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Lyapunov-based Model Predictive Control of Nonlinear Systems Subject to Time-Varying Measurement Delays

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Abstract— **In this work, we focus on model predictive control of nonlinear systems subject to time-varying measurement delays. The motivation for studying this control problem is provided by networked control problems and the presence of time-varying delays in measurement sampling in chemical processes. We propose a Lyapunov-based model predictive controller which is designed taking time-varying measurement delays explicitly into account, both in the optimization problem formulation and in the controller implementation. The proposed predictive controller guarantees that the closed-loop system is ultimately bounded in a region that contains the origin if the maximum delay is smaller than a given constant. The theoretical results are illustrated through a chemical process example.**

I. Introduction

The problem of designing feedback control systems for nonlinear systems subject to time-varying measurement delays is a fundamental one and its solution can find significant application in a number of control engineering problems including, for example, design of networked control systems (NCS). NCS are control systems which operate over a communication network (wired or wireless) and can lead to significant improvements in the efficiency, flexibility, robustness and fault-tolerance of industrial control systems as well as to reduction of the installation, reconfiguration and maintenance costs. However, the design of NCS has to account for the dynamics introduced by the communication network which may include time-varying delays, data quantization or data losses. In addition to NCS, another source of time-varying delays in the feedback loop is measurement sensor delays, which are particularly important during the measurement of species concentrations and particle size distributions in process control applications.

Model predictive control (MPC) is an advanced method of process control that has been in use in the process industries such as chemical plants and oil refineries since the 1980s; see for example [1], [2] for a review of results in this area. In order to guarantee stability of the closed-loop system, MPC schemes must include a set of stability constraints. Different MPC schemes can be found in the literature, see [3] for a review on MPC stability results. In a recent series of papers, we proposed Lyapunov-based model predictive control (LMPC) schemes for nonlinear systems [4], [5] based on uniting receding horizon control with control Lyapunov functions as a way of guaranteeing closed-loop stability. The main idea is to formulate appropriate constraints in the predictive controller's optimization problem based on an existing Lyapunov-based controller, in such a way that the MPC controller guarantees closed-loop practical stability. Lyapunov-based model predictive control has been applied with success to constrained non-linear systems, switched systems and to fault-tolerant control problems [4], [5], [6], [7]. However, the available results on Lyapunov-based MPC do not account for the effect of time-varying measurement delays, and when time-varying measurement delays are taken into account, these schemes are not guaranteed to maintain the desired closed-loop stability properties. In terms of other research work pertaining to the control problem studied in this manuscript, we note that most of the available results on MPC of systems with delays deal with linear systems (e.g., [8], [9]). Finally, the importance of time delays in the context of networked control systems has motivated significant research effort in modeling such delays and designing control systems to deal with them, primarily in the context of linear systems (e.g., [10], [11], [12], [13], [14], [15]).

Motivated by the above, this work deals with the design of predictive controllers for nonlinear system subject to time-varying measurement delay in the feedback loop. In particular, we propose to design the controller in a Lyapunov-based MPC scheme. While there are several Lyapunov-based MPC schemes that have been proposed in the literature (see for example [16]), in this work we design the proposed Lyapunov-based MPC based on the results developed in [4], [5], [6] by our group. In the LMPC scheme proposed in the present work, when measurement delays occur, the nominal model of the system is used together with the delayed measurement to estimate the current state, and the resulting estimate is used to evaluate the LMPC controller; at sampling times where no measurements are available due to the delay, the actuator implements the last optimal input trajectory evaluated by the controller. The proposed LMPC scheme allows for an explicit characterization of the stability region and guarantees that the closed-loop system in the presence of time-varying measurement delays is ultimately bounded in a region that contains the origin if the maximum delay is smaller than a constant that depends on the parameters of the system and the Lyapunov-based controller that is used to formulate the optimization problem. The theoretical results are illustrated through a chemical process example.

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II. Preliminaries

A. Problem formulation

In this work, we consider a nonlinear system subject to disturbances with the following state-space description

$$\dot{x}(t) = f(x(t), u(t), w(t)) \quad (1)$$

where $x(t) \in R^{n_x}$ denotes the vector of state variables, $u(t) \in R^{n_u}$ denotes the vector of manipulated input variables, $w(t) \in R^{n_w}$ denotes the vector of disturbance variables, and f is a locally Lipschitz vector function on $R^{n_x} \times R^{n_u} \times R^{n_w}$. The disturbance vector is bounded, i.e., $w(t) \in W$ where

$$W := \{w \in R^{n_w} \text{ s.t. } |w| \leq \theta, \theta > 0\}^1.$$

We assume that the nominal system (system (1) with $w(t) \equiv 0$ for all t) has an asymptotically stable equilibrium at the origin $x = 0$ for a given feedback control $h : R^{n_x} \to R^{n_u}$ which satisfies $h(0) = 0$ (this assumption is equivalent to the existence of a control Lyapunov function (CLF) for the system $\dot{x} = f(x, u, 0)$). This feedback law will be used in the design of the LMPC controller. Using converse Lyapunov theorems (see [17]), this assumption implies that there exist functions $\alpha_i(\cdot)$, $i = 1, 2, 3, 4$ of class $\mathcal{K}^2$ and a Lyapunov function V which is continuous and bounded in R^{n_x}, that satisfy the following inequalities

$$\begin{aligned} \alpha_1(|x|) \leq V(x) &\leq \alpha_2(|x|) \\ \frac{\partial V(x)}{\partial x} f(x, h(x), 0) &\leq -\alpha_3(|x|) \\ \left|\frac{\partial V(x)}{\partial x}\right| &\leq \alpha_4(|x|) \end{aligned} \quad (2)$$

for all $x \in O \subseteq R^{n_x}$ where O is any open neighborhood of the origin, see [17]. We denote the region $\Omega_\rho{}^3 \subseteq O$ as the stability region of the closed-loop system under controller $h(x)$. In the remainder, we will refer to the controller $h(x)$ as the Lyapunov-based controller.

