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# Stability for Semidiscrete Galerkin Approximation of Neutral Delay Equations

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***Abstract*— We consider the issue of stability for semidiscrete Galerkin approximations of neutral delay-differential equations. We recall recent results which show how a renorming of the energy state space can be used to obtain a dissipative inequality which implies exponential stability of the solution semigroup associated with the delay differential equation. We then show in detail how the norm is used to construct finite dimensional semidiscrete Galerkin approximations which preserve the stability behavior of the original neutral equation. In applications to optimal control problems, it is important that semidiscrete approximation schemes have this property.**

## I. INTRODUCTION

Consider the following system of neutral delay-differential equations

$$\begin{aligned} \frac{d}{dt}[x(t)+Cx(t-r)] &= Ax(t)+Bx(t-r), \\ x(0)+Cx(-r) &= \eta_0, \\ x(\theta)=\phi_0(\theta), \qquad & -r\le\theta<0, \end{aligned} \tag{1}$$

where $0<r$, $\eta_0\in\mathbb{C}^n$, $\phi_0\in L_2(-r,0;\mathbb{C}^n)$, and $A$, $B$, and $C$ are $n\times n$ matrices with complex entries. These equations arise in many applications (see the examples in [1]). In the last several decades there has been ongoing research into the question of sufficient conditions (specifically, conditions on the matrices $A$, $B$, and $C$ and sometimes on the delay $r$) for exponential stability of the solution semigroup for the system (1). A related question of interest is - if the solution semigroup associated with (1) is exponentially stable, do the solution semigroups of finite dimensional semidiscrete approximations of (1) preserve the stability behavior uniformly in the discretization parameter? We shall investigate this question for a linear spline-based Galerkin approximation scheme.

The preservation of stability under approximation is an important issue for applications in control and optimization of dynamics governed by infinite dimensional evolution equations in general, including delay equations, partial differential equations, etc. We refer to [2], [3], [4], and [5] for a discussion of this issue in general, and [6], [7], [8], and [9] for discussion of this issue as it relates to delay equations.

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## II. PRELIMINARIES

Roughly speaking, the analysis of the stability of (1) involves either using a Liapunov-type function, or a direct analysis of the associated characteristic equation

$$\Delta(\lambda)=\det(A-\lambda I+Be^{-\lambda r}-\lambda Ce^{-\lambda r})=0. \tag{2}$$

It is known that exponential stability of (1) is equivalent to the condition that

$$\sup\{\mathrm{Re}\,\lambda:\ \Delta(\lambda)=0\}=\omega<0, \tag{3}$$

and it is the difficulty of determining the location of the roots of $\Delta(\lambda)$ which makes this an interesting problem. Recently in [10] the authors have taken the approach of renorming the underlying state space so as to obtain a dissipative inequality which implies the desired exponential stability. We shall briefly describe this approach, and then in the next sections discuss how the new norm may be used to construct a spline-based, finite dimensional semidiscrete Galerkin approximation scheme.

In a standard fashion (1) can be reformulated as an abstract Cauchy problem on a Hilbert space. In particular, define the Hilbert space $Z=\mathbb{C}^n\times L^2(-r,0;\mathbb{C}^n)$ endowed with the norm

$$\|(\eta,\phi)\|_Z^2=\|\eta\|_n^2+\int_{-r}^0\|\phi(\theta)\|_n^2\,d\theta, \tag{4}$$

and compatible inner product

$$\langle(\eta,\phi),(\xi,\psi)\rangle_Z=\eta^T\overline{\xi}+\int_{-r}^0\phi(\theta)^T\overline{\psi(\theta)}\,d\theta. \tag{5}$$

Here $\|\cdot\|_n$ is the standard Euclidean norm on $\mathbb{C}^n$. Next define the linear operator $\mathcal{A}:\operatorname{dom}\mathcal{A}\subset Z\to Z$ on the domain

$$\begin{aligned}\operatorname{dom}\mathcal{A}=\{(\eta,\phi)\in Z:\ &\phi\in H^1(-r,0),\\ &\eta=\phi(0)+C\phi(-r)\},\end{aligned}$$

by

$$\mathcal{A}(\eta,\phi)=(A\phi(0)+B\phi(-r),\phi'). \tag{6}$$

It is well known that $\mathcal{A}$ is the infinitesimal generator of a strongly continuous semigroup $T(t)$ on $Z$, and if we make the identification $z(t) = (x(t) + Cx(t - r), x(t+\theta))$ then as introduced in [11] equation (1) can be reformulated as the Cauchy problem

$$\begin{aligned} \frac{d}{dt}z(t) &= \mathcal{A}z(t), \\ z(0) &= (\eta_0, \phi_0), \end{aligned} \tag{7}$$

on $Z$.

