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Jet bundles and algebro-geometric characterisations for controllability of affine systems

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***Abstract*— A geometric setting for studying control-affine systems is presented and a feedback-invariant approach to studying local controllability is introduced. The principal geometric object in this construction is a system-independent linear map which, when restricted to system-dependent data, describes a class of variations for the system. The resulting conditions for controllability take the form of algebraic equations on the jets of sections of certain vector bundles.**

I. Introduction and literature review

Controllability for linear systems has been well understood since the 1960's [1]. For nonlinear systems, however, controllability is now well known to be far more delicate than for linear systems. A weak form of controllability known as accessibility—wherein one requires that the set of states reachable from a fixed initial state has a nonempty interior—has been understood since the work of Sussmann and Jurdjevic [2] and Krener [3]. This work, consistent with a result of Nagano [4] and its generalisation by Sussmann [5], establishes the essential rôle of the Lie algebra generated by the set of vector fields defining the system trajectories. A stronger form of controllability, small-time local controllability (STLC), requires that the initial state itself be in the interior of the set of states reachable from it in small time. Conditions for STLC have proven to be challenging to understand. Indeed, the decidability of STLC has been shown to be NP-hard [6], [7]. Nonetheless, the problem of controllability is so fundamental, particularly given its connections to optimal control, that it remains worthy of study.

Much of the work on controllability has been carried out for so-called control-affine systems, i.e., those with governing equations

$$\dot{x}(t) = f_0(x(t)) + \sum_{a=1}^{m} u^a(t) f_a(x(t)),$$

where $t \mapsto x(t)$ is a curve in the state manifold M, f_0 is a vector field on M called the ***drift vector field***, and $f_1, \dots, f_m$ are vector fields on M called ***control vector fields***. Motivated by the understanding that the Lie algebra generated by the vector fields $f_0, f_1, \dots, f_m$ is fundamental, many of the existing results on controllability of control-affine systems have the flavour of providing conditions on Lie brackets of these vector fields that are either necessary or sufficient for controllability. The use of Lie bracket conditions on the vector fields $f_0, f_1, \dots, f_m$ often provides computable conditions for STLC, conditions that apply to many examples. Representative of this approach is the paper of Sussmann [8] which additionally contains many useful technical tools for studying control-affine systems.

One of the difficulties with the approach of deriving Lie bracket characterisations using the vector fields $f_0, f_1, \dots, f_m$ is that these characterisations behave under feedback transformations in ways that are difficult to understand. Let us explain what we mean by this. The idea of a feedback transformation for control-affine systems is that it transforms one control-affine system into another one in such a way that the two systems have the same trajectories. If one has two control-affine systems with governing equations

$$\dot{x}(t) = f_0(x(t)) + \sum_{a=1}^{m} u^a(t) f_a(x(t)),$$
$$\dot{x}(t) = g_0(x(t)) + \sum_{b=1}^{n} v^b(t) g_b(x(t)),$$

a ***feedback transformation*** from the first system to the second system is typically of the form

$$g_0(x) = \sum_{a=0}^{m} \lambda^a(x) f_a(x), \quad g_b(x) = \sum_{a=1}^{m} \Lambda^a_b(x) f_a(x), \quad b \in \{1, \dots, n\},$$

for functions λ^a, $a \in \{0, 1, \dots, m\}$, and Λ^a_b, $a \in \{1, \dots, m\}$, $b \in \{1, \dots, n\}$, on M for which $\lambda^0 = 1$. To ensure that the transformation is indeed trajectory-preserving, some conditions need to be imposed on these functions. However, since we have no intention of expounding on this point of view here, we simply refer to the monograph of Elkin [9] for details. Since feedback transformations preserve trajectories, they also preserve controllability. Therefore, one way to think about Lie bracket-type controllability results for a fixed drift and control vector fields is that they *should* be read as, "Given a control-affine system defined by the vector fields $f_0, f_1, \dots, f_m$, if there exist vector fields $g_0, g_1, \dots, g_n$ related by a feedback transformation to $f_0, f_1, \dots, f_m$ and which satisfy [list of conditions], then the system is STLC." By bundling a feedback transformation into the picture, the simple, computable conditions on $f_0, f_1, \dots, f_m$ for STLC are now difficult to compute and difficult to interpret geometrically. An example is illustrative.

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I.1 EXAMPLE: We take $\mathsf{M} = \mathbb{R}^m \times \mathbb{R}^{n-m}$ and consider a control-affine system with the following governing equations:

$$\dot{\boldsymbol{x}}_1(t) = \boldsymbol{u}(t), \quad \dot{\boldsymbol{x}}_2(t) = \boldsymbol{Q}(\boldsymbol{x}_1(t)), \tag{1}$$

where $(\boldsymbol{x}_1, \boldsymbol{x}_2) \in \mathbb{R}^m \times \mathbb{R}^{n-m}$, $\boldsymbol{u} \in \mathbb{R}^m$, and $\boldsymbol{Q} \colon \mathbb{R}^m \to \mathbb{R}^{n-m}$ is a quadratic map. Let us write

$$\boldsymbol{Q}(\boldsymbol{x}_1) = (Q_1(\boldsymbol{x}_1), \dots, Q_{n-m}(\boldsymbol{x}_1))$$

for scalar-valued quadratic functions $Q_1, \dots, Q_{n-m}$. If one applies the "generalised Hermes condition" of Sussmann [8] then one sees that the system is STLC from $(\boldsymbol{0}_m, \boldsymbol{0}_{n-m})$ if the diagonal entries of each of the symmetric matrices corresponding to the quadratic functions $Q_1, \dots, Q_{n-m}$ is zero. This condition, however, is easily seen by virtue of simple counterexamples to not be necessary. However, one may show that we have the equivalence of the following statements (see [10], [11]):

1) the system (1) is STLC from $(\boldsymbol{0}_m, \boldsymbol{0}_{n-m})$;
2) there exists a feedback transformation taking the system (1) into one satisfying the generalised Hermes condition;
3) $\boldsymbol{0}_{n-m}$ is in the interior of the convex hull of $\mathrm{image}(\boldsymbol{Q})$.

The third of these conditions contains what one would like to think of as the "geometric essence" of the controllability of this system. However, the verifiability of this condition is known to be NP-complete. This illustrates that when one wraps the prefix "there exist vector fields $g_1, g_1, \dots, g_n$ related by a feedback transformation to $f_0, f_1, \dots, f_m$" around the computable generalised Hermes condition, one arrives at a condition that is computationally difficult to verify. However, the resulting condition nonetheless provides a geometrically satisfying characterisation of controllability. It is geometrically satisfying results that we are pursuing in the work we initiate here. •

The point of the preceding example is not that one should go about attempting to provide the feedback-invariant versions of existing controllability results. The point, rather, is that one should not work with vector fields $f_0, f_1, \dots, f_m$, but with geometric object represented by these vector fields, i.e., with the affine distribution generated by these vector fields (see Definition II.1). By doing this, and by making sure that all constructions are made in terms of the affine distribution rather than a specific choice of generators, we are guaranteed to arrive at results that are feedback-invariant. One should imagine our approach as being analogous to the relationship between a manifold and local coordinates. While a manifold always possesses local coordinates (just as an affine distribution is assumed to always possesses local generators $f_0, f_1, \dots, f_m$), one should always make constructions that do not depend on a specific choice of such coordinates.

II. AFFINE SYSTEMS

We describe in this section our abstraction from control-affine systems to affine systems. This provides the framework we need to study controllability in a feedback-invariant manner. As we shall see, the shift in terminology from "control-affine" to "affine" is literal, since in our formulation the notion of a "control" is actually removed.

A. Affine distributions and affine systems

We recall that an ***affine subspace*** of a vector space V is a subset of the form

$$\mathsf{A} = v_0 + \mathsf{U} \triangleq \{v_0 + u \mid u \in \mathsf{U}\},$$

where $v_0 \in \mathsf{V}$ and where $\mathsf{U} \subset \mathsf{V}$ is a subspace. The smallest affine subspace containing a set $S \subset \mathsf{V}$ is the ***affine hull*** of S and is denoted $\mathrm{aff}(S)$. By $\mathrm{conv}(S)$ we denote the ***convex hull*** of S, i.e., the smallest convex set containing S, and by $\mathrm{conv}^+(S)$ we denote the ***coned convex hull*** of S, i.e., the smallest convex cone containing S.