By continuity and the local Lipschitz property assumed for the vector field $f(x, u, w)$, there exist positive constants M, L_w and L_x such that

$$|f(x, h(x), w)| \leq M \quad (3)$$

$$|f(x, h(x), w) - f(x', h(x), 0)| \leq L_w|w| + L_x|x - x'| \quad (4)$$

for all $x, x' \in \Omega_\rho$ and $w \in W$. These constants will be used to prove the main results of this work.

B. Model of measurement delays

Although system (1) is defined in continuous time, we focus on a sample and hold implementation of the controller subject to time-varying measurement delays because we work within a model predictive control framework. State are sampled with a sampling time Δ at times $t_k = t_0 + k\Delta$ where $k = 0, 1 \ldots$ and t_0 is the initial time. However, due to the time-varying measurement delays, the controller may not receive the latest measurement, but a previous one. We assume that each measurement is time-labeled, so the controller is able to discard non-relevant information; i.e., the controller discards earlier measurements if it has already received more recent information. We do not consider delays in the computation and implementation of the control actions. This model is of relevance to systems subject to asynchronous delayed measurements and to networked control systems, where the delay is introduced by the communication network connecting the sensor and the controller. Figure 1 shows a schematic of the closed-loop system for the control problem considered.

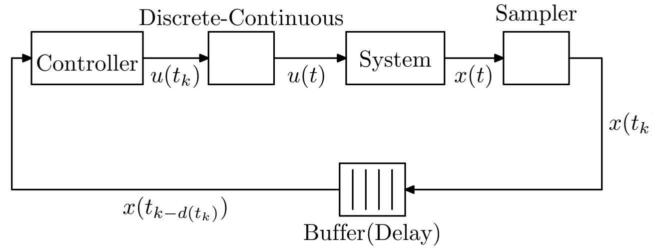


Fig. 1. Sampled-data closed-loop system subject to time-varying measurement delays.

To model measurement delays, an auxiliary random variable $d(t_k)$ is introduced to indicate the number of sampling times that the measurement received at time t_k is delayed. That is, at sampling time t_k, the controller receives the delayed measurement $x(t_{k-d(t_k)})$. When more than one measurements are received, the controller only uses the latest one and discards the rest. For example, if at sampling time t_k, three new measurements are received with a delay of 5, 4 and 3 sampling times respectively, the controller only uses the measurement with a delay of 3 sampling times (recall that the measurements are time-labeled). The sequence $\{d(t_k) \geq 0\}$ characterizes the time needed to obtain a new measurement in the case of asynchronous measurements or the quality of the feedback link in the case of NCS. Because of the delay, it is possible that there exist sampling times t_k in which a new measurement is not received. In this case, the controller must decide the control action in open-loop, for example, setting the control input to zero or to the last implemented value. In the next section, we present an LMPC controller in which time-varying delays are taken into account explicitly both in the controller design and in the implementation procedure.

In general, if the sequence $\{d(t_k) \geq 0\}$ is modeled using a random process, there exists the possibility of arbitrarily large delays. In this case, it is not possible to provide guaranteed stability properties, because there exists a non-zero probability that the controller operates in open-loop for a period of time large enough for the state to leave the stability region or even escape to infinity (i.e., finite escape time). In order to study the stability properties in a deterministic framework, in this paper we consider systems where there exists an upper bound D on the delay of the state measurement that we receive at each sampling time,

[1] $|\cdot|$ denotes Euclidean norm of a vector.

[2] Class $\mathcal{K}$ functions are strictly increasing functions of their argument and satisfy $\alpha_i(0) = 0$.

[3] We use Ω_r to denote the set $\Omega_r := \{x \in R^{n_x} | V(x) \leq r\}$.

i.e. $d(t_k) \leq D, k = 0, 1 \ldots$.

III. Lyapunov-based Model Predictive Control

A. LMPC design

Lyapunov-based MPC is based on uniting receding horizon control with control Lyapunov functions and computes the control input trajectory solving a finite horizon constrained optimal control problem. The control input trajectory (i.e., the free variable of the LMPC optimization problem) is constrained to belong to the family of piece-wise constant functions $S(\Delta)$ with sampling period Δ and length equal to the prediction horizon. As mentioned in the introduction, LMPC is characterized by a set of constraints based on an existing Lyapunov-based controller.

Previous LMPC schemes (see [4], [5], [6] for standard results and [7] for systems subject to asynchronous measurements and data losses) do not consider time-varying measurement delays. Under certain assumptions these controllers guarantee practical stability of the closed-loop system, however, when time-varying measurement delays are present, these results do not hold. In this section, a Lyapunov-based model predictive controller for system (1) which takes into account time-varying measurement delays explicitly, both in the constraints imposed in the optimization problem and in the implementation procedure, is proposed. The proposed controller guarantees that, under certain conditions, the closed-loop system subject to time-varying measurement delays is ultimately bounded in a set that contains the origin.

