar{r^*}, Q_0^*[r^*, \bar{r^*}]]\right\} < 0 \tag{23}$$

Then the quadratic and cubic terms of the HJB are given by

$$H_{(2)} = a^T\left\{\alpha(L_0^{*T}\Pi + \Pi L_0^*) + \frac{2\alpha^2}{\eta^2}\Pi bb^T\Pi + \frac{1}{2}c^Tc\right\}a = a^TMa \tag{24}$$

$$\begin{aligned} H_{(3)} = \alpha\Big\{&d^3\left\{2\bar{\ell^*}Q_0^*[r^*, r^*] + 3j\omega_c\rho[r^*, r^*, r^*]\right\} \\ &+ d^2\bar{d}\left\{2\ell^*Q_0^*[r^*, r^*] + 4\bar{\ell^*}Q_0^*[r^*, \bar{r^*}] + 3j\omega_c\rho[r^*, r^*, \bar{r^*}]\right\} + d^2\left\{4\bar{\ell^*}Q_0^*[r^*, a] + 2Q_0^{*T}[r^*, r^*]\Pi a + 3\rho[r^*, r^*, L_0^*a] + 6j\omega_c\rho[r^*, r^*, a]\right\} \\ &+ d\bar{d}\left\{4\bar{\ell^*}Q_0^*[\bar{r^*}, a] + 4\ell^*Q_0^*[r^*, a] + 4Q_0^{*T}[r^*, \bar{r^*}]\Pi a + 6\rho[r^*, \bar{r^*}, L_0^*a]\right\} + O(H_{(3)})\Big\} \end{aligned}$$

where $O(H_{(3)})$ represents the terms that have no impact on the local definiteness of the HJB equation, from the Proposition (3.1)'s point of view; and the coefficients of $\bar{d}^3$, $\bar{d}^2d$ and $\bar{d}^2$ are not listed since they are simply the conjugate of the coefficients of d^3, $d^2\bar{d}$ and d^2 respectively. Substituting Equations (14), (15), (19) and (20) into the above equation, we have

$$H_{(3)} = \alpha O(H_{(3)})$$

The quartic term of the HJB is given by

$$H_{(4)} = \alpha\left\{d^4\left\{2\bar{\ell^*}C_0^*[r^*, r^*, r^*] + 3\rho[r^*, r^*, Q_0^*[r^*, r^*]]\right.\right.$$

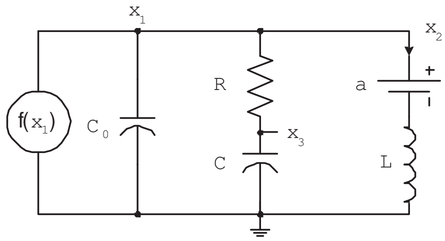


Fig. 2. A modified Van der Pol circuit.

$$+4j\omega_c\varphi[r^*,r^*,r^*,r^*] + 2d^3\bar{d}\{\ell^* C_0^*[r^*,r^*,r^*]\}$$
$$+3\bar{\ell}^* C_0^*[r^*,r^*,\bar{r^*}] + 3\rho[r^*,r^*,Q_0^*[r^*,\bar{r^*}]]$$
$$+3\rho[r^*,\bar{r^*},Q_0^*[r^*,r^*]] + 4j\omega_c\varphi[r^*,r^*,r^*,\bar{r^*}]\}$$
$$+6d^2\bar{d}^2\{2\ell^* C_0^*[r^*,r^*,\bar{r^*}] + \rho[r^*,r^*,Q_0^*[\bar{r^*},\bar{r^*}]]$$
$$+2\rho[r^*,\bar{r^*},Q_0^*[r^*,\bar{r^*}]]\} + O(H_{(4)})\}$$

where $O(H_{(4)})$ has the same meaning as $O(H_{(3)})$, and the coefficients of $\bar{d}^4$ and $\bar{d}^3 d$ are not listed since they are simply the conjugate of the coefficients of d^4 and $d^3\bar{d}$ respectively. Substituting Equations (12) and (13) into above equation, we obtain

$$H_{(4)} = \alpha\left\{6d^2\bar{d}^2\{\ell^* C_0^*[r^*,r^*,\bar{r^*}] + \rho[r^*,r^*,Q_0^*[\bar{r^*},\bar{r^*}]]\right.$$
$$\left.+2\rho[r^*,\bar{r^*},Q_0^*[r^*,\bar{r^*}]]\} + O(H_{(4)})\right\}$$

Let $s \equiv a$, $t \equiv d$, $c_{20} = M$, $c_{12} = 0$, and c_{04} be the coefficient of $d^2\bar{d}^2$ in $H_{(4)}$, then from inequality (23) and Proposition (3.1), we conclude that the HJB is indeed negative definite.