We denote the tangent bundle projection by $\pi_{\mathsf{TM}} \colon \mathsf{TM} \to \mathsf{M}$. The set of smooth vector fields is denoted by $\Gamma^\infty(\pi_{\mathsf{TM}})$.

II.1 DEFINITION: Let M be a manifold. A ***smooth affine distribution*** on M is a subset $\mathsf{A} \subset \mathsf{TM}$ such that, for each $x_0 \in \mathsf{M}$, there exists a neighbourhood $\mathcal{U}$ of x_0 and $X_0, X_1, \dots, X_k \in \Gamma^\infty(\pi_{\mathsf{TM}})$ such that

$$\mathsf{A}_x \triangleq \mathsf{A} \cap \mathsf{T}_x\mathsf{M} = X_0(x) + \mathrm{span}(X_1(x), \dots, X_k(x))$$

for each $x \in \mathcal{U}$. The vector fields $X_0, X_1, \dots, X_k$ are ***local generators*** for A about x_0. •

The restriction that there be a finite number of local generators is made without loss of generality for analytic affine distributions; this is due to the Noetherian property of the ring of germs of analytic functions. However, for C^∞-affine distributions, this finiteness is a real restriction, albeit one that is not important in applications.

For a control-affine system

$$\dot{x}(t) = f_0(x(t)) + \sum_{a=1}^{m} u^a(t) f_a(x(t)),$$

one often restricts the controls $\boldsymbol{u}$ to take values in a subset $U \subset \mathbb{R}^m$. This has the effect of restricting the set of tangent vectors to system trajectories. We adapt this restriction to our setting as follows.

II.2 DEFINITION: Let A be a smooth affine distribution on M.

(i) An ***affine system*** in A is a map $\mathscr{A} \colon \mathsf{M} \to \mathbf{2}^{\mathsf{TM}}$ with values in the power set of TM having the following properties:
   (a) $\mathscr{A}(x) \subset \mathsf{T}_x\mathsf{M}$;
   (b) $\mathrm{aff}(\mathscr{A}(x)) = \mathsf{A}_x$.
(ii) An affine system $\mathscr{A}$ in A is ***smooth*** if, for each $x_0 \in \mathsf{M}$, there exists a neighbourhood $\mathcal{U}$ of x_0 such that, if $v \in \mathscr{A}(x_0)$, then there exists $\xi \in \Gamma^\infty(\pi_{\mathsf{TM}})$ such that $\xi(x_0) = v$ and $\xi(x) \in \mathscr{A}(x)$ for every $x \in \mathcal{U}$.
(iii) A ***trajectory*** for an affine system $\mathscr{A}$ is a locally absolutely continuous curve $\gamma \colon I \to \mathsf{M}$ defined on an interval $I \subset \mathbb{R}$ such that $\gamma'(t) \in \mathscr{A}(\gamma(t))$ for almost every $t \in I$.

(iv) An ***𝒜-vector field*** is a map $\xi\colon \mathsf{M} \to \mathsf{TM}$ such that $\xi(x) \in \mathscr{A}(x)$ for every $x \in \mathsf{M}$. •

The condition that $\mathrm{aff}(\mathscr{A}(x)) = \mathsf{A}_x$ is a nondegeneracy condition, and is the analogue of requiring that $\mathrm{aff}(U) = \mathbb{R}^m$ in the control-affine case. For controllability of a control-affine system from a point x_0, a necessary condition is that $f_0(x_0) = 0_{x_0}$ and that $\mathbf{0}_m \in \mathrm{conv}(U)$. We shall strengthen this by requiring, in our setting, the analogue of the requirement that $\mathbf{0}_m \in \mathrm{int}(\mathrm{conv}(U))$.

II.3 DEFINITION: Let A be an affine distribution on M and let $x_0 \in \mathsf{M}$. An affine system $\mathscr{A}$ in A is ***proper*** at x_0 if $0_{x_0} \in \mathrm{int}(\mathrm{conv}(\mathscr{A}(x_0)))$. •

B. Controllability definitions for affine systems

In this section we propose our definitions for controllability. These definitions are a little subtle since what we want at the end of the day is a definition of controllability for an affine distribution. What is most natural, however, is the notion of controllability of an affine system. So this is where we begin.

For an affine system $\mathscr{A}$ in a smooth affine distribution A, we denote

$$\mathcal{R}_{\mathscr{A}}(x_0, T) = \{\gamma(T) \mid \gamma\colon [0, T] \to \mathsf{M} \text{ a trajectory for } \mathscr{A}\}$$

and $\mathcal{R}_{\mathscr{A}}(x_0, \leq T) = \cup_{t\in[0,T]}\mathcal{R}_{\mathscr{A}}(x_0, t)$. We also denote $\mathcal{R}_{\mathscr{A}}(x_0) = \cup_{T\in\mathbb{R}_{>0}}\mathcal{R}_{\mathscr{A}}(x_0, T)$.

II.4 DEFINITION: An affine system $\mathscr{A}$ in a smooth affine distribution A on M is ***small-time locally controllable*** (***STLC***) from $x_0 \in \mathsf{M}$ if there exists $T \in \mathbb{R}_{>0}$ such that $x_0 \in \mathrm{int}(\mathcal{R}_{\mathscr{A}}(x_0, \leq t))$ for every $t \in]0, T]$. •

This definition is, of course, the usual one from the controllability theory for control-affine systems, but adapted to our setting. This notion of STLC for affine systems will be of use for us. However, geometrically speaking, the definition is problematic since it involves the affine system $\mathscr{A}$, whereas we are really interested in the geometry of the affine distribution A. The following definitions address this. These definitions are essentially new, and have not been explicitly explored in the existing literature.

II.5 DEFINITION: Let A be a smooth affine distribution on M and let $x_0 \in \mathsf{M}$.

(i) A is ***properly small-time locally controllable*** (***PSTLC***) from x_0 if every smooth affine system $\mathscr{A}$ in A that is proper at x_0 is STLC from x_0.

(ii) A is ***small-time locally uncontrollable*** (***STLUC***) from x_0 if every smooth affine system $\mathscr{A}$ in A having the property that $\mathscr{A}(x_0)$ is compact is not STLC from x_0.

(iii) A is ***conditionally small-time locally controllable*** (***CSTLC***) from x_0 if it is neither PSTLC nor STLUC from x_0. •

III. Jet bundle notation and structure

In this brief paper we do not have the space to provide a full accounting of jet bundle structure needed to understand our approach at any level of detail. Therefore, we simply define enough jet bundle concepts and notation that we can describe the essence of what we are doing. We are interested in two sorts of jet bundles: (1) jet bundles of sections of vector bundles; (2) jet bundles of maps between manifolds. We refer the reader to the books of Saunders [12] and Kolář, Michor, and Slovák [13, Chapter 4] for further details of jet bundle structure.

We need some algebraic notation. For a $\mathbb{R}$-vector space V, denote the k-fold tensor product of V with itself by $\mathrm{T}^k(\mathsf{V})$. The subset of $\mathrm{T}^k(\mathsf{V})$ invariant under the symmetrisation operation is denoted by $\mathrm{S}^k(\mathsf{V})$. We shall also use the notation $\mathrm{T}^{\leq k}(\mathsf{V}) = \oplus_{j=1}^k \mathrm{T}^j(\mathsf{V})$ and $\mathrm{S}^{\leq k}(\mathsf{V}) = \mathrm{T}^{\leq k}(\mathsf{V}) \cap \mathrm{S}^k(\mathsf{V})$.