The following stability result has recently been obtained in [12] (for the definition of matrix measure, see [13]).

*Theorem 1: Consider the neutral system (1). Define the matrices $G = -(A + A^T)/2$ and $H = G + A = (A - A^T)/2$. If the matrix measure of $A$ satisfies $\mu(A) < 0$, and if*

$$\begin{aligned} &G - C^T GC - \frac{1}{k}C^T H^T HC \\ &\quad - \frac{1}{|\mu(A)| - k}B^T B > 0 \end{aligned} \tag{8}$$

*for some constant $0 < k < |\mu(A)|$, then the semigroup $T(t)$ associated with (7) is exponentially stable.*

In the proof of this theorem a new norm on $Z$ is constructed, which is equivalent to the original norm. Observe that $G$ is a positive-definite, self-adjoint matrix, so one can define a norm on $Z$ by

$$\|(\eta, \phi)\|_e^2 = \eta^T \overline{\eta} + \int_{-r}^{0} e^{-2\gamma\theta}\phi(\theta)^T G\overline{\phi(\theta)}\, d\theta$$

for all $(\eta, \phi) \in Z$, with a compatible inner product given by

$$\langle(\eta, \phi), (\xi, \psi)\rangle_e = \xi^T \overline{\eta} + \int_{-r}^{0} e^{-2\gamma\theta}\psi(\theta)^T G\overline{\phi(\theta)}\, d\theta,$$

for all $(\eta, \phi), (\xi, \psi) \in Z$. It is shown that with this norm one obtains the dissipative inequality

$$\mathrm{Re}\, \langle \mathcal{A}z, z\rangle_e \leq \alpha\, \|z\|_e^2 \tag{9}$$

for all $z \in \mathrm{dom}\, \mathcal{A}$, where $\alpha < 0$. Our main result in this paper will be to show that a similar inequality can be obtained, uniformly in the discretization parameter, for a convergent, finite dimensional, linear spline-based Galerkin approximation scheme for (1).

## III. Stability and Approximation for Delay Equations

A typical semi discrete approximation scheme for a Cauchy problem like (7) consists of a sequence $\{\mathcal{A}^N, Z^N\}_{N=0}^{\infty}$ of finite dimensional subspaces $Z^N \subset Z$ and operators $\mathcal{A}^N : Z^N \to Z^N$. The operators $\mathcal{A}^N$ define semigroups $T^N(t) = e^{t\mathcal{A}^N}$ on $Z^N$, and the subspaces $Z^N$ define orthogonal projections $P^N : Z \to Z^N$. Such an approximation scheme defines a finite dimensional Cauchy problem

$$\begin{aligned} \frac{d}{dt}z^N(t) &= \mathcal{A}^N z^N(t), \\ z^N(0) &= P^N z_0, \end{aligned} \tag{10}$$

on $Z^N$. A typical convergence result involves showing that $P^N \to I$ strongly and that $T^N(t)P^N \to T(t)$ in the Trotter-Kato sense. Such a convergence result justifies using (10) to approximate the dynamics of (7), and being finite dimensional, (10) can be solved on the computer. However for certain applications in optimization and optimal control, it is known that the approximation scheme should also have the property that the approximating semigroups $T^N(t)$ for (10) preserve the stability behavior of the semigroup $T(t)$ for (7). The so-called 'averaging' scheme (a finite difference scheme with characteristic functions as basis functions) is one popular approximation scheme for which convergence has been shown for both retarded [14] and neutral delay equations [15]. For retarded equations it is known that the approximating semigroups $T^N(t)$ preserve the stability behavior of the semigroup $T(t)$, uniformly in the discretization parameter $N$ (see [16]), and recently in [17] the present authors showed that this is also true for the neutral equations. Here we would like to obtain a similar result for spline-based approximations. An early use of spline-based approximation for a retarded delay equation is found in [18]. It was observed numerically in [6] that the scheme did not provide convergence of the approximate adjoint semigroups (which is desirable particularly in optimal control problems), and an improved scheme was constructed in [8]. The issue of semigroup convergence for spline-based approximations of neutral equations has been dealt with in [19], but we are unaware of any preservation of stability results for neutral equations (certainly for retarded equations this has been fairly well studied).