Since only $H_{(2)}$ contains η, therefore if $M \leq 0$, then there exists a small neighborhood $D \subset R^n$ of the origin such that, $\forall x \in D$, the local $\mathcal{L}_2$ gain achieved by $u = Kx$ will be less than or equal to η. ∎

It is clear from this theorem that, although only the second order term of x contains η (in E^s), we may need to compute up to the fourth order to verify the local definiteness of the HJB equation. In other words, we may construct a family of Lyapunov functions containing cubic and quartic terms, but only quadratic term contributes to the analysis of local $\mathcal{L}_2$ gain of the bifurcated systems. Thus, the cumbersome computation of the cubic and quartic terms may be omitted. Also note that for all state feedback stabilizing bifurcation controllers synthesized from theorems in [1], [7], the $\mathcal{L}_2$ gain could be calculated by applying Theorem 3.3. Hence, the best robust controller could be determined by comparing all of these $\mathcal{L}_2$ gain values and the controller corresponding to the smallest value will be the one.

IV. Application to a Van der Pol Oscillator

Van der Pol oscillator and its modified forms have been widely studied for their rich nonlinear dynamical behaviors [11]. The one used in this paper is shown in Fig. 2 [5], where a battery with voltage a volts is added, and the nonlinear negative conductance has a voltage-current characteristic $f(x_1) = -a_1x_1 + a_3x_1^3$, $a_1, a_3 > 0$. Applying the Kirchhoff's laws, we have

$$\begin{aligned} C_0\dot{x}_1 &= -f(x_1) + \tfrac{x_3-x_1}{R} - x_2 \\ L\dot{x_2} &= x_1 - a, \quad C\dot{x_3} = -\tfrac{x_3-x_1}{R} \end{aligned}$$

Using the dimensionless variables $x_1 = V_0\bar{x}_1$, $x_2 = \frac{V_0}{\omega L}\bar{x}_2$, $x_3 = V_0\bar{x}_3$, $\tau = \omega t$ with $\omega = \frac{1}{\sqrt{LC}}$, $V_0 = \sqrt{\frac{a_1}{3a_3}}$; defining new parameters $\epsilon = \frac{C_0}{C}$, $\bar{a} = \frac{a}{V_0}$, $\varrho = \frac{a_1}{\omega C}$, $\bar{R} = R\omega C$; and omitting the bars, we can re-write the above equations as:

$$\begin{aligned} \epsilon\dot{x}_1 &= -g(x_1) + \tfrac{x_3-x_1}{R} - x_2 \\ \dot{x_2} &= x_1 - a, \quad \dot{x_3} = -\tfrac{x_3-x_1}{R} \end{aligned}$$

where $g(x_1) = \varrho(x_1^3/3 - x_1)$. we take a as bifurcation parameter with $\epsilon = 0.95$, $\varrho = 2$ and $R = 1$. It can be shown that subcritical Hopf bifurcation is born at $a = 0.897$ [5]. Once again, taking the transformation $\hat{x} = [\hat{x}_1, \hat{x}_2, \hat{x}_3]^T = [x_1 - a, \quad x_2 + g(a), \quad x_3 - a]^T$; letting $u = K\hat{x}$ be the stabilization controller and taking the disturbance w into consideration, we can re-write the above system equations into the following form by omitting the hats:

$$\begin{aligned} \dot{x} &= f(x) + u + bw = f^*(x) + bw \\ z &= cx \end{aligned} \tag{25}$$

with $f^*(x) = f(x) + u$. Applying theorems in [1] or [7], a class of stabilization controllers can be synthesized as:

$$K = k\begin{bmatrix} 0.2386 & 0.1530 & -0.3916 \\ 0.2350 & 0.1507 & -0.3857 \\ -0.3721 & -0.2386 & 0.6107 \end{bmatrix}$$

where $0.0797 < k < 1.6413$. It can be verified that the critical modes remain unchanged under these controllers. In this example, we take $b = [0.3156, -0.1923, -0.4921]^T$ and $c = [1, 0.6413, -1.6413]$. Following Algorithm 3.2 to construct a Lyapunov function, we have:

$$\Pi = p_{33}\begin{bmatrix} 1 & 0.6413 & -1.6413 \\ 0.6413 & 0.4113 & -1.0526 \\ -1.6413 & -1.0526 & 2.6940 \end{bmatrix}$$