Let $\pi_{\mathsf{V}}\colon \mathsf{V} \to \mathsf{M}$ be a vector bundle and denote by $\Gamma^\infty(\pi_{\mathsf{V}})$ the set of its smooth sections. Let $k \in \mathbb{Z}_{\geq 0}$ and let $x \in \mathsf{M}$. Sections ξ and η ***agree to order k at x*** if, in vector bundle coordinates about x, ξ and η and their first k derivatives agree when evaluated at x. This defines an equivalence relation on the set of sections. An equivalence class is called a ***k-jet*** of sections and the set of equivalence classes is called the ***bundle of k-jets*** and we denote it by $\mathsf{J}^k\pi_{\mathsf{V}}$. For $k, l \in \mathbb{Z}_{\geq 0}$ with $k \geq l$ we have the natural projection $(\pi_{\mathsf{V}})_l^k\colon \mathsf{J}^k\pi_{\mathsf{V}} \to \mathsf{J}^l\pi_{\mathsf{V}}$ sending the k-jet to the l-jet. Note that $\mathsf{J}^0\pi_{\mathsf{V}}$ is naturally identified with V. The composition $(\pi_{\mathsf{V}})_k \triangleq \pi_{\mathsf{V}} \circ (\pi_{\mathsf{V}})_1^k$ can be shown to give a vector bundle structure to $(\pi_{\mathsf{V}})_k\colon \mathsf{J}^k\pi_{\mathsf{V}} \to \mathsf{M}$. If $\xi \in \Gamma^\infty(\pi_{\mathsf{V}})$ then we denote by $j^k\xi \in \Gamma^\infty((\pi_{\mathsf{V}})_k)$ the corresponding section of the k-jet bundle. For $x \in \mathsf{M}$ we shall denote by $\mathsf{J}_x^k\pi_{\mathsf{V}}$ the set of k-jets of the form $j^k\xi(x)$. It turns out that we have an exact sequence of vector bundles shown in Figure 1, where ϵ_k is the injection defined by

$$\epsilon_k((\boldsymbol{d}f_1(x) \odot \cdots \odot \boldsymbol{d}f_k(x)) \otimes \eta(k)) = j^k((f_1 \cdots f_k)\eta)(x)$$

and where $\odot$ denotes the symmetric tensor product.

Next we let M and N be manifolds and denote by $C^\infty(\mathsf{M}, \mathsf{N})$ the set of smooth maps from M to N. Let $k \in \mathbb{Z}_{\geq 0}$ and let $x \in \mathsf{M}$. Maps $\phi, \psi \in C^\infty(\mathsf{M}, \mathsf{N})$ ***agree to order k at x*** if $\phi(x) = \psi(x)$ and if, in a chart for M about x and a chart for N about $\phi(x) = \psi(x)$, the first k derivatives of ϕ and ψ agree when evaluated at x. This defines at equivalence relation on $C^\infty(\mathsf{M}, \mathsf{N})$, and we call an equivalence class a ***k-jet*** of maps. The set of equivalence classes we denote by $\mathsf{J}^k(\mathsf{M}; \mathsf{N})$. For $k, l \in \mathbb{Z}_{\geq 0}$ with $k \geq l$ we again have a natural projection, this denoted by $\sigma_l^k\colon \mathsf{J}^k(\mathsf{M}; \mathsf{N}) \to \mathsf{J}^l(\mathsf{M}; \mathsf{N})$. We have a natural identification of $\mathsf{J}^0(\mathsf{M}; \mathsf{N})$ with $\mathsf{M} \times \mathsf{N}$. If $\phi \in C^\infty(\mathsf{M}, \mathsf{N})$ then we denote by $j^k\phi\colon \mathsf{M} \to \mathsf{J}^k(\mathsf{M}; \mathsf{N})$ the map assigning to x the k-jet of ϕ at x. For $x \in \mathsf{M}$ and $y \in \mathsf{N}$ we denote by $\mathsf{J}_{(x,y)}^k(\mathsf{M}; \mathsf{N})$ the set of k-jets of the form $j^k\phi(x)$ where $\phi(x) = y$. Let us quickly indicate the algebraic structure of $\mathsf{J}_{(x,y)}^k(\mathsf{M}; \mathsf{N})$. Let us abbreviate

$$\mathsf{T}_{x_0}^{*k}\mathsf{M} = \mathsf{J}_{(x_0,0)}^k(\mathsf{M}; \mathbb{R}).$$

One can show that $\mathsf{T}_{x_0}^{*k}\mathsf{M}$ is a $\mathbb{R}$-algebra (using the $\mathbb{R}$-algebra structure of $C^\infty(\mathsf{M})$) and that there is a natural cor-

$$0 \longrightarrow \mathsf{S}^k(\mathsf{T}^*\mathsf{M}) \otimes \mathsf{V} \xrightarrow{\epsilon_k} \mathsf{J}^k\pi_{\mathsf{V}} \xrightarrow{(\pi_{\mathsf{V}})^k_{k-1}} \mathsf{J}^{k-1}\pi_{\mathsf{V}} \longrightarrow 0$$

Fig. 1.

$$0 \longrightarrow \mathsf{S}^k(\mathsf{T}^*_x\mathsf{M}) \otimes \mathsf{T}_y\mathsf{N} \longrightarrow \mathsf{J}^k_{(x,y)}(\mathsf{M};\mathsf{N}) \xrightarrow{\sigma^k_{k-1}} \mathsf{J}^{k-1}_{(x,y)}(\mathsf{M};\mathsf{N}) \longrightarrow 0$$

Fig. 2.

respondence (via the pull-back) between $\mathsf{J}^k_{(x_0,y_0)}(\mathsf{M};\mathsf{N})$ and the set $\mathrm{Hom}(\mathsf{T}^{*k}_{y_0}\mathsf{N};\mathsf{T}^{*k}_{x_0}\mathsf{M})$ of $\mathbb{R}$-algebra homomorphisms. Moreover, it turns out that we have an exact sequence of vector spaces as shown in Figure 2. Space does not permit an explicit description of the second arrow from the left, so we refer to [13].

We shall also briefly make use of infinite jets, and we refer to the references for these definition.

IV. A JET BUNDLE SETTING FOR STUDYING THE REACHABLE SET

In this section we illustrate how jet bundle geometry may be used to investigate controllability problems for affine systems. We let A be an affine distribution. In our definition of the reachable set for an affine system $\mathscr{A}$ in A we considered trajectories that were absolutely continuous. For the purposes of providing sufficient conditions for controllability, however, it is enough to consider trajectories that are concatenations of smooth trajectories. This follows from, for example, a theorem of Grasse [14]. Thus, to describe the reachable set from x_0, it is sufficient to consider smooth curves $\nu\colon [0,\epsilon] \to \mathsf{M}$ having the two properties

1) $\nu(0) = x_0$ and
2) for each $s \in [0,\epsilon]$, $\nu(s)$ is the endpoint of a concatenation of a finite number of smooth trajectories.

The example of Kawski [15] indicates that we have to allow for the possibility of the number of smooth curve segments to become infinite as $s \to 0$.

A. Multitrajectories and variations

Our basic construction for investigating controllability is the following. By Φ^ξ_t we denote the flow of a vector field ξ. Thus $t \mapsto \Phi^\xi_t(x_0)$ is the integral curve of ξ through x_0.

IV.1 DEFINITION: Let M be a manifold, let $x_0 \in \mathsf{M}$, and let $\boldsymbol{\xi} = (\xi_1,\dots,\xi_p) \subset \Gamma^\infty(\pi_{\mathsf{TM}})$ be such that $\boldsymbol{\xi}_j$ is complete for every $j \in \{1,\dots,p\}$.

(i) The C^∞-map

$$\Phi^{\boldsymbol{\xi}}_{x_0}\colon \mathbb{R}^p \to \mathsf{M}$$
$$(t_1,\dots,t_p) \mapsto \Phi^{\xi_1}_{t_1} \circ \dots \circ \Phi^{\xi_p}_{t_p}(x_0)$$

is the ***$\boldsymbol{\xi}$-multitrajectory***.

(ii) A ***positive p-end-time variation*** is a C^∞-map $\boldsymbol{\tau}\colon \mathbb{R}_{\geq 0} \to \mathbb{R}^p_{\geq 0}$ with the property that $\boldsymbol{\tau}(0) = \mathbf{0}_p$. The set of positive p-end-time variations is denoted by ET^+_p.

(iii) Let $\boldsymbol{\tau}$ be a positive p-end-time variation. The ***order*** of the pair $(\boldsymbol{\xi},\boldsymbol{\tau})$ at x_0, denoted $\mathrm{ord}_{x_0}(\boldsymbol{\xi},\boldsymbol{\tau})$, is the smallest positive integer k such that

$$j^k(\Phi^{\boldsymbol{\xi}}_{x_0} \circ \boldsymbol{\tau})(0) \neq 0_{x_0}$$

(derivatives are assumed to be taken from the right). If no such k exists then the order is taken to be ∞.