Thus in the remainder of the paper we will do the following:

1) define an approximation scheme $\{\mathcal{A}^N, Z^N\}_{N=0}^{\infty}$ in which the finite dimensional spaces $Z^N$ are constructed using linear splines,
2) prove the semigroup convergence $T^N(t)P^N \to T(t)$,
3) verify that the operators $\mathcal{A}^N$ satisfy a dissipative inequality similar to (9) uniformly in the discretization parameter $N$.

We shall make use of the following Trotter-Kato semigroup convergence result.

*Theorem 2: Suppose that $V$ and $H$ are Hilbert* *spaces, with $V$ densely and continuously embedded in $Z$, and suppose that $\mathcal{A}$ : dom $\mathcal{A} \subset V \subset Z \to Z$ is the infinitesimal generator of a $C_0$-semigroup $T(t)$ on $Z$. Further assume that there is a sesquilinear form $\sigma : V \times V \to \mathbb{C}$ and a fixed $\alpha \in \mathbb{R}$ satisfying*

$$\sigma(u,v) = \langle \mathcal{A}u, v\rangle_Z \quad \forall u \in \mathrm{dom}\mathcal{A}, v \in V \tag{11}$$

*and*

$$\mathrm{Re}\,\sigma(u,u) \le \alpha \|u\|_Z^2 \quad \forall u \in V. \tag{12}$$

*Let $\{Z^N\}_{N=1}^{\infty}$ be a sequence of finite dimensional subspaces of $V$, and let $P^N$ denote the orthogonal projection of $Z$ onto $Z^N$. For each $N$ define the operator $\mathcal{A}^N : Z^N \to Z^N$ by*

$$\langle \mathcal{A}^N u, v\rangle_Z = \sigma(u,v) \qquad \forall u, v \in Z^N. \tag{13}$$

*If there exists $s \ge 1$ and $L > 0$ such that for all $v \in \mathrm{dom}\,\mathcal{A}^s$, there exists $v^N \in Z^N$ satisfying*

C1 $|\sigma(u, v - v^N)| \le L\,\|u\|_Z\,\|v - v^N\|_V$ *for all $u \in V$, and*

C2 $\|v - v^N\|_V \to 0$ *as $N \to \infty$,*

*then $T^N(t)P^N \to T(t)$ strongly on $Z$.*

This useful convergence theorem, which was applied to retarded delay equations in [20], is similar to those found in [21], [22].

## IV. Linear Spline Approximation for a Neutral Delay Equation

In this section we discuss in detail a linear spline-based approximation scheme which is both convergent (in the Trotter-Kato sense) and stability preserving. For clarity of the presentation we describe the construction for the case of a scalar neutral equation, and note that similar results can be obtained for a system of neutral equations by using a similar construction. Therefore in (1) we assume that $\eta_0 \in \mathbb{C}$, $\phi_0 \in L_2(-r, 0; \mathbb{C})$, and $A$, $B$, and $C$ are complex scalars. We assume that the assumptions of Theorem 1 hold. In the scalar case, this means we assume that $A < 0$, $|C| < 1$, and there exists a constant $k$ such that $0 < k < |A|$ and

$$|A| - |A|\,|C|^2 - \frac{1}{|A| - k}|B|^2 > 0. \tag{14}$$

Note that in the scalar case $\mu(A) = A$. Equation (14) implies that there exists $\gamma < 0$ such that

$$|A|e^{2\gamma r} - |A|\,|C|^2 - \frac{1}{|A| - k}|B|^2 = 0. \tag{15}$$

Now we endow the state space $Z$ with the norm

$$\|(\eta, \phi\|_e^2 = |\eta|^2 + \int_{-r}^{0} g(\theta)\,|\phi(\theta)|^2\,d\theta, \tag{16}$$

where $g(\theta) = |A|\,e^{-2\gamma\theta}$. This is equivalent to the norm $\|\cdot\|_Z$, and has a compatible inner product

$$\langle(\eta,\phi),(\xi,\psi)\rangle_e = \eta\overline{\xi} + \int_{-r}^{0} g(\theta)\phi(\theta)\overline{\psi(\theta)}\,d\theta.$$