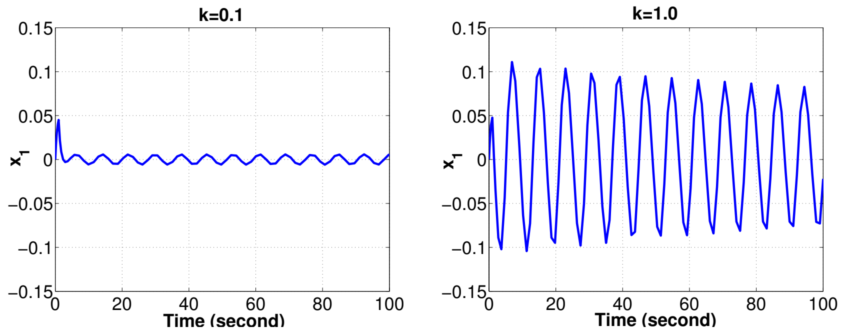
where $p_{33} > 0$ is a dummy constant that satisfies the criteria in Step 2 of Algorithm 3.2 but has no effect on our next computation .

From Theorem 3.3 and after some algebra, we can reduce the matrix inequality $M \leq 0$ to the following inequality

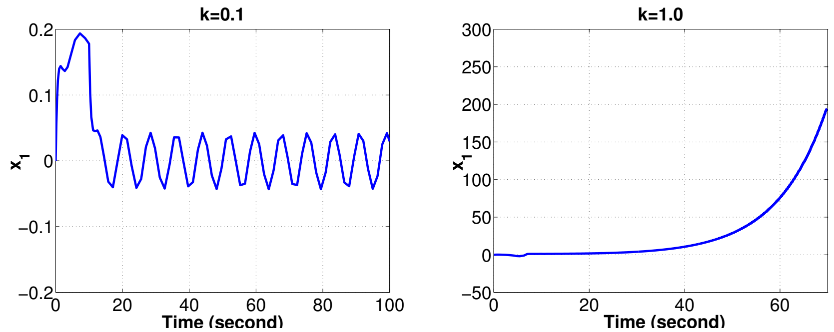
$$2\alpha p_{33}(k - 1.6413) + \tfrac{2\alpha^2}{\eta^2}p_{33}^2 + \tfrac{1}{2} \leq 0$$

The minimum value of η can be obtained as $\eta_0 = \frac{1}{1.6413-k}$ when $\alpha = \frac{1}{2p_{33}1.6413-k}$. From Theorem 3.3, we conclude that the local $\mathcal{L}_2$ gain of the system (25) is less than or equal to $\frac{1}{1.6413-k}$. Since $0.0797 < k < 1.6413$, it is obvious that the most robust controller, which has the lowest $\mathcal{L}_2$ gain, is $k = 0.0797$.

To illustrate the effectiveness of $\mathcal{L}_2$-gain analysis, we compare two controllers under different uncertainties: $k = 0.1$ and $k = 1.0$, which represent approximately the upper bound and medium of all stabilizing controllers. Assume that $x(0) = 0$. First, a comparatively small uncertainty w is



(a)w=0.1



(b)w=1

Fig. 3. Comparison of state x_1 between two controllers with different uncertainties: (a) Small uncertainty $w = 0.1$ lasting for 1 second. (b) Large uncertainty $w = 1.0$ lasting for 10 seconds. System with the controller $k = 0.1$ that has a smaller $\mathcal{L}_2$ gain is stable.

set as a rectangular signal with amplitude 0.1 and duration lasting for 1 second. Transient responses of the state x_1 of the controllers are depicted in Fig. 3(a). It is clear that oscillation of the controller $k = 0.1$ is very small, while oscillation of $k = 1.0$ with a larger $\mathcal{L}_2$ gain is considerably big. Second, comparison under a comparatively large uncertainty $w = 1$ for a duration of 10 seconds is also plotted in Fig. 3(b). Apparently, the system under the controller $k = 0.1$ that has the smaller $\mathcal{L}_2$ gain is stable, while another becomes unstable.

V. Conclusion

In this paper, robust analysis results based on the $\mathcal{L}_2$ gain approach are derived for a class of stabilizing controllers for nonlinear bifurcated/critical systems with the linearized system at the equilibrium point possessing a pair of pure imaginary eigenvalues. Simplified sufficient conditions for computing the local $\mathcal{L}_2$ gain achieved by the controllers is synthesized. Robust design can be conducted through searching the most robust controller among all the given controllers based on the $\mathcal{L}_2$ gain value. The theoretical results provide useful guidance for selecting a robust controller from a given class of Hopf bifurcation stabilizing control. The general idea presented in this paper is also applicable to an entire family of feedback controllers developed recently for Hopf bifurcations with two uncontrollable modes [8].

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