(iv) Let $\boldsymbol{\tau}$ be a positive p-end-time variation such that $(\boldsymbol{\xi},\boldsymbol{\tau})$ has finite order $k = \mathrm{ord}_{x_0}(\boldsymbol{\xi},\boldsymbol{\tau})$. The ***$(\boldsymbol{\xi},\boldsymbol{\tau})$-variation*** at x_0 is the curve $\nu_{\boldsymbol{\xi},\boldsymbol{\tau}}(x_0)\colon s \mapsto \Phi^{\boldsymbol{\xi}}_{x_0} \circ \boldsymbol{\tau}(s)$ and the ***$(\boldsymbol{\xi},\boldsymbol{\tau})$-infinitesimal variation*** is the tangent vector $V_{\boldsymbol{\xi},\boldsymbol{\tau}}(x_0) \in \mathsf{T}_{x_0}\mathsf{M}$ defined by

$$V_{\boldsymbol{\xi},\boldsymbol{\tau}}(x_0) = j^k\nu_{\boldsymbol{\xi},\boldsymbol{\tau}}(x_0)(0) \in \mathsf{S}^k(\mathbb{R}^*) \otimes \mathsf{T}_{x_0}\mathsf{M} \simeq \mathsf{T}_{x_0}\mathsf{M}$$

(derivatives are assumed to be taken from the right). •

The assumption of completeness of the vector fields $\boldsymbol{\xi}$ is made without loss of generality (e.g., by making all vector fields have compact support). Thus we shall not always state this condition.

Let $\boldsymbol{\xi} = (\xi_1,\dots,\xi_p) \subset \Gamma^\infty(\pi_{\mathsf{TM}})$ and let $\boldsymbol{\tau}$ be a positive p-end-time variation with $\mathrm{ord}_{x_0}(\boldsymbol{\xi},\boldsymbol{\tau}) = k$. Then $j^r(\Phi^{\boldsymbol{\xi}}_{x_0} \circ \boldsymbol{\tau})(0) = 0_{\mathsf{J}^r_{x_0}\pi_{\mathsf{M}}}$ for each $r \in \{1,\dots,k-1\}$. By the exact sequence in Figure 1 it follows that $j^k(\Phi^{\boldsymbol{\xi}}_{x_0} \circ \boldsymbol{\tau})(0) \in \mathsf{S}^k(\mathbb{R}^*) \otimes \mathsf{T}_{x_0}\mathsf{M}$. Since $\mathsf{S}^k(\mathbb{R}^*)$ is essentially the collection of $\mathbb{R}$-valued polynomial functions on $\mathbb{R}$ of homogeneous degree k, it is canonically isomorphic to $\mathbb{R}$ (by evaluating all polynomials at 1). Thus we can think of $j^k(\Phi^{\boldsymbol{\xi}}_{x_0} \circ \boldsymbol{\tau})(0)$ as being an element of $\mathsf{T}_{x_0}\mathsf{M}$, as indicated in the definition above.

The idea of these constructions is evident. First of all, if $\xi_1,\dots,\xi_p$ are $\mathscr{A}$-vector fields then clearly $\Phi^{\boldsymbol{\xi}}_{x_0}(\mathbb{R}^p_{\geq 0}) \subset \mathcal{R}_{\mathscr{A}}(x_0)$. Therefore, if $\boldsymbol{\xi} \in \mathrm{ET}^+_p$, it follows that the variation $\nu_{\boldsymbol{\xi},\boldsymbol{\tau}}(x_0)$ is a curve in $\mathcal{R}_{\mathscr{A}}(x_0)$. The infinitesimal variation $V_{\boldsymbol{\xi},\boldsymbol{\tau}}(x_0)$ is a tangent vector, possibly of higher-order, to the reachable set. The relationship with controllability is given by the following result whose proof is a standard argument in degree theory, cf. [16, Proposition 2.1].

IV.2 THEOREM: *If $\mathscr{A}$ is an affine system in a smooth affine distribution A and if there exists*

(i) families $\boldsymbol{\xi}_j = (\xi_{j1},\dots,\xi_{jp_j})$, $j \in \{1,\dots,r\}$, of smooth $\mathscr{A}$-vector fields and

(ii) positive p_j-end-time variations $\boldsymbol{\tau}_j\colon \mathbb{R}_{\geq 0} \to \mathbb{R}^{p_j}_{\geq 0}$, $j \in \{1,\dots,r\}$,

such that

$$0_{x_0} \in \mathrm{int}(\mathrm{conv}(\{V_{\boldsymbol{\xi}_1,\boldsymbol{\tau}_1},\dots,V_{\boldsymbol{\xi}_r,\boldsymbol{\tau}_r}\})),$$

then $\mathscr{A}$ is locally controllable from x_0.

B. A system-independent algebraic construction

In this section, motivated by our constructions above, we make two purely algebraic constructions involving jet bundles, and which do not depend on the system.

The first observation we make is that for $k \in \mathbb{Z}_{>0}$ we have

$$j^k(\Phi^{\boldsymbol{\xi}}_{x_0} \circ \boldsymbol{\tau})(0) = j^k\boldsymbol{\tau}(0) \circ j^k\Phi^{\boldsymbol{\xi}}_{x_0}(\mathbf{0}_p),$$

where we think of

$$j^k\boldsymbol{\tau}(0) \in \mathrm{Hom}((\mathbb{R}^p)^{*k}; (\mathbb{R})^{*k}),$$
$$j^k\Phi^{\boldsymbol{\xi}}_{x_0}(\mathbf{0}_p) \in \mathrm{Hom}(\mathsf{T}^{*k}_{x_0}\mathsf{M}; (\mathbb{R}^p)^{*k})$$

as homomorphisms of $\mathbb{R}$-algebras, and where we use the abbreviation $(\mathbb{R}^p)^{*k} = \mathsf{J}^k_{(\mathbf{0}_p,0)}(\mathbb{R}^p; \mathbb{R})$. This shows that it is important to know the character of $j^k\Phi^{\boldsymbol{\xi}}_{x_0}(\mathbf{0}_p)$. Indeed, this object, when restricted to the case when the vector fields $\boldsymbol{\xi}$ are A-valued, encodes fundamental information concerning the structure of the affine distribution. Moreover, this object has an elegant characterisation that we now present as the main construction in this paper.

Let us denote by TM^p the p-fold Whitney sum of TM with itself, and denote by $\pi^p_{\mathsf{TM}}\colon \mathsf{TM}^p \to \mathsf{M}$ the canonical projection. For a family $\boldsymbol{\xi} = (\xi_1, \dots, \xi_p)$ of C^∞-vector fields on M, let us denote by $\boldsymbol{\xi}$ the corresponding section of TM^p, accepting a convenient abuse of notation. We define a map

$$\begin{aligned} \Delta_k\colon \mathsf{V} &\to \mathsf{T}^{\leq k}(\mathsf{V}) \\ v &\mapsto v \oplus (v \otimes v) \oplus \cdots \oplus (v \otimes \cdots \otimes v). \end{aligned}$$

For $\mathbb{R}$-algebras A and B we recall that $\mathrm{Hom}(\mathsf{A}; \mathsf{B}) \subset \mathrm{L}(\mathsf{A}; \mathsf{B})$—i.e., homomorphisms of algebras are linear maps—but $\mathrm{Hom}(\mathsf{A}; \mathsf{B})$ is not a subspace in general.

IV.3 THEOREM: *For each $k, p \in \mathbb{Z}_{>0}$ there exists a unique map*

$$\mathscr{T}^k_p(x_0) \in \mathrm{L}(\mathsf{S}^{\leq k}(\mathsf{J}^{k-1}_{x_0}\pi^p_{\mathsf{TM}}); \mathrm{L}(\mathsf{T}^{*k}_{x_0}\mathsf{M}; (\mathbb{R}^p)^{*k}))$$

such that

$$\mathscr{T}^k_p(x_0)(\Delta_k(j^{k-1}\boldsymbol{\xi}(x_0))) = j^k\Phi^{\boldsymbol{\xi}}_{x_0}(\mathbf{0}_p).$$

for every family $\boldsymbol{\xi} = (\xi_1, \dots, \xi_p)$ of C^∞-vector fields. Moreover, the diagram in Figure 3 commutes, where the horizontal arrows are the canonical projections.

Proof: We sketch the important elements of the proof; space does not permit the development of the somewhat uninteresting details.