Next define the Hilbert space $V = \mathbb{C} \times H^1(-r, 0)$ with the usual norm $\|(\eta,\phi)\|_V^2 = |\eta|^2 + \|\phi\|_{H^1}^2$. For $u = (\eta,\phi), v = (\xi,\psi)$ define the sesquilinear form $\sigma : V \times V \to \mathbb{C}$ by

$$\begin{aligned}\sigma(u,v) = {}& [A(\eta - C\phi(-r)) + B\phi(-r)]\overline{\xi} \\ &+ \int_{-r}^{0} g(\theta)\,\phi'(\theta)\,\overline{\psi(\theta)}\,d\theta \\ &+ |A|[\eta - \phi(0) - C\phi(-r)]\overline{\psi(0)}.\end{aligned}$$

It is straightforward to check that

$$\sigma(u,v) = \langle \mathcal{A}u, v\rangle_e$$

for all $u \in \mathrm{dom}\mathcal{A}$, $v \in V$, so (11) holds.

To see that (12) holds, observe that for $u = (\eta,\phi) \in V$,

$$\begin{aligned}\sigma(u,u) = {}& [A(\eta - C\phi(-r)] + B\phi(-r)]\overline{\eta} \\ &+ \int_{-r}^{0} g(\theta)\,\phi'(\theta)\,\overline{\phi(\theta)}\,d\theta \\ &+ |A|[\eta - \phi(0) - C\phi(-r)]\overline{\phi(0)}.\end{aligned}$$

Thus

$$\begin{aligned}\mathrm{Re}\,\sigma(u,u) = {}& A|\eta|^2 + \mathrm{Re}(B - AC)\phi(-r)\overline{\eta} \\ &+ \frac{1}{2}g(0)|\phi(0)|^2 - \frac{1}{2}g(-r)|\phi(-r)|^2 \\ &+ \gamma\int_{-r}^{0} g(\theta)|\phi(\theta)|^2\,d\theta - |A||\phi(0)|^2 \\ &+ |A|\,\mathrm{Re}\,[\eta - C\phi(-r)]\overline{\phi(0)}\end{aligned}$$

where we used that $-\frac{1}{2}g'(\theta) = \gamma g(\theta)$. Since $A = -|A|$, $g(0) = |A|$ and $g(-r) = |A|\,e^{2\gamma r}$, we may continue and get

$$\begin{aligned}\mathrm{Re}\,\sigma(u,u) = {}& -|A|\,|\eta|^2 + \mathrm{Re}(B + |A|C)\phi(-r)\overline{\eta} \\ &- \frac{1}{2}|A|\,|\phi(0)|^2 - \frac{1}{2}|A|\,e^{2\gamma r}|\phi(-r)|^2 \\ &+ \gamma\int_{-r}^{0} g(\theta)|\phi(\theta)|^2\,d\theta \\ &+ |A|\,\mathrm{Re}\,[\eta - C\phi(-r)]\overline{\phi(0)}.\end{aligned}$$

Now use the Cauchy-Schwarz inequality

$$\mathrm{Re}\,[\eta - C\phi(-r)]\overline{\phi(0)} \le \frac{1}{2}|\eta - C\phi(-r)|^2 + \frac{1}{2}|\phi(0)|^2$$

to get

$$\begin{aligned}\operatorname{Re}\sigma(u,u) \leq & -|A|\,|\eta|^2 + \operatorname{Re}(B+|A|C)\phi(-r)\overline{\eta} \\ & -\frac{1}{2}|A|\,e^{2\gamma r}|\phi(-r)|^2 \\ & +\gamma\int_{-r}^{0} g(\theta)|\phi(\theta)|^2\,d\theta \\ & +\frac{1}{2}|A|\,|\eta - C\phi(-r)|^2.\end{aligned}$$

Next use the fact that

$$|\eta - C\phi(-r)|^2 = |\eta|^2 + |C|^2\,|\phi(-r)|^2 - 2\operatorname{Re} C\phi(-r)\overline{\eta}$$

to get

$$\begin{aligned}\operatorname{Re}\sigma(u,u) \leq & -\frac{1}{2}|A|\,|\eta|^2 + \operatorname{Re} B\phi(-r)\overline{\eta} \\ & -\frac{1}{2}|A|\,(e^{2\gamma r} - |C|^2)|\phi(-r)|^2 \\ & +\gamma\int_{-r}^{0} g(\theta)|\phi(\theta)|^2\,d\theta.\end{aligned}$$