A controller for a system subject to time-varying measurement delays must take into account two important issues. First, when a new measurement is received, this measurement may not correspond to the current state of the system. This implies that in this case, the controller has to take a decision using an estimate of the current state. Second, because the delays are time-varying, the controller may not receive new information every sampling time. This implies that in this case, the controller has to operate in open-loop using the last received measurements. In order to deal with these two issues, we propose to take advantage of the model predictive control scheme to decide the control input based on a prediction obtained using the nominal model of the system. This prediction is used both for estimating the current state from previous measurements and for deciding the input when the controller does not receive new information.

The proposed Lyapunov-based MPC that takes into account time-varying measurement delays in an explicit way is based on the following finite horizon constrained optimal control problem

$$u_k^*(t) = \arg \min_{u_k \in S(\Delta)} J \tag{5}$$

subject to

$$\dot{\tilde{x}}(t) = f(\tilde{x}(t), u_k(t), 0) \tag{6a}$$

$$u_k(t) = u_{k-1}^*(t), \forall t \in [t_{k-d(t_k)}, t_k] \tag{6b}$$

$$\tilde{x}(t_{k-d(t_k)}) = x(t_{k-d(t_k)}) \tag{6c}$$

$$\dot{\hat{x}}(t) = f(\hat{x}(t), h(\hat{x}(t_j)), 0),\ t \in [t_j, t_{j+1}] \tag{6d}$$

$$\hat{x}(t_k) = \tilde{x}(t_k) \tag{6e}$$

$$V(\tilde{x}(t)) \leq V(\hat{x}(t)),\ \forall t \in [t_k, t_{k+D+1-d(t_k)}] \tag{6f}$$

with

$$J = \int_{t_{k-d(t_k)}}^{t_{k+N}} [\tilde{x}(\tau)^T Q_c \tilde{x}(\tau) + u_k(\tau)^T R_c u_k(\tau)] d\tau \tag{7}$$

where $S(\Delta)$ is the family of piece-wise constant functions with sampling period Δ, $\tilde{x}(t)$ is the predicted sampled trajectory of the nominal system for the input trajectory computed by the LMPC (5), $j = k, \ldots, k+N-1$, $x(t_{k-d(t_k)})$ is the delayed measurement that is received at t_k, $u_{k-1}^*(t)$ is the optimal control input trajectory computed at time t_{k-1}, $\hat{x}(t)$ is the nominal sampled trajectory under the Lyapunov-based controller $u = h(\hat{x}(t))$ along the prediction horizon with initial state the estimated state $\tilde{x}(t_k)$, and Q_c, R_c are weight matrices that define the cost.

If at a sampling time, a new measurement $x(t_{k-d(t_k)})$ is received, an estimate of the current state $\tilde{x}(t_k)$ is obtained using the nominal model of the system (constraint (6a)) and the control inputs applied to the system from $t_{k-d(t_k)}$ to t_k (constraint (6b)) with the initial condition being $\tilde{x}(t_{k-d(k)}) = x(t_{k-d(t_k)})$ (constraint (6c)). Note that this implies that the controller has to store the past control input trajectory. The estimated state $\tilde{x}(t_k)$ is then used to obtain the optimal future control input trajectory solving the finite horizon constrained optimal control problem (5). The Lyapunov-based MPC scheme uses the nominal model to predict the future trajectory $\tilde{x}(t)$ for a given input trajectory $u_k(t) \in S(\Delta)$ with $t \in [t_k, t_{k+N}]$. A cost function is minimized (equation (7)), while assuring that the value of the Lyapunov function along the predicted trajectory $\tilde{x}(t)$ satisfies a Lyapunov-based contractive constraint (constraint (6f)) where $\hat{x}(t)$ is the state trajectory corresponding to the nominal system in closed-loop with the Lyapunov-based controller (constraint (6d)) with the initial condition being $\hat{x}(t_k) = \tilde{x}(t_k)$ (constraint (6e)). Note that the contractive constraint (6f) depends on the current delay $d(t_k)$. If the controller does not receive any new measurement at a sampling time, it keeps implementing the last evaluated optimal trajectory. This strategy is a receding horizon strategy, which takes time-varying measurement delays explicitly into account. The receding horizon scheme is modified as follows to take into account time-varying delays:

1) If a new measurement is received, then solve (5) and obtain $u_k^*(t)$, else $u_k^*(t) = u_{k-1}^*(t)$.
2) Apply $u(t) = u_k^*(t)$ for all $t \in [t_k, t_{k+1}]$.
3) Obtain a new sample and go to 1.

This modification states that if no measurement is received, then the previous optimal control input trajectory is applied.

This is equivalent to using the model to predict the future evolution of the state of the system and update the input accordingly as in [7], [18], [11], [19], [20] In addition, the modified receding horizon scheme and constraint (6b) guarantee that $u_{k-1}(t)$ stores the control input applied to the system from $t_{k-d(t_k)}$ to t_k.

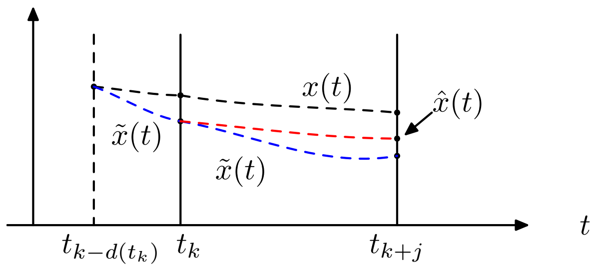


Fig. 2. Possible scenario of the measurements received by the LMPC and the corresponding state trajectories defined in problem (5).