Let $\boldsymbol{\xi} = (\xi_1, \dots, \xi_p)$ be a family of C^∞-vector fields and denote by $\mathrm{BCH}_k(t_1\xi_1, \dots, t_p\xi_p)$ the vector field defined by truncating the Baker–Campbell–Hausdorff formula in indeterminates $t_1\xi_1, \dots, t_p\xi_p$ to order k. Define $\beta^k_{\boldsymbol{\xi}}\colon \mathbb{R}^p \to \mathsf{M}$ by

$$\beta^k_{\boldsymbol{\xi}}(t_1, \dots, t_p) = \Phi_1^{\mathrm{BCH}_k(t_1\xi_1, \dots, t_p\xi_p)}(x_0).$$

According to known asymptotic estimates for the Baker–Campbell–Hausdorff formula (see, e.g., [17]) we have

$$j^k\Phi^{\boldsymbol{\xi}}_{x_0}(\mathbf{0}_p) = j^k\beta^k_{\boldsymbol{\xi}}(\mathbf{0}_p).$$

Let $f \in C^\infty(\mathsf{M})$ and note that $j^kf(x_0) \in \mathsf{T}^{*k}_{x_0}\mathsf{M}$ so that, thinking of $j^k\beta^k_{\boldsymbol{\xi}}(\mathbf{0}_p)$ as an algebra homomorphism from $\mathsf{T}^{*k}_{x_0}\mathsf{M}$ to $(\mathbb{R}^p)^{*k}$,

$$j^k\beta^k_{\boldsymbol{\xi}}(\mathbf{0}_p)(j^kf(x_0)) \in (\mathbb{R}^p)^{*k}.$$

Explicitly,

$$j^k\beta^k_{\boldsymbol{\xi}}(\mathbf{0}_p)(j^kf(x_0)) = j^k(f \circ \beta^k_{\boldsymbol{\xi}})(\mathbf{0}_p).$$

Note that elements of $(\mathbb{R}^p)^{*k}$ can be thought of as polynomial functions of degree k on $\mathbb{R}^p$; we think of $j^k(f \circ \beta^k_{\boldsymbol{\xi}})(\mathbf{0}_p)$ in this way. On doing so, an application of the Chain Rule gives

$$j^k\beta^k_{\boldsymbol{\xi}}(\mathbf{0}_p)(j^kf(x_0))(\boldsymbol{v}) = \frac{\mathrm{d}(f \circ \beta^k_{\boldsymbol{\xi}})(s\boldsymbol{v})}{\mathrm{d}s}\Big|_{s=0} + \cdots + \frac{1}{k!}\frac{\mathrm{d}^k(f \circ \beta^k_{\boldsymbol{\xi}})(s\boldsymbol{v})}{\mathrm{d}s^k}\Big|_{s=0} \quad (2)$$

for $\boldsymbol{v} \in \mathbb{R}^p$.

Note that

$$\mathrm{BCH}_k(t_1\xi_1, \dots, t_p\xi_p) = \sum_{j=1}^k \sum_{\substack{(j_1,\dots,j_p)\in\mathbb{Z}^p_{\geq 0}\setminus\{\mathbf{0}_p\} \\ j_1+\cdots+j_p=j}} t_1^{j_1}\cdots t_p^{j_p} B_{j_1\cdots j_p},$$

where the vector field $B_{j_1\cdots j_p}$ is a linear combination of Lie brackets of degree $j_1 + \cdots + j_p$ of the vector fields $\boldsymbol{\xi}$. For $\boldsymbol{v} \in \mathbb{R}^p$ it follows that

$$\mathrm{BCH}_k(sv_1\xi_1, \dots, sv_p\xi_p) = s\sum_{j=1}^k s^{j-1}v_1^{j_1}\cdots v_p^{j_p}B_{j_1\cdots j_p}.$$

If we define a family of vector fields $s \mapsto \eta^k_{\boldsymbol{\xi},\boldsymbol{v}}(s)$ by

$$\eta^k_{\boldsymbol{\xi},\boldsymbol{v}}(s) = \sum_{j=1}^k s^{j-1}v_1^{j_1}\cdots v_p^{j_p}B_{j_1\cdots j_p}$$

we then have

$$\beta^k_{\boldsymbol{\xi}}(s\boldsymbol{v}) = \Phi_1^{s\eta^k_{\boldsymbol{\xi},\boldsymbol{v}}(s)}(x_0) = \Phi_s^{\eta^k_{\boldsymbol{\xi},\boldsymbol{v}}(s)}(x_0).$$

By (2) we then have

$$j^k\beta^k_{\boldsymbol{\xi}}(\mathbf{0}_p)(j^kf(x_0))(\boldsymbol{v}) = \frac{\mathrm{d}(f \circ \Phi_s^{\eta^k_{\boldsymbol{\xi},\boldsymbol{v}}(s)}(x_0))}{\mathrm{d}s}\Big|_{s=0} + \cdots + \frac{1}{k!}\frac{\mathrm{d}^k(f \circ \Phi_s^{\eta^k_{\boldsymbol{\xi},\boldsymbol{v}}(s)}(x_0))}{\mathrm{d}s^k}\Big|_{s=0}.$$

The remainder of the proof consists of carefully considering the derivatives in the preceding expression and determining the manner in which they depend on (1) the components of $\boldsymbol{v}$, (2) the derivatives of f, and (3) the derivatives of the vector fields $\boldsymbol{\xi}$.

The commutativity of the diagram in the theorem statement follows from (2) along with the analysis of the derivatives carried out in the first part of the proof. ■

IV.4 REMARKS: 1. The proof of the theorem is constructive in that it gives a means of determining a coordinate

$$\begin{array}{ccccccc}
\Delta_1(\mathsf{J}^0_{x_0}\pi^p_{\mathsf{TM}}) & \longleftarrow & \Delta_2(\mathsf{J}^1_{x_0}\pi^p_{\mathsf{TM}}) & \longleftarrow & \Delta_3(\mathsf{J}^2_{x_0}\pi^p_{\mathsf{TM}}) & \longleftarrow & \cdots \\
\Big\downarrow{\scriptstyle \mathscr{T}^1_p(x_0)} & & \Big\downarrow{\scriptstyle \mathscr{T}^2_p(x_0)} & & \Big\downarrow{\scriptstyle \mathscr{T}^3_p(x_0)} & & \\
\mathrm{Hom}(\mathsf{T}^{*1}_{x_0}\mathsf{M};(\mathbb{R}^p)^{*1}) & \longleftarrow & \mathrm{Hom}(\mathsf{T}^{*2}_{x_0}\mathsf{M};(\mathbb{R}^p)^{*2}) & \longleftarrow & \mathrm{Hom}(\mathsf{T}^{*3}_{x_0}\mathsf{M};(\mathbb{R}^p)^{*3}) & \longleftarrow & \cdots
\end{array}$$

Fig. 3.

formula for $\mathscr{T}^k_p(x_0)$ from the Campbell–Baker–Hausdorff formula.

2. Note that we are actually not interested in the behaviour of $\mathscr{T}^k_p(x_0)$ off the image of Δ_k. Thus, while $\mathscr{T}^k_p(x_0)$ is linear, we are only interested in its restriction to the algebraic variety $\mathrm{image}(\Delta_k)$. We shall use the notation $\hat{\mathscr{T}}^k_p(x_0) = \mathscr{T}^k_p(x_0) \circ \Delta_k$.
3. $\mathscr{T}^k_p(x_0)$ is system independent, depending on k, p, $\dim(\mathsf{M})$, and the Baker–Campbell–Hausdorff formula.

The diagram from the Theorem IV.3 allows us to take the limit as $k \to \infty$ in a natural (i.e., projective) way. To do this explicitly (i.e., without just writing $\mathrm{proj}\lim_{k\to\infty}$ in front of everything) requires us to introduce some notation. For $k, l \in \mathbb{Z}_{>0}$ with $l \leq k$ we have a projection

$$\Pi^k_l\colon \mathsf{S}^{\leq k}(\mathsf{J}^{k-1}_{x_0}\pi^p_{\mathsf{TM}}) \to \mathsf{S}^{\leq l}(\mathsf{J}^{l-1}_{x_0}\pi^p_{\mathsf{TM}})$$

defined by

$$\Pi^k_l(\Delta_k(j^{k-1}\boldsymbol{\xi}(x_0))) = \Delta_l(j^{l-1}\boldsymbol{\xi}(x_0)).$$

By [18, A.IV.54], the preceding equation uniquely determines the linear map Π^k_l. Let us denote by $\mathsf{S}^{\leq\infty}(\mathsf{J}^\infty_{x_0}\pi^p_{\mathsf{TM}})$ the set of maps

$$\phi\colon \mathbb{Z}_{>0} \to \cup_{k\in\mathbb{Z}_{>0}}\mathsf{S}^{\leq k}(\mathsf{J}^{k-1}_{x_0}\pi^p_{\mathsf{TM}})$$

having the properties

1) $\phi(k) \in \mathsf{S}^{\leq k}(\mathsf{J}^{k-1}_{x_0}\pi^p_{\mathsf{TM}})$, $k \in \mathbb{Z}_{>0}$, and
2) $\phi(l) = \Pi^k_l\phi(k)$ for $k, l \in \mathbb{Z}_{>0}$ satisfying $l \leq k$.