Finally, use the Cauchy-Schwarz inequality

$$\operatorname{Re} B\phi(-r)\overline{\eta} \leq \frac{\epsilon}{2}\,|A|\,|\eta|^2 + \frac{1}{2\epsilon}\frac{1}{|A|}|B|^2|\phi(-r)|^2$$

with $\epsilon = (|A|-k)/|A|$ to get

$$\begin{aligned}\operatorname{Re}\sigma(u,u) \leq & -\frac{k}{2}\,|\eta|^2 + \gamma\int_{-r}^{0} g(\theta)|\phi(\theta)|^2\,d\theta \\ & -\frac{1}{2}\left(|A|\,e^{2\gamma r} - |A|\,|C|^2 - \frac{|B|^2}{|A|-k}\right) \\ \leq & \ \alpha\,\|u\|_e^2,\end{aligned}$$

where $\alpha = \max\{-k/2, \gamma\} < 0$. Thus (12) is true, and since $\alpha < 0$, it follows from (11) that the semigroup $T(t)$ generated by $\mathcal{A}$ is exponentially stable with a decay rate $\alpha$. Furthermore, once we construct the spaces $Z^N$ and use (13) to define $\mathcal{A}^N$, it will similarly follow that the semigroups $T^N(t)$ are exponentially stable, with a decay rate $\alpha$ as well, uniformly in $N$. All that will then remain will be to verify (C1) and (C2).

To proceed, let us now construct the linear spline-based spaces $Z^N$. For each $N$ define a partition of $[-r,0]$ by $\theta_j^N = -jr/N$ for $j = 0,1,\ldots,N$. Define the piecewise linear functions (the so-called 'hat' functions)

$$e_0^N(\theta) = \begin{cases} \frac{N}{r}(\theta - \theta_1^N) & \text{if } \theta_1^N \leq \theta \leq 0 \\ 0 & \text{elsewhere} \end{cases}$$

$$e_N^N(\theta) = \begin{cases} -\frac{N}{r}(\theta - \theta_{N-1}^N) & \text{if } \theta_N^N \leq \theta \leq \theta_{N-1}^N \\ 0 & \text{elsewhere} \end{cases}$$

and, for $j = 1,2,\ldots,N-1$,

$$e_j^N(\theta) = \begin{cases} -\frac{N}{r}(\theta - \theta_{j-1}^N) & \text{if } \theta_j^N \leq \theta \leq \theta_{j-1}^N \\ \frac{N}{r}(\theta - \theta_{j+1}^N) & \text{if } \theta_{j+1}^N \leq \theta \leq \theta_j^N \\ 0 & \text{elsewhere.} \end{cases}$$

Now set

$$\begin{aligned} B_0^N &= (1,0) \\ B_j^N &= (0, e_{j-1}^N) \quad j = 1,2,\ldots,N+1, \end{aligned}$$

and define $Z^N = \operatorname{span}\{B_j^N\}_{j=0}^{N+1}$. Thus $Z^N$ is a finite dimensional subspace of $V$. For each $N$ we use (13) to define $\mathcal{A}^N : Z^N \to Z^N$. Notice that these basis functions and spaces $Z^N$ are the same as those used in [8] for a retarded delay equation. What is new here is not just that we are considering a neutral instead of a retarded delay equation, but that we are constructing the operators $\mathcal{A}^N$ via Galerkin projections in our new norm. A significant reason for doing this is that Galerkin projections in the original norm lead to approximating operators which fail to preserve the stability of the original system, which has been well documented for retarded delay equations.

It remains for us to verify conditions (C1) and (C2). For $\psi \in H^3(-r,0)$, let $\psi_I^N(\theta)$ be the linear spline which interpolates $\psi$. Thus $\psi_I^N(\theta)$ is continuous, piecewise linear, and takes the values $\psi_I^N(\theta_j^N) = \psi(\theta_j^N)$ for $j = 0,1,\ldots,N$. The interpolating spline has the property that

$$\|\psi - \psi_I^N\|_{L^2(-r,0)} \leq \mathcal{O}(\frac{1}{N^2})$$

and

$$\|\frac{d}{d\theta}(\psi - \psi_I^N)\|_{L^2(-r,0)} \leq \mathcal{O}(\frac{1}{N})$$