Figure 2 shows a possible scenario for a system of dimension 1. A delayed measurement $x(t_{k-d(t_k)})$ is received at time t_k and the next new measurement is not obtained until t_{k+j}. This implies that at time t_k we solve problem (5) and we apply the optimal input $u_k^*(t)$ from t_k to t_{k+j}. The solid vertical lines are used to indicate sampling times in which a new measurement is obtained (that is, t_k and t_{k+j}) and the dashed vertical line is used to indicate the time corresponding to the measurement obtained in t_k (that is, $t_{k-d(t_k)}$).

B. Stability properties of the proposed LMPC

In this section, we present the stability properties of the proposed LMPC controller for systems subject to time-varying measurement delays. To this end, we need to introduce several auxiliary results that will be used in the proof of the main theorem of the paper. We first investigate the properties of the Lyapunov-based controller $h(x)$ applied in a sample-and-hold fashion without considering uncertainty or time-varying measurement delays. These properties are important because the proposed LMPC scheme is based on the nominal model of system (1).

Proposition 1 (c.f. [7]): Consider the nominal sampled trajectory $\hat{x}(t)$ *of system (1) for a controller* $h(x)$ *that satisfies (2) obtained by solving recursively*

$$\dot{\hat{x}}(t) = f(\hat{x}(t), h(\hat{x}(t_k)), 0),\ t \in [t_k, t_{k+1}]$$

where $t_k = t_0 + k\Delta,\ k = 0, 1, \ldots$ *Let* $\Delta, \epsilon_s > 0$ *and* $\rho > \rho_s > 0$ *satisfy*

$$-\alpha_3(\alpha_2^{-1}(\rho_s)) + \alpha_4(\alpha_1^{-1}(\rho))L_x M\Delta \leq -\epsilon_s/\Delta. \quad (8)$$

Then, if $\rho_{\min} \leq \rho$ *where*

$$\rho_{min} = \max\{V(\hat{x}(t+\Delta)) : V(\hat{x}(t)) \leq \rho_s\}$$

and $\hat{x}(t_0) \in \Omega_\rho$*, the following inequalities hold*

$$\begin{array}{l} V(\hat{x}(t_k)) \leq \max\{V(\hat{x}(t_0)) - k\epsilon_s, \rho_{min}\} \\ V(\hat{x}(t)) \leq \max\{V(\hat{x}(t_k)), \rho_{min}\},\ \forall t \in [t_k, t_{k+1}]. \end{array} \quad (9)$$

Proposition 1 guarantees that if system (1) with $w(t) \equiv 0$ for all t under the control law $u = h(x)$, implemented in a sample-and-hold fashion, starts in Ω_ρ, then it is ultimately bounded in $\Omega_{\rho_{min}}$. The following proposition provides an upper bound on the deviation of the state trajectory obtained using the nominal model, from the real state trajectory when the same control input trajectory is applied.

Proposition 2: Consider the following state trajectories

$$\begin{array}{ll} \dot{x}_a(t) & = f(x_a(t), u(t), w(t)) \\ \dot{x}_b(t) & = f(x_b(t), u(t), 0) \end{array} \quad (10)$$

with initial states $x_a(t_0) = x_b(t_0) \in \Omega_\rho$*, then the following inequality holds,*

$$|x_a(t) - x_b(t)| \leq f_W(t - t_0), \quad (11)$$

for all $x_a(t), x_b(t) \in \Omega_\rho$ *and all* $w(t) \in W$ *where*

$$f_W(\tau) = \frac{L_w\theta}{L_x}(e^{L_x\tau} - 1).$$

Proof: Define the error vector as $e(t) = x_a(t) - x_b(t)$. The time derivative of the error is given by

$$\dot{e}(t) = f(x_a(t), u(t), 0) - f(x_b(t), u(t), 0).$$

Applying (3), the following inequality holds

$$|\dot{e}(t)| \leq L_w|w(t) - 0| + L_x|x_a(t) - x_b(t)| \leq L_w\theta + L_x|e(t)|$$

for all $x_a(t), x_b(t) \in \Omega_\rho$ and $w(t) \in W$. Integrating $|\dot{e}(t)|$ with initial condition $e(t_0) = 0$ (recall that $x_a(t_0) = x_b(t_0)$), the following bound on the norm of the error vector is obtained

$$|e(t)| \leq \frac{L_w\theta}{L_x}(e^{L_x(t-t_0)} - 1).$$

This implies that (11) holds for

$$f_W(\tau) = \frac{L_w\theta}{L_x}(e^{L_x\tau} - 1).$$

■

The following proposition bounds the difference between the magnitudes of the Lyapunov function of two different states in Ω_ρ. This proposition is used together with Propositions 1 and 2 to obtain an upper bound on the value of the Lyapunov function of the real state if a given control input trajectory is applied.