The set $\mathsf{S}^{\leq\infty}(\mathsf{J}^\infty_{x_0}\pi^p_{\mathsf{TM}})$ has the obvious $\mathbb{R}$-vector space structure defined by

$$(\phi+\psi)(k) = \phi(k)+\psi(k), \quad (a\phi)(k) = a(\phi(k)),$$

for $\phi, \psi \in \mathsf{S}^{\leq\infty}(\mathsf{J}^\infty_{x_0}\pi^p_{\mathsf{TM}})$ and $a \in \mathbb{R}$. We then have linear maps

$$\Pi^\infty_k\colon \mathsf{S}^{\leq\infty}(\mathsf{J}^\infty_{x_0}\pi^p_{\mathsf{TM}}) \to \mathsf{S}^{\leq k}(\mathsf{J}^{k-1}_{x_0}\pi^p_{\mathsf{TM}}), \qquad k \in \mathbb{Z}_{>0},$$

defined by $\Pi^\infty_k(\phi) = \phi(k)$. Let us define

$$\Delta_\infty\colon \mathsf{J}^\infty_{x_0}\pi^p_{\mathsf{TM}} \to \mathsf{S}^{\leq\infty}(\mathsf{J}^\infty_{x_0}\pi^p_{\mathsf{TM}})$$

by $\Delta_\infty(\phi)(k) = \Delta_k(\phi(k-1))$. We also define

$$\mathsf{T}^{*\infty}_{x_0}\mathsf{M} = \operatorname*{proj\,lim}_{k\to\infty}\mathsf{T}^{*k}_{x_0}\mathsf{M} = \mathsf{J}^\infty_{(x_0,0)}(\mathsf{M};\mathbb{R}),$$
$$(\mathbb{R}^p)^{*k} = \operatorname*{proj\,lim}_{k\to\infty}(\mathbb{R}^p)^{*k} = \mathsf{J}^\infty_{(\mathbf{0}_p,0)}(\mathbb{R}^p;\mathbb{R}),$$

referring to [13, Chapter 4] for these definitions.

With all of this notation we have the following result.

IV.5 PROPOSITION: *For $p \in \mathbb{Z}_{>0}$ there exists a unique map*

$$\mathscr{T}^\infty_p(x_0) \in \mathrm{L}(\mathsf{S}^{\leq\infty}(\mathsf{J}^\infty_{x_0}\pi^p_{\mathsf{TM}});\mathrm{L}(\mathsf{T}^{*\infty}_{x_0}\mathsf{M};(\mathbb{R}^p)^{*\infty}))$$

such that

$$\mathscr{T}^\infty_p(x_0)(\Delta_\infty(j^\infty\boldsymbol{\xi}(x_0))) = j^\infty\Phi^{\boldsymbol{\xi}}_{x_0}(\mathbf{0}_p)$$

for every family $\boldsymbol{\xi} = (\xi_1,\dots,\xi_p)$ of C^∞-vector fields. Moreover, for each $k \in \mathbb{Z}_{>0}$, the following diagram commutes:

$$\begin{array}{ccc}
\Delta_k(\mathsf{J}^{k-1}_{x_0}\pi^p_{\mathsf{TM}}) & \xleftarrow{\;\Pi^\infty_k\;} & \Delta_\infty(\mathsf{J}^\infty_{x_0}\pi^p_{\mathsf{TM}}) \\
\Big\downarrow{\scriptstyle \mathscr{T}^k_p(x_0)} & & \Big\downarrow{\scriptstyle \mathscr{T}^\infty_p(x_0)} \\
\mathrm{Hom}(\mathsf{T}^{*k}_{x_0}\mathsf{M};(\mathbb{R}^p)^{*k}) & \xleftarrow[\;\sigma^\infty_k\;]{} & \mathrm{Hom}(\mathsf{T}^{*\infty}_{x_0}\mathsf{M};(\mathbb{R}^p)^{*\infty})
\end{array}$$

Proof: This follows from the definitions of the objects involved, along with applications of [18, Proposition IV.2]. ■

C. Algebraic representation of infinitesimal variations

Now let us explicitly relate the map $\mathscr{T}^k_p(x_0)$ to infinitesimal variations at x_0. Given $A \in \mathsf{J}^k_{(x_0,\mathbf{0}_p)}(\mathbb{R}^p;\mathsf{M}) \simeq \mathrm{Hom}(\mathsf{T}^{*k}_{x_0}\mathsf{M};(\mathbb{R}^p)^{*k})$ we define a map

$$\begin{aligned}\Psi^k_{p,A}\colon \mathrm{Hom}((\mathbb{R}^p)^{*k};\mathbb{R}^{*k}) &\to \mathrm{Hom}(\mathsf{T}^{*k}_{x_0}\mathsf{M};\mathbb{R}^{*k}) \\ B &\mapsto B\circ A.\end{aligned}$$

We make the following definition.

IV.6 DEFINITION: For $x_0 \in \mathsf{M}$ denote

$$\mathscr{Z}^{+,k}_p(x_0) = \{A \in \mathrm{Hom}(\mathsf{T}^{*k}_{x_0}\mathsf{M};(\mathbb{R}^p)^{*k}) \mid \text{there exists } \boldsymbol{\tau} \in \mathrm{ET}^+_p \text{ such that } j^k\boldsymbol{\tau}(0) \neq 0,\ \Psi^k_{p,A}(j^k\boldsymbol{\tau}(0)) = 0\}.$$

For $A \in \mathscr{Z}^{+,k}_p(x_0)$ let us denote

$$Z^+(A) = \{\boldsymbol{\tau} \in \mathrm{ET}^+_p \mid \Psi^k_{p,A}(j^k\boldsymbol{\tau}(0)) = 0\}. \qquad \bullet$$

The subset $\mathscr{Z}^{+,k}_p(x_0)$ is an algebraic subvariety of the algebraic variety $\mathrm{Hom}(\mathsf{T}^{*k}_{x_0}\mathsf{M};(\mathbb{R}^p)^{*k})$ in the vector space $\mathrm{L}(\mathsf{T}^{*k}_{x_0}\mathsf{M};(\mathbb{R}^p)^{*k})$. Note that these varieties are canonical, as they depend only on k, p, and $\dim(\mathsf{M})$.

The following result summarises how one should interpret the maps $\mathscr{T}^k_p(x_0)$ in our context.

IV.7 PROPOSITION: *If $\boldsymbol{\xi} = (\xi_1,\dots,\xi_p)$ is a family of C^∞-vector fields on M, if $\boldsymbol{\tau}$ is a positive p-end-time variation, if $x_0 \in \mathsf{M}$, and if $k, p \in \mathbb{Z}_{>0}$, then*

$$j^k(\Phi^{\boldsymbol{\xi}}_{x_0}\circ\boldsymbol{\tau})(0) = j^k\boldsymbol{\tau}(0)\circ\hat{\mathscr{T}}^k_p(x_0)(j^{k-1}\boldsymbol{\xi}(x_0)).$$

Moreover, if $\hat{\mathscr{T}}^k_p(x_0)(j^{k-1}\boldsymbol{\xi}(x_0)) \in \mathscr{Z}^k_p(x_0)$ and if $\boldsymbol{\tau} \in Z^+(\hat{\mathscr{T}}^k_p(x_0)(j^{k-1}\boldsymbol{\xi}(x_0)))$ then $\mathrm{ord}(\boldsymbol{\xi},\boldsymbol{\tau}) \geq k$.

While the notation $j^k(\Phi^{\boldsymbol{\xi}}_{x_0}\circ\boldsymbol{\tau})(0)$ is more compact, the notation $j^k\boldsymbol{\tau}(0)\circ\hat{\mathscr{T}}^k_p(x_0)(j^{k-1}\boldsymbol{\xi}(x_0))$ better represents the structure of the problem.