(see [23]). Thus we take $s = 3$ in Theorem 2 and observe that if

$$v = (\psi(0) + C\psi(-r), \psi) \in \operatorname{dom}\mathcal{A}^3,$$

then $\psi \in H^3(-r,0)$ and we can define

$$v^N = (\psi_I^N(0) + C\psi_I^N(-r), \psi_I^N) \in Z^N.$$

Notice that $v - v^N = (0, \psi - \psi_I^N)$, which implies that

$$\|v - v^N\|_V^2 = \|\psi - \psi_I^N\|_{H^1(-r,0)}^2 \to 0$$

as $N \to \infty$, so (C2) holds. Also

$$(\psi - \psi_I^N)(0) = (\psi - \psi_I^N)(-r) = 0,$$

so for any $u = (\eta, \phi) \in V$ we have

$$\begin{aligned} |\sigma(u, v - v^N)| &= |\sigma((\eta,\phi),(0,\psi - \psi_I^N))| \\ &= \left| \int_{-r}^{0} g(\theta)\phi'(\theta)[\overline{\psi(\theta)} - \overline{\psi_I^N(\theta)}]\, d\theta \right| \\ &= \left| \int_{-r}^{0} \phi(\theta) \frac{d}{d\theta}(g(\theta)[\overline{\psi(\theta)} - \overline{\psi_I^N(\theta)}])\, d\theta \right| \\ &\le K \|\phi\|_{L^2(-r,0)} \, \|\psi - \psi_I^N\|_{H^1(-r,0)} \\ &\le L\, \|u\|_e \, \|v - v^N\|_V . \end{aligned}$$

Thus (C1) holds, and we obtain the desired Trotter-Kato convergence for this approximation scheme.

## V. EXAMPLE

In this section we illustrate the theory in an example. Consider the scalar neutral delay-differential equation

$$\frac{d}{dt}[x(t) + Cx(t-r)] = Ax(t) + Bx(t-r), \qquad (17)$$

with appropriate initial conditions (the initial conditions are not relevant to the construction of the operators $\mathcal{A}$ and $\mathcal{A}^N$, which is all that we consider here. They would become relevant if we were intent on solving the Cauchy problems (10) forward in time.) Fix the following parameter values:

$$A = -2, \quad B = 1/2, \quad C = 3/4, \quad r = 1. \qquad (18)$$

The theory in Section IV is applied to construct finite dimensional operators $\mathcal{A}^N$, where $N$ is the discretization parameter related to the number of linear spline basis functions used. This discretization is based upon the use of the new norm defined in (16). We note that in the scalar case under consideration here, it is possible to determine quite accurately the optimal decay rate $\alpha$ as follows (it is more difficult to make this calculation for systems of delay-differential equations). Since $-k/2$ is a decreasing function of $k$, and from (15) $\gamma$ is an increasing function of $k$, and $\alpha = \max\{-k/2, \gamma\}$, then the optimal value of the decay rate $\alpha$ occurs when $\alpha = \gamma = -k/2$. From (15), the value of $k$ which yields this optimal rate is the root of

$$e^{-rk} = |C|^2 + \frac{|B|^2}{|A|^2 - k|A|}.$$

For our choice of parameters this gives

$$\alpha = -0.22101.$$

It is also possible to construct finite dimensional operators $\mathcal{A}^N$ using the same basis functions but with the Galerkin projections in the original energy norm. In this case there would not be a guaranteed decay rate (even though there would be Trotter-Kato convergence). In Figure 1 and Figure 2 we compare the eigenvalues of the operators $\mathcal{A}^N$ arising from the