Proposition 3: Consider the Lyapunov function $V(\cdot)$ *of system (1). Given any positive constant* $\rho > 0$*, there exists a quadratic function* $f_V(\cdot)$ *such that*

$$V(x) \leq V(\hat{x}) + f_V(|x - \hat{x}|) \quad (12)$$

for all $x, \hat{x} \in \Omega_\rho$*.*

Proof: Because the Lyapunov function $V(x)$ is continuous and bounded on compact sets, we can find a positive constant β such that a Taylor series expansion of V around $\hat{x}$ yields

$$V(x) \leq V(\hat{x}) + \frac{\partial V}{\partial x}|x - \hat{x}| + \beta|x - \hat{x}|^2, \forall x, \hat{x} \in \Omega_\rho.$$

Note that the term $\beta|x - \hat{x}|^2$ bounds the high order terms of the Taylor series of $V(x)$ for all $x, \hat{x} \in \Omega_\rho$. Taking into account (2), the following bound for $V(x)$ is obtained

$$V(x) \leq V(\hat{x}) + \alpha_4(\alpha_1^{-1}(\rho))|x - \hat{x}| + \beta|x - \hat{x}|^2, \forall x, \hat{x} \in \Omega_\rho.$$

This implies that (12) holds for $f_V(x) = \alpha_4(\alpha_1^{-1}(\rho))x + \beta x^2$. ■

Theorem 1 provides sufficient conditions under which the LMPC scheme (5) guarantees stability of the nonlinear closed-loop system in the presence of time-varying measurement delays.

Theorem 1: Consider system (1) in closed-loop with the LMPC scheme (5) based on a controller $h(x)$ that satisfies (2). Let constants $\epsilon_s, \rho_s > 0$, Δ, θ and D satisfy (8) and the following constraint:

$$-\epsilon_s + f_V(f_W(D\Delta)) + f_V(f_W(D+1)\Delta) < 0. \quad (13)$$

If $D + 1 \leq N$, $x(t_0) \in \Omega_\rho$ and $d(t_0) = 0$, then $x(t)$ is ultimately bounded in Ω_{ρ_c} where

$$\rho_c = \rho_{\min} + f_V(f_W(D\Delta)) + f_V(f_W(D+1)\Delta).$$

Proof: In order to prove that system (1) in closed-loop with the proposed LMPC is ultimately bounded in a region that contains the origin we will prove that the Lyapunov function $V(x)$ is a decreasing function of time with a lower bound on its magnitude. We assume that the delayed measurement $x(t_{k-d(t_k)})$ is received at time t_k and that a new measurement is not obtained until t_{k+j}. The optimization problem (5) is solved at t_k and the optimal input $u_k^*(t)$ is applied from t_k to t_{k+j}.

The trajectory $\hat{x}(t)$ corresponds to the nominal system in closed-loop with the Lyapunov-based controller implemented in a sample-and-hold fashion with initial condition $\tilde{x}(t_k)$ (constraints (6d) and (6e)). Applying Proposition 1, we obtain the following inequality:

$$V(\hat{x}(t_{k+j})) \leq \max\{V(\hat{x}(t_k)) - j\epsilon_s, \rho_{\min}\}.$$

The contractive constraint (6f) of the proposed LMPC guarantees that

$$V(\tilde{x}(\tau)) \leq V(\hat{x}(\tau)), \forall \tau \in [t_k, t_{k+D+1-d(t_k)}]$$

and constraint (6e) guarantees that $V(\hat{x}(t_k)) = V(\tilde{x}(t_k))$. This implies that

$$V(\tilde{x}(t_{k+j})) \leq \max\{V(\tilde{x}(t_k)) - j\epsilon_s, \rho_{\min}\}.$$

Assuming that $x(t) \in \Omega_\rho$ for all times, we can apply Proposition 3 to obtain the following inequalities

$$V(\tilde{x}(t_k)) \leq V(x(t_k)) + f_V(|x(t_k) - \tilde{x}(t_k)|).$$

and

$$V(x(t_{k+j})) \leq V(\tilde{x}(t_{k+j})) + f_V(|x(t_{k+j}) - \tilde{x}(t_{k+j})|)$$

This assumption is automatically satisfied if the system is proved to be ultimately bounded. Constraints (6a), (6c), (6b) and the implementation procedure allow us to apply Proposition 2 because it is guaranteed that the real state $x(t)$ and the state estimated using the nominal model $\tilde{x}(t)$ are obtained using the same input trajectory. Applying Proposition 2 we obtain the following upper bounds on the deviation of $\tilde{x}(t)$ from $x(t)$

$$|x(t_k) - \tilde{x}(t_k)| \leq f_W(d(t_k)\Delta)$$

$$|x(t_{k+j}) - \tilde{x}(t_{k+j})| \leq f_W((d(t_k) + j)\Delta)$$

Using these inequalities the following upper bound on $V(x(t_{k+j}))$ is obtained:

$$\begin{aligned} V(x(t_{k+j})) \leq \; & \max\{V(x(t_k)) - j\epsilon_s, \rho_{\min}\} \\ & + f_V(f_W(d(t_k)\Delta)) \\ & + f_V(f_W((d(t_k) + j)\Delta)) \end{aligned} \quad (14)$$

In order to prove that for all possible sequences $d(t_k)$ the Lyapunov function is guaranteed to decrease between two consecutive new measurements until a lower bound is obtained we will consider the worst case scenario; that is, after a new measurement is obtained at sampling time t_k, the controller has to operate in open-loop for the maximum possible time due to the time-varying delays. Taking into account that the maximum allowable delay is D, it holds that the maximum number of sampling times in which the system will operate in open-loop is $D + 1 - d(t_k)$. This implies that the worst case is given for $j = D + 1 - d(t_k)$.

$$\begin{aligned} V(x(t_{k+D+1-d(t_k)})) \leq \; & \max\{V(x(t_k)) \\ & -(D + 1 - d(t_k))\epsilon_s, \rho_{\min}\} \\ & + f_V(f_W(d(t_k)\Delta)) \\ & + f_V(f_W((D + 1)\Delta)) \end{aligned}$$