D. Neutralisable families and infinitesimal variations

We now use the above definitions to characterise some properties of affine distributions. Specifically, we see that infinitesimal variations can be described using the restriction of $\mathscr{T}_p^k(x_0)$ to a subset defined by the affine system one is considering. In this way one separates the system independent from the system dependent constructions.

Let us first introduce a C^∞-affine distribution A on M. We denote by π_{A} the restriction of π_{TM} to A. Note that $\pi_{\mathsf{A}}\colon \mathsf{A} \to \mathsf{M}$ is not assumed to be a subbundle of π_{TM} since the rank of $L(\mathsf{A})$ might change locally. Nonetheless, we shall denote by $\Gamma^\infty(\pi_{\mathsf{A}})$ the set of A-valued vector fields of class C^∞. We can also unambiguously define

$$\mathsf{J}^k\pi_{\mathsf{A}} = \{j^k\xi(x) \mid \xi \in \Gamma^\infty(\pi_{\mathsf{A}}),\ x \in \mathsf{M}\} \subset \mathsf{J}^k\pi_{\mathsf{TM}}.$$

If $\mathscr{A}$ is a C^∞-affine system in A then we can also denote $\Gamma^\infty(\mathscr{A})$ as the set of C^∞-vector fields taking values in $\operatorname{image}(\mathscr{A})$, i.e., $\mathscr{A}$-vector fields that are of class C^∞. Similarly, $\mathsf{J}^k\mathscr{A}$ denotes the set of k-jets of $\mathscr{A}$-vector fields.

We also denote

$$\mathsf{A}^p = \{\xi_1(x) \oplus \cdots \oplus \xi_p(x) \in \mathsf{TM}^p \mid \xi_1(x), \dots, \xi_p(x) \in \mathsf{A}_x,\ x \in \mathsf{M}\},$$

and denote by $\pi^p_{\mathsf{A}}\colon \mathsf{A}^p \to \mathsf{M}$ the projection. We denote

$$\Gamma^\infty(\pi^p_{\mathsf{A}}) = \{(\xi_1, \dots, \xi_p) \in \Gamma^\infty(\pi^p_{\mathsf{TM}}) \mid \xi_1, \dots, \xi_p \in \Gamma^\infty(\pi_{\mathsf{A}})\}$$

and

$$\mathsf{J}^k\pi^p_{\mathsf{A}} = \{j^k\boldsymbol{\xi}(x) \mid \boldsymbol{\xi} \in \Gamma^\infty(\pi^p_{\mathsf{A}}),\ x \in \mathsf{M}\} \subset \mathsf{J}^k\pi^p_{\mathsf{TM}}.$$

For an affine system $\mathscr{A}$ in A we denote

$$\mathscr{A}^p = \{\xi_1(x) \oplus \cdots \oplus \xi_p(x) \in \mathsf{TM}^p \mid \xi_1(x), \dots, \xi_p(x) \in \mathscr{A}(x),\ x \in \mathsf{M}\},$$

and by $\Gamma^\infty(\mathscr{A}^p)$ the p-tuples $\boldsymbol{\xi} = (\xi_1, \dots, \xi_p)$ of $\mathscr{A}$-vector fields. We let $\mathsf{J}^k\mathscr{A}^p$ be the k-jets of elements of $\Gamma^\infty(\mathscr{A}^p)$.

The next step is the following idea, adapting to our setting the common notion of neutralisability encountered in the controllability literature.

IV.8 DEFINITION: Let $k \in \mathbb{Z}_{>0}$. A family $\boldsymbol{\xi} = (\xi_1, \dots, \xi_p)$ of C^∞-vector fields is ***positively neutralisable to order k*** at $x_0 \in \mathsf{M}$ if

$$\hat{\mathscr{T}}_p^k(x_0)(j^{k-1}\boldsymbol{\xi}(x_0)) \in \mathscr{Z}_p^{+,k}(x_0). \qquad \bullet$$

The condition of positive neutralisability of $\boldsymbol{\xi} = (\xi_1, \dots, \xi_p)$ to order k is a condition on the $(k-1)$-jets of the vector fields $\xi_1, \dots, \xi_p$.

Infinitesimal variations are defined by taking appropriate order jets of positively neutralisable elements. Thus the following notion is useful.

IV.9 DEFINITION: Let A be a C^∞-affine distribution on M with $\mathscr{A}$ a C^∞-affine system in A. For $k, p \in \mathbb{Z}_{>0}$ with $k \geq 2$ denote

$$\mathscr{N}_p^{+,k}(\mathscr{A}, x_0) = \{j^{k-1}\boldsymbol{\xi}(x_0) \in \mathsf{J}^{k-1}_{x_0}\mathscr{A}^p \mid \boldsymbol{\xi} = (\xi_1, \dots, \xi_p) \text{ is positively neutralisable to order } k-2 \text{ at } x_0\}. \qquad \bullet$$

Note that the condition for membership in $\mathscr{N}_p^{+,k}(\mathscr{A}, x_0)$ is a condition on $(\pi^p_{\mathsf{TM}})^{k-1}_{k-2}$ applied to the element.

Infinitesimal variations of order k associated with $\mathscr{A}$-vector fields are then essentially k-jets of elements that are neutralisable to order $k-1$. For $k = 1$ there is no neutralisation to be done, so we directly define

$$\mathscr{V}_p^{+,1}(\mathscr{A}, x_0) = \{j^1\boldsymbol{\tau}(0) \circ \hat{\mathscr{T}}_p^1(x_0)(\boldsymbol{\xi}(x_0)) \mid \boldsymbol{\xi} = (\xi_1, \dots, \xi_p) \subset \Gamma^\infty(\mathscr{A}),\ \boldsymbol{\tau} \in \mathrm{ET}_p^+\}.$$

Then, for $k \geq 2$, we denote

$$\mathscr{V}_p^{+,k}(\mathscr{A}, x_0) = \{j^k\boldsymbol{\tau}(0) \circ \hat{\mathscr{T}}_p^k(x_0)(j^{k-1}\boldsymbol{\xi}(x_0)) \mid j^{k-1}\boldsymbol{\xi}(x_0) \in \mathscr{N}_p^k(\mathscr{A}, x_0),\ \boldsymbol{\tau} \in Z^+(j^{k-1}\boldsymbol{\xi}(x_0))\}.$$

Note that

$$\mathscr{V}_p^{+,k}(\mathscr{A}, x_0) \subset \mathsf{S}^k(\mathbb{R}^*) \otimes \mathsf{T}_{x_0}\mathsf{M} \simeq \mathsf{T}_{x_0}\mathsf{M}.$$

Let us also denote

$$\mathscr{V}^{+,k}(\mathscr{A}, x_0) = \cup_{p \in \mathbb{Z}_{>0}} \mathscr{V}_p^{+,k}(\mathscr{A}, x_0).$$

These subsets of $\mathsf{T}_{x_0}\mathsf{M}$ are (possibly high-order) tangent vectors to the reachable set.

Let us record some of the more basic properties of our tangent vectors to the reachable set.

IV.10 PROPOSITION: *For a C^∞-affine distribution A on M, for a C^∞-affine system $\mathscr{A}$ in A, and for $x_0 \in \mathsf{M}$, the following statements hold:*

(i) $\mathscr{V}^{+,1}(\mathscr{A}, x_0) = \operatorname{conv}^+(\mathscr{A}(x_0))$;

(ii) $\mathscr{V}^{+,k}(\mathscr{A}, x_0)$ *is a convex cone.*

Proof: These results easily proved directly using the higher-order Chain Rule [19, Supplement 2.4A]. ■

It is very often the case that the variational cones constructed in the literature have a "nesting" property whereby cones of lower-order variations are subsets of the cones of higher-order variations, cf. [20, Proposition 2.5]. This is not quite the case for our setup since we require the composition $\Phi^{\boldsymbol{\xi}}_{x_0} \circ \boldsymbol{\tau}$ to be of class C^∞. We could relax this condition to achieve the nesting property. However, since the nesting property is not necessary for controllability, we elect not to do this. Nonetheless, we have the following result which essentially captures the desired behaviour.