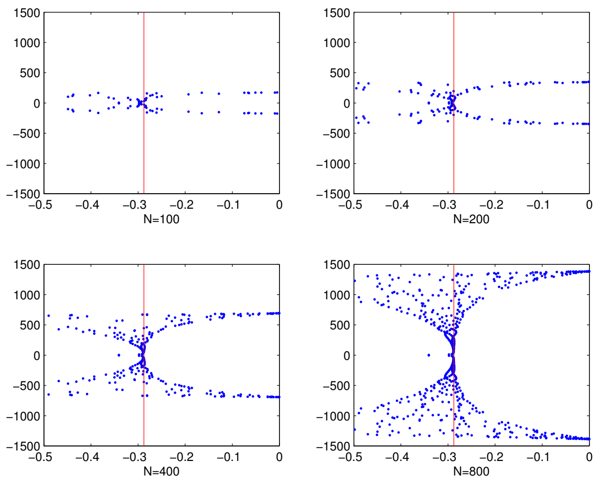


Fig. 1. Eigenvalues of $\mathcal{A}^N$, energy norm

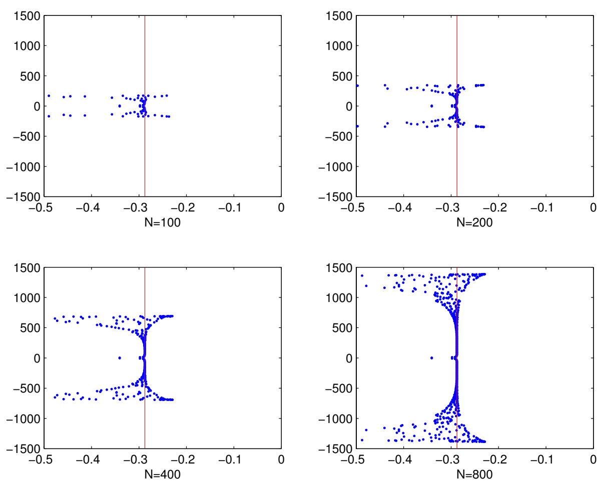


Fig. 2. Eigenvalues of $\mathcal{A}^N$, new norm

two different Galerkin constructions, for values $N = 100, 200, 400, 800$. The figures illustrate that only the discretization using the new norm maintains a uniform stability behavior, justifying the theory in the paper.

We note that for such neutral delay-differential equations there will exist a so-called 'neutral chain' of eigenvalues of the operator $\mathcal{A}$. This is a sequence of eigenvalues asymptotic to the vertical line located at $\frac{1}{r}\log|C| = -.28768$ on the real axis. We have included this line in the figures, and clearly both discretization schemes 'capture' this behavior. This capturing of the neutral chain behavior is better seen in Figure 3 and Figure 4, in which we repeat the plots for $N = 800$ but with a different scale on the real axis. (We note that due to scaling, some eigenvalues have been left off all of the plots).

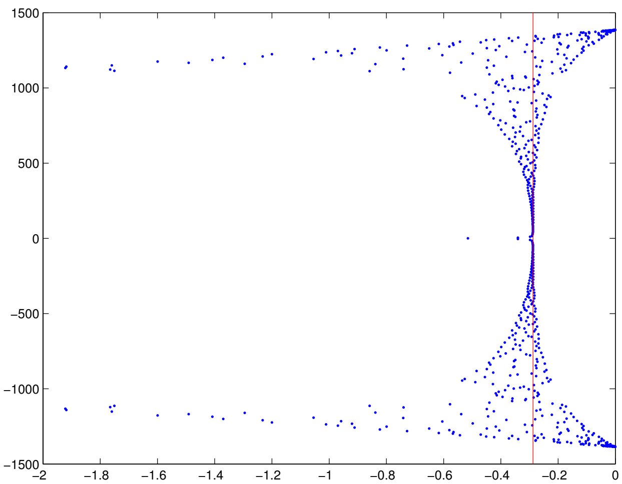


Fig. 3. Eigenvalues of $\mathcal{A}^N$, energy norm, $N = 800$

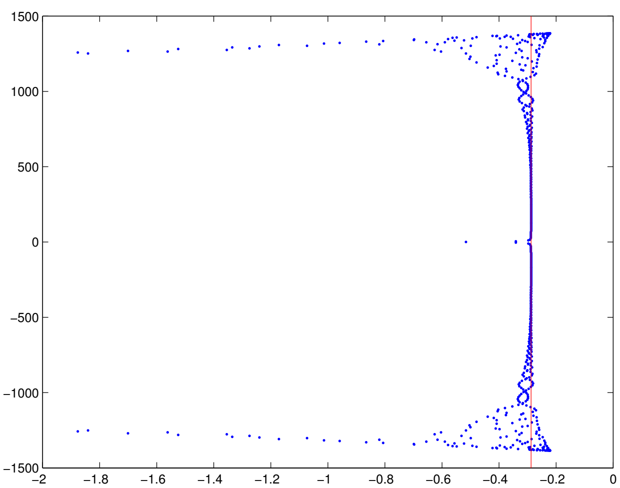


Fig. 4. Eigenvalues of $\mathcal{A}^N$, new norm, $N = 800$

## VI. Conclusion

We have provided a detailed construction of a linear spline-based semidiscrete approximation scheme for a scalar neutral equation, which is convergent in the Trotter-Kato sense and which preserves the decay rate uniformly in the discretization parameter. The ideas extend and apply easily to a neutral system.

## References


[1] S.-I. Niculescu, *Delay Effects on Stability*, ser. Lecture Notes in Control and Information Sciences. Springer-Verlag, 2001, vol. 269.