In order to prove that the Lyapunov function is decreasing between two consecutive new measurements for the worst possible case the following inequality must hold

$$(D+1-d(t_k))\epsilon_s > f_V(f_W(d(t_k)\Delta)) + f_V(f_W((D+1)\Delta))$$

for all $d(t_k) = 0, \ldots, D$. The worst possible case is $d(t_k) = D$. This implies that if condition (13) is satisfied, then for all $d(t_k) = 0, 1, \ldots, D$ and all $j = 1, \ldots, D + 1 - d(t_k)$ (where j indicates when a new measurement is received after sampling time t_k) there exists $\epsilon_w > 0$ such that the following inequality holds

$$V(x(t_{k+j})) \leq \max\{V(x(t_k)) - \epsilon_w, \rho_c\} \quad (15)$$

which implies that when $x(t_k) \in \Omega/\Omega_{\rho_c}$[4], $V(x(t))$ will decrease until the state converges to Ω_{ρ_c} for all $t \in [t_k, t_{k+j}]$, and when $x(t_k) \in \Omega_{\rho_c}$, it remains inside Ω_{ρ_c} for all $t \in [t_k, t_{k+j}]$.

If $x(t_0) \in \Omega$ and it is known, using (15) recursively it is proved that the closed-loop trajectories of system (1) subject to time-varying measurements delays satisfy

$$\limsup_{t\to\infty} V(x(t)) \leq \rho_c.$$

for all possible sequences $\{d(t_k)\}$. This proves that the closed-loop system is ultimately bounded in Ω_{ρ_c} ■

Remark 1: The main difference between the original LMPC controller [4], [5], [6] and the proposed scheme (apart from the modified receding horizon implementation technique) is that the contractive constraint (6f) in the original LMPC optimization problem has to hold only in the first predicted step. This implies that even if the same

[4]We use the operator "/" to denote set substraction, i.e., $A/B := \{x \in R^{n_x} | x \in A, x \notin B\}$

TABLE I

PROCESS PARAMETERS

F	4.998 $[m^3/h]$	k_{10}	3*10^6 $[h^{-1}]$
V_r	1$[m^3]$	k_{20}	3*10^5 $[h^{-1}]$
R	8.314 $[KJ/kmol \cdot K]$	k_{30}	3*10^5 $[h^{-1}]$
T_{A0}	300 $[K]$	E_1	5*10^4 $[KJ/kmol]$
C_{A0}	4 $[kmol/m^3]$	E_2	7.53*10^4 $[KJ/kmol]$
ΔH_1	-5.0*10^4 $[KJ/kmol]$	E_3	7.53*10^4 $[KJ/kmol]$
ΔH_2	-5.2*10^4 $[KJ/kmol]$	σ	1000 $[kg/m^3]$
ΔH_3	-5.4*10^4 $[KJ/kmol]$	c_p	0.231 $[KJ/kg \cdot K]$

implementation procedure is used with the original LMPC, stability cannot be guaranteed. This point will be illustrated in the next section.

Remark 2: The main difference between the LMPC controller for systems subject to data losses [7] and the proposed scheme (apart from the modified receding horizon implementation technique) is that the contractive constraint (6f) in the LMPC optimization problem for systems subject to data losses has to hold for the whole prediction horizon. This constraint makes the computed control action more conservative (and thus less optimal). This implies that even if the same implementation procedure is used with the LMPC for systems subject to data losses, the resulting controller will be, in general, less optimal than the proposed LMPC controller (5). This point will be illustrated in the next section.

IV. Application to a Chemical Reactor

Consider a well mixed, non-isothermal continuous stirred tank reactor where three parallel irreversible elementary exothermic reactions take place of the form $A \to B$, $A \to C$ and $A \to D$. B is the desired product and C and D are byproducts. The feed to the reactor consists of pure A at flow rate F, temperature T_{A0} and molar concentration $C_{A0} + \Delta C_{A0}$ where ΔC_{A0} is an unknown time-varying uncertainty. Due to the non-isothermal nature of the reactor, a jacket is used to remove/provide heat to the reactor. Using first principles and standard modeling assumptions the following mathematical model of the process is obtained

$$\frac{dT}{dt} = \frac{F}{V_r}(T_{A0} - T) - \sum_{i=1}^{3} \frac{\Delta H_i}{\sigma c_p} k_{i0} e^{\frac{-E_i}{RT}} C_A + \frac{Q}{\sigma c_p V_r}$$
$$\frac{dC_A}{dt} = \frac{F}{V_r}(C_{A0} + \Delta C_{A0} - C_A) + \sum_{i=1}^{3} k_{i0} e^{\frac{-E_i}{RT}} C_A \tag{16}$$

where C_A denotes the concentration of the reactant A, T denotes the temperature of the reactor, Q denotes the rate of heat input/removal, V_r denotes the volume of the reactor, $\Delta H_i, k_{i0}, E_i, i = 1, 2, 3$ denote the enthalpies, pre-exponential constants and activation energies of the three reactions, respectively, and c_p and σ denote the heat capacity and the density of the fluid in the reactor. The values of the process parameters are shown in Table I.

System (16) has three steady-states (two locally asymptotically stable and one unstable). The control objective is to stabilize the system at the open-loop unstable steady state $T_s = 388\ K$, $C_{As} = 3.59\ mol/l$. The manipulated input is the rate of heat input Q and the allowable input is bounded by $|Q| \leq 10^5\ KJ/h$. We consider a time-varying uncertainty in the concentration of the inflow $|\Delta C_{A0}| \leq 0.2\ mol/l$. The control system is subject to time-varying measurement delays in the measurements of the concentration of the reactant, C_A, and in the measurements of the temperature, T.