IV.11 PROPOSITION: *Let A be a C^∞-affine distribution on M, let $\mathscr{A}$ be a C^∞-affine system in A, let $\boldsymbol{\xi} = (\xi_1, \dots, \xi_p)$ be $\mathscr{A}$-vector fields, let $\boldsymbol{\tau}$ be a positive p-end-time variation, and let $x_0 \in \mathsf{M}$. If $\operatorname{ord}_{x_0}(\boldsymbol{\xi}, \boldsymbol{\tau}) = l$ and if $k \in \mathbb{Z}_{>0}$ satisfies $k \geq l$, then there exists a curve $s \mapsto \sigma(s) \in \mathsf{M}$ having the following properties:*

(i) σ *is of class* C^k;

(ii) $\operatorname{image}(\sigma) \subset \operatorname{image}(\Phi^{\boldsymbol{\xi}}_{x_0})$;

(iii) $j^r\sigma(0) = 0$ *for* $r \in \{1, \dots, k-1\}$;

(iv) $j^k\sigma(0) = V_{\boldsymbol{\xi},\boldsymbol{\tau}}(x_0)$.

Proof: By choosing a coordinate chart about x_0, let us suppose that M is a neighbourhood of $\mathbf{0}_n \in \mathbb{R}^n$ and that $x_0 = \mathbf{0}_n$. Our conclusions do not depend on the chart chosen.

Since the curve $s \mapsto \Phi^{\boldsymbol{\xi}}_{x_0} \circ \boldsymbol{\tau}(s)$ is of class C^∞ and its first l derivatives vanish, we have

$$\Phi^{\boldsymbol{\xi}}_{x_0} \circ \boldsymbol{\tau}(s) = \frac{s^l}{l!} V_{\boldsymbol{\xi},\boldsymbol{\tau}}(x_0) + o(s^l),$$

Now define

$$\sigma(s) = \Phi^{\boldsymbol{\xi}}_{x_0} \circ \boldsymbol{\tau}\Big(\frac{s^{k/l}}{(k!/l!)^{1/l}}\Big).$$

Note that

$$\begin{aligned}\sigma(s) &= \frac{1}{l!}\Big(\frac{s^{k/l}}{(k!/l!)^{1/l}}\Big)^l V_{\boldsymbol{\xi},\boldsymbol{\tau}}(x_0) + o\Big(\Big(\frac{s^{k/l}}{(k!/l!)^{1/l}}\Big)^l\Big) \\ &= \frac{s^k}{k!} V_{\boldsymbol{\xi},\boldsymbol{\tau}}(x_0) + o(s^k),\end{aligned}$$

giving $j^r\sigma(0) = 0$ for $r \in \{1, \dots, k-1\}$ and $j^k\sigma(0) = V_{\boldsymbol{\xi},\boldsymbol{\tau}}(x_0)$. It is also evident that $\operatorname{image}(\sigma) \subset \operatorname{image}(\Phi^{\boldsymbol{\xi}}_{x_0})$. ■

V. Future work

It is our belief that the approach to local controllability initiated in this paper will offer new insights into the mechanisms of local controllability, mainly by virtue of its emphasis on feedback-invariance. However, this work is in its very early stages, and so the nature of the outcome of any investigations into this approach are very much not clear at present. In this section we present some questions and directions that we speculate might, at least in the short term, clarify where our ideas may lead.

1. One should first understand better the details of the structure of the maps $\mathscr{T}^k_p(x_0)$. These maps should have some basic algebro-geometric structure that could help to better understand the structure of infinitesimal variations. In particular, by better understanding the structure of these maps, it should be possible to arrive at simple and geometrically attractive sufficient conditions for local controllability.
2. As mentioned in the introduction, many existing sufficient or necessary conditions for local controllability rely on characterisations involving Lie brackets of a fixed set of generators for the affine distribution under consideration. The resulting conditions, then, are typically not feedback-invariant, and this contributes to the gap between sufficient and necessary conditions for local controllability being "fuzzy." It might be useful to use our feedback-invariant approach here to clarify this gap, and possibly narrow it.
3. Our formulation of the structure of an affine distribution makes explicit use of jets as homomorphisms of algebras of $\mathbb{R}$-valued functions. This seems to have much in common with series expansions such as the Chen–Fliess–Sussmann series. It would seem to be a valuable exercise to explore this is a systematic way.
4. Of course, one of the important ingredients missing from this paper is any sort of example illustrating our ideas and how to use them. It will certainly be the case that well-chosen examples will do a great deal to illuminate what we are doing. To this end, it is extremely desirable to have developed symbolic software for doing computations in this framework. Indeed, Baker–Campbell–Hausdorff computations such as are embedded in the computation of our map $\mathscr{T}^k_p(x_0)$ quickly become unmanageable if done "by hand."

References

[1] R. E. Kalman, Y.-C. Ho, and K. S. Narendra, "Controllability of linear dynamical systems", *Contrib. Diff. Eqs.*, vol. 1, pp. 189–213, 1963.
[2] H. J. Sussmann and V. Jurdjevic, "Controllability of nonlinear systems", *J. Differential Equations*, vol. 12, pp. 95–116, 1972.
[3] A. J. Krener, "A generalization of Chow's theorem and the bang-bang theorem to non-linear control problems", *J. Soc. Indust. Appl. Math. Ser. A Control*, vol. 12, pp. 43–52, 1974.
[4] T. Nagano, "Linear differential systems with singularities and an application to transitive Lie algebras", *J. Math. Soc. Japan*, vol. 18, pp. 398–404, 1966.
[5] H. J. Sussmann, "An extension of a theorem of Nagano on transitive Lie algebras", *Proc. Amer. Math. Soc.*, vol. 45, no. 3, pp. 349–356, 1974.
[6] E. D. Sontag, "Controllability is harder to decide than accessibility", *SIAM J. Control Optim.*, vol. 26, no. 5, pp. 1106–1118, 1988.
[7] M. Kawski, "The complexity of deciding controllability", *Systems Control Lett.*, vol. 15, no. 1, pp. 9–14, 1990.
[8] H. J. Sussmann, "A general theorem on local controllability", *SIAM J. Control Optim.*, vol. 25, no. 1, pp. 158–194, 1987.
[9] V. I. Elkin, "Affine control systems: Their equivalence, classification, quotient systems, and subsystems", *J. Math. Sci. (New York)*, vol. 88, no. 5, pp. 675–721, 1998.
[10] F. Bullo and A. D. Lewis, "Low-order controllability and kinematic reductions for affine connection control systems", *SIAM J. Control Optim.*, vol. 44, no. 3, pp. 885–908, 2005.
[11] J. Basto-Gonçalves, "Second-order conditions for local controllability", *Systems Control Lett.*, vol. 35, no. 5, pp. 287–290, 1998.
[12] D. J. Saunders, *The Geometry of Jet Bundles*, Number 142 in London Mathematical Society Lecture Note Series. Cambridge University Press, New York/Port Chester/Melbourne/Sydney, 1989.
[13] I. Kolář, P. W. Michor, and J. Slovák, *Natural Operations in Differential Geometry*, Springer-Verlag, New York-Heidelberg-Berlin, 1993.
[14] K. A. Grasse, "On the relation between small-time local controllability and normal self-reachability", *Math. Control Signals Systems*, vol. 5, no. 1, pp. 41–66, 1992.
[15] M. Kawski, "High-order small-time local controllability", in *Nonlinear Controllability and Optimal Control*, vol. 133 of *Monographs and Textbooks in Pure and Applied Mathematics*, pp. 431–467. Dekker Marcel Dekker, New York, 1990.
[16] R. M. Bianchini, "Variations of a control process at the initial point", *J. Optim. Theory Appl.*, vol. 81, no. 2, pp. 249–258, 1994.
[17] R. S. Strichartz, "The Campbell–Baker–Hausdorff–Dynkin formula and solutions of differential equations", *J. Funct. Anal.*, vol. 72, pp. 320–345, 1987.
[18] N. Bourbaki, *Algebra II*, Elements of Mathematics. Springer-Verlag, New York-Heidelberg-Berlin, 1990.
[19] R. Abraham, J. E. Marsden, and T. S. Ratiu, *Manifolds, Tensor Analysis, and Applications*, Number 75 in Applied Mathematical Sciences. Springer-Verlag, 2 edition, 1988.
[20] R. M. Bianchini and G. Stefani, "Controllability along a trajectory: A variational approach", *SIAM J. Control Optim.*, vol. 31, no. 4, pp. 900–927, 1993.