[2] H. Banks and R. Fabiano, "Approximation issues for applications in optimal control and parameter estimation," in *Modelling and Computation for Applications in Mathematics, Science, and Engineering*, ser. Numerical Mathematics and Scientific Computation, J. J. Jerome, Ed. Oxford University Press, 1998, pp. 141–165.

[3] K. Ito, "Finite-dimensional compensators for infinite-dimensional systems via galerkin-type approximation," *SIAM J. Control and Optimization*, vol. 28, pp. 1251–1269, 1990.

[4] F. Kappel and D. Salamon, "An approximation theorem for the algebraic riccati equation," *SIAM J. Control and Optimization*, vol. 28, pp. 1136–1147, 1990.

[5] I. Lasiecka, "Finite element approximations of compensator design for analytic generators with fully unbounded controls/observations," *SIAM J. Control and Optimization*, vol. 33, pp. 67–88, 1995.

[6] H. T. Banks, K. Ito, and I. G. Rosen, "A spline based technique for computing Riccati operators and feedback controls in regulator problems for delay equations," *SIAM J. Scientific and Statistical Computing*, vol. 5, pp. 830–855, 1984.

[7] J. Gibson, "Linear-quadratic optimal control of hereditary differential systems: Infinite-dimensional riccati equations and numerical approximations," *SIAM J. Control and Optimization*, vol. 21, pp. 95–139, 1983.

[8] F. Kappel and D. Salamon, "Spline approximation for retarded systems and the Riccati equation," *SIAM J. Control and Optimization*, vol. 25, pp. 1082–1117, 1987.

[9] ——, "On the stability properties of spline approximations for retarded systems," *SIAM J. Control and Optimization*, vol. 27, pp. 407–431, 1989.

[10] R. Fabiano and J. Turi, "Stability for retarded delay equations with multiple delays," in *Proceedings of the 45th IEEE Conference on Decision and Control*, 2006, pp. 4712–4716.

[11] J. A. Burns, T. L. Herdman, and H. W. Stech, "Linear functional differential equations as semigroups on product spaces," *SIAM J. Math. Anal.*, vol. 14, pp. 98–116, 1983.

[12] R. Fabiano and J. Turi, "Stability conditions for differential-difference systems of retarded and neutral type: The single delay case," *International Journal of Qualitative Theory of Differential Equations and Applications*, vol. 1, pp. 59–75, 2007.

[13] C. Desoer and M. Vidyasagar, *Feedback Systems: Input-Ouput Properties*. Academic Press, 1975.

[14] H. T. Banks and J. A. Burns, "Hereditary control problems: Numerical methods based on averaging approximations," *SIAM J. Control and Optimization*, vol. 16, pp. 169–208, 1978.

[15] F. Kappel, "Approximation of neutral functional differential equations in the state space $R \times L^2$," in *Qualitative Theory of Differential Equations*, ser. Colloquia Mathematica Societatis Bolyai, vol. 30. North Holland, 1979, pp. 463–506.

[16] D. Salamon, "Structure and stability of finite dimensional approximations for functional differential equations," *SIAM J. Control and Optimization*, vol. 23, pp. 928–951, 1985.

[17] R. Fabiano and J. Turi, "Preservation of stability in neutral systems," Preprints of 3rd IFAC Symposium on System, Structure and Control, SSSC07, Foz do Iguassu, Brazil, October 2007.

[18] H. T. Banks and F. Kappel, "Spline approximations for functional differential equations," *Journal of Differential Equations*, vol. 34, pp. 496–522, 1979.

[19] F. Kappel and K. Kunisch, "Spline approximations for neutral functional differential equations," *SIAM J. Numerical Analysis*, vol. 18, pp. 1058–1080, 1981.

[20] R. H. Fabiano, "Uniformly stable spline approximations for scalar delay problems," in *Optimal Design and Control*, ser. Progress in Systems and Control Theory, J. Borggaard, J. Burkardt, M. Gunzberger, and J. Peterson, Eds., vol. 19. Birkhäuser, 1995, pp. 151–157.

[21] K. Ito, F. Kappel, and D. Salamon, "A variational approach to approximation of delay systems," *Differential and Integral Equations*, vol. 4, pp. 51–72, 1991.

[22] K. Ito and F. Kappel, *Evolution Equations and Approximations*, ser. Advances in Mathematics for Applied Sciences. World Scientific, 2002, vol. 61.

[23] M. Schultz, *Spline Analysis*. Prentice Hall, 1973.