To illustrate the theoretical results, we first design a Lyapunov based feedback law using the method presented in [21]. System (16) belongs to the following class of nonlinear systems

$$\dot{x}(t) = f(x(t)) + g(x(t))u(t) + w(x(t))\theta(t)$$

where $x^T = [T - T_s\ C_A - C_{As}]$ is the state, $u = Q$ is the input and $\theta = \Delta C_{A0}$ is a time varying bounded disturbance. Consider the control Lyapunov function $V(x) = x^T P x$ with

$$P = \begin{bmatrix} 1 & 0 \\ 0 & 10^4 \end{bmatrix}.$$

The values of the weights have been chosen to account for the different range of numerical values for each state. The following feedback law [21] asymptotically stabilizes the open-loop unstable steady-state of the nominal system (i.e., $\theta(t) = 0$) and is of the form (2):

$$h(x) = \begin{cases} -\frac{L_f V + \sqrt{(L_f V)^2 + (L_g V)^4}}{L_g V} & \text{if } L_g V \neq 0 \\ 0 & \text{if } L_g V = 0 \end{cases} \tag{17}$$

where $L_f V = \frac{\partial V(x)}{\partial x} f$ and $L_g V = \frac{\partial V(x)}{\partial x} g$ denote the Lie derivatives of the scalar function V with respect to the vectors fields f and g respectively. This controller will be used to design the LMPC controller. The stability region Ω_ρ is defined as $V(x) \leq 700$, i.e., $\rho = 700$.

In order to choose an appropriate sampling time we resort to extensive off-line closed-loop simulations under the Lyapunov-based controller of (17). After trying different sampling times, we choose $\Delta = 0.025h$. For this sampling time, the closed-loop system without measurement delays under $u = h(x)$ is practically stable and the performance is similar to the closed-loop system with continuous measurements. We chose the maximum allowable measurement delay equal to 6Δ (i.e., $D = 6$). The cost function is defined by the weight matrices $Q_c = P$ and $R_c = 10^{-6}$. The values of the weights have been tuned in such a way that the values of the control inputs are comparable to the ones computed by the Lyapunov-based controller.

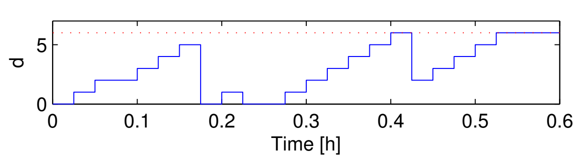
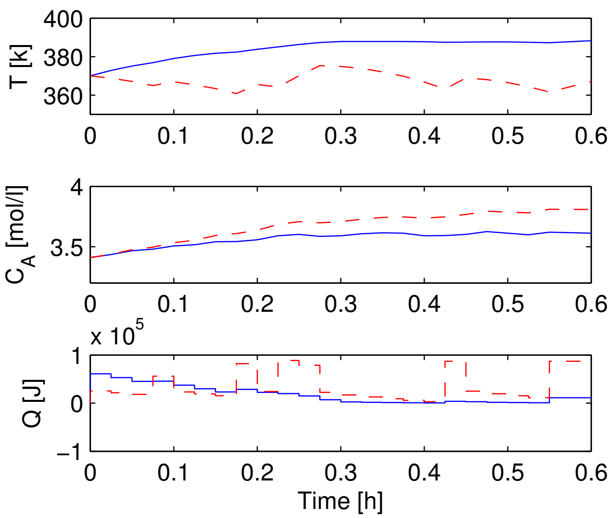
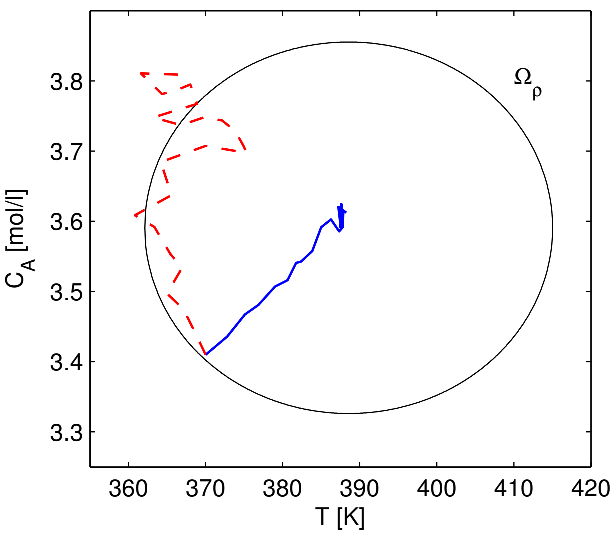


Fig. 3. Delay sequence $d(t_k)$ used in the simulation shown in Figure 4.

We will first compare the proposed LMPC scheme (5) with the original LMPC scheme (see Remark 1). For this



(a)



(b)

Fig. 4. (a)(b) Trajectories of system (16) with the proposed LMPC scheme (5) (plain line) and the original LMPC scheme (dashed line) when the maximum allowable measurement delay D is 6.

simulation, we choose the prediction horizon of the two LMPC controllers N equal to 7 ($N \geq D+1$). The same weights, sampling time and prediction horizon are used. We implement the original LMPC scheme using the modified receding horizon scheme, that is, the current state is estimated using the nominal model of system (16) when a delayed measurement is received and the last optimal input is applied when no new measurement is received. In order to simulate the process in the presence of measurement delays, we use a random process to generate the delay sequence $d(t_k)$. Figure 3 shows the delay sequence $d(t_k)$ used in these simulations. Note that when $d(t_{k+1}) = d(t_k) + 1$, the controller does not receive any new measurement. In Figure 4, the trajectories of the closed-loop system under both controllers are shown in the presence of measurement delay with $D = 6$. It can be seen that the original LMPC controller can not stabilize the system at the desired open-loop unstable steady-state and the trajectories leave the stability region, while the proposed LMPC scheme keeps the trajectories inside the stability region.

TABLE II

PERFORMANCE COSTS ALONG THE CLOSED-LOOP TRAJECTORIES.

sim.	Proposed LMPC	LMPC for data losses
1	1.8295×10^4	2.4428×10^4
2	4.2057×10^4	6.0522×10^4
3	3.2481×10^3	1.0428×10^4
4	7.4328×10^2	7.3961×10^2
5	1.4229×10^3	2.7798×10^5
6	4.9435×10^4	6.1596×10^4
7	3.2519×10^4	3.4319×10^4
8	2.7590×10^4	4.7075×10^4
9	9.4216×10^2	9.4866×10^2
10	5.4505×10^2	5.4322×10^2
11	1.9723×10^4	3.1282×10^4
12	2.7235×10^4	3.8772×10^4
13	1.8671×10^3	1.9200×10^3
14	3.7789×10^4	4.0050×10^4
15	2.1839×10^3	2.1392×10^3
16	4.2920×10^4	4.4594×10^4
17	1.5153×10^2	1.7190×10^2
18	4.9955×10^3	9.9094×10^3
19	3.2086×10^4	4.8838×10^4
20	1.5420×10^3	1.5197×10^3

We have also carried out a set of simulations to compare the proposed LMPC scheme with the LMPC scheme for systems subject to data losses (see Remark 2) from a performance index point of view. We also implement the LMPC controller for systems subject to data losses using the modified receding horizon scheme. Table II shows the total cost computed for 20 different closed-loop simulations under the proposed LMPC and the LMPC for systems subject to data losses. To carry out this comparison, we have computed the total cost of each simulation based on the following performance index

$$\sum_{i=0}^{M} x(t_i)^T Q_c x(t_i) + u(t_i)^T R_c u(t_i)$$

where t_0 is the initial time of the simulations and $t_M = 2\ h$ is the final simulation time. The prediction horizon in this set of simulations is $N = 10$. For each pair of simulations (one for each controller) a different initial state inside the stability region, a different uncertainty trajectory and a different random measurement delay sequence is chosen. As it can be seen in Table II, the proposed LMPC controller has a cost lower than the corresponding total cost under the LMPC controller designed for system subject to data losses in 16 out of 20 simulations. These simulations illustrate that the proposed LMPC controller is, in general, more optimal from a performance point of view.

We have also carried out a set of simulations to study the dependence on the value of the maximum delay D of the set in which the trajectory of system (16) under the proposed

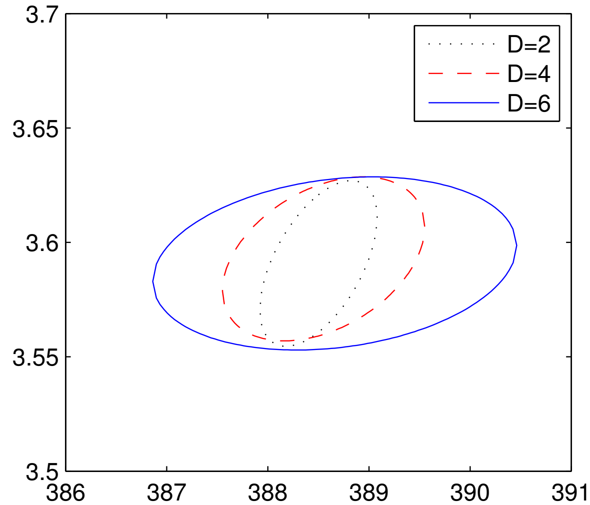


Fig. 5. Estimates of the set in which the trajectories of the system (16) with the proposed LMPC scheme are ultimately bounded when the maximum allowable measurement delay D is 2, 4 and 6.

LMPC scheme is ultimately bounded. In order to estimate the size of each set for a given D, we start the system very close to the equilibrium state and run it for a sufficient long time. In this set of simulations, we set $\Delta C_{A0} = 0.1\ kmol/m^3$ and $N = 7$. The simulation time is 25 h. Figure 5 shows the location of the states, (C_A, T), at each sampling time and the estimated regions for $D = 2, 4, 6$. Three ellipses are used to estimate the boundaries of the sets, and they are chosen to be as small as possible but still include all the corresponding points indicating the states. From Figure 5, we see that the size of these sets becomes larger as D increases. Note that all the sets for $D = 2, 4, 6$ are included in the stability region of the closed loop system under the proposed LMPC ($\Omega_\rho,\ \rho = 700$).

The simulations have been carried out using Matlab in a Pentium 3.20G Hz. The nonlinear optimization problem has been solved using the function `fmincom` with an initial feasible solution provided by the Lyapunov based controller. To solve the CSTR ODEs, both in the simulations and in the optimization algorithm, an Euler method with a fixed integration time of 0.001hr has been implemented in a mex DLL using the C Programming Language. The mean time of 100 runs to solve the LMPC optimizaion problem of this set of simulations has been 1.32s for $N = 7$ and 1.98s for $N = 10$.

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