Proceedings of the
47th IEEE Conference on Decision and Control
Cancun, Mexico, Dec. 9-11, 2008

Adaptive Nonlinear Regulation for Uncertain Minimum Phase Systems with Unknown Exosystem

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Abstract— **The theory of regulation aims at zero output regulation error under the assumption that the order of the exosystem, which generates the disturbance and the output reference signals, or at least its upper bound is known; this is a crucial assumption since the regulator which achieves exact regulation should incorporate a possibly adaptive internal model which includes the exosystem. We design an adaptive internal model to characterize the output regulation error obtained on the basis of the features of the ideal unknown input reference signal capable of exactly zeroing the regulation error. We show that if the designed adaptive internal model can generate the ideal zeroing reference input, then the regulation error tends to zero while if the adaptive internal model cannot generate the ideal zeroing reference input, then the regulation error exponentially tends to a residual which decreases with the number of unmodeled terms in the ideal reference input.**

I. INTRODUCTION

The theory of regulation aims at zero output regulation error under the assumption that the order of the exosystem, which generates the disturbance and the output reference signals, or at least its upper bound is known; this is a crucial assumption since the regulator which achieves exact regulation should incorporate a possibly adaptive internal model which includes the exosystem. The theory of regulators started with linear systems with known linear exosystems [1], [2] and was then extended to nonlinear systems with known nonlinear exosystems [3] to explore and establish the necessary and sufficient conditions under which the regulator problem is exactly solvable. Adaptive regulators were then proposed for linear and nonlinear systems with unknown exosystems with known order [4], [5]; in the case of sinusoidal disturbances this assumption amounts to know the maximum number of allowed sinusoids. However in the presence of unknown disturbances the maximum number of sinusoidal components may be very large and possibly infinite in the case of periodic disturbances. In a recent paper [6] the assumption of known order for the uncertain linear exosystem was removed in the context of regulation of uncertain minimum phase linear systems allowing for unmodeled exosystem dynamics: exponential convergence of the output error into a region which decreases with the order of the unmodeled system dynamics is obtained; when the regulator can exactly model all of the exosystem excited frequencies the regulation error tends exponentially to zero while asymptotic regulation is achieved when the regulator overmodels the actual exosystem.

The goal of this paper is to explore the extension of these techniques to a class of nonlinear systems with uncertain output dependent nonlinearities, for which the assumption of knowing the maximum number of sinusoids to be generated by the internal model is certainly more critical, since the nonlinearities may generate many higher order harmonics to be compensated by the controller even from a single sinusoidal disturbance. For this reason, very often the nonlinear regulation problem is solved under the 'immersion' assumption [7], [8], [9], [10] that the dynamical system generating all possible feedforward inputs which can ensure an identically zero regulation error be immersed into a linear observable system or more general systems of predefined structure [11], [12]. We do not require the immersion assumption since we design an adaptive internal model to characterize the output regulation error obtained on the basis of the features of the ideal unknown input reference signal capable of exactly zeroing the regulation error. We basically show that if the designed adaptive internal model can generate the ideal zeroing reference input, then the regulation error tends to zero while if the adaptive internal model cannot generate the ideal zeroing reference input, then the regulation error exponentially tends to a residual which decreases with the number of unmodeled terms in the ideal reference input.

II. MAIN RESULT

Consider the following class of uncertain nonlinear systems in output feedback form of relative degree ρ, with $1 \leq \rho \leq n$,

$$\begin{aligned} \dot{x} &= A_c x + \phi(y, w) + bu, \quad x \in \mathbb{R}^n, \quad u \in \mathbb{R} \\ \dot{w} &= R(w), \quad w \in \Omega_w \subset \mathbb{R}^r \\ e &= C_c x - q(w) \end{aligned} \tag{1}$$

in which: $\phi(\cdot, \cdot)$ is an unknown smooth vector field such that $\forall y_1, y_2 \in \mathbb{R}$, $\forall w \in \mathbb{R}^r$

$$\|\phi(y_1, w) - \phi(y_2, w)\| \leq \bar{\phi}(y_1, y_2, w)|y_1 - y_2| \tag{2}$$

with $\bar{\phi}$ a known function; $b = [0, \ldots, 1, b_{\rho+1}, \ldots, b_n]^T$ is a known vector such that the polynomial $s^{n-\rho} + b_{\rho+1}s^{n-\rho-1} + \cdots + b_n$ is Hurwitz; $w(t)$ is the vector, generated by an unknown exosystem, containing both the references to be tracked by the output $C_c x(t)$ and the disturbance to be

This work was supported by Ministero della Università e Ricerca.

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978-1-4244-3124-3/08/$25.00 ©2008 IEEE

rejected, which belongs to the known region Ω_w; the output tracking error $e(t)$ is the only measurement available for feedback; A_c and C_c are in the observer canonical form given by

$$\begin{array}{rcl} A_c & = & \left[\begin{array}{ccccc} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ldots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{array}\right] \\ C_c & = & \left[\begin{array}{cccc} 1 & 0 & \cdots & 0 \end{array}\right] . \end{array}$$

Assume that there exists a (unknown) bounded solution ($x_r = \Gamma(w)$, $u_r = \gamma(w)$) to the regulator equations

$$\begin{array}{rcl} \dot{x}_r & = & A_c x_r + \phi(y_r, w) + b u_r \\ y_r & = & C_c x_r \\ C_c x_r - q(w) & = & 0 . \end{array} \qquad (3)$$

Defining the regulation error $\tilde{x} = x - x_r$, from (1) and (3) we have

$$\begin{array}{rcl} \dot{\tilde{x}} & = & A_c \tilde{x} + \phi(y, w) - \phi(y_r, w) + b(u - u_r) \\ \dot{w} & = & R(w) \\ e & = & C_c \tilde{x} . \end{array} \qquad (4)$$

Problem 2.1: Let $\bar{u}_r(t)$ be an estimate of $u_r(t)$ given by a biased linear combination of m distinct sinusoids, i.e.

$$\begin{array}{rcl} \dot{\bar{\eta}} & = & \bar{R}_c \bar{\eta}, \ \bar{\eta} \in \mathbb{R}^{2m+1}, \ \bar{\eta}(0) = \bar{\eta}_0 \\ \bar{u}_r & = & \bar{\eta}_1 \end{array} \qquad (5)$$

in which $\bar{R}_c = \left[\begin{array}{ccccc} 0 & 1 & 0 & \cdots & 0 \\ -\bar{\theta}_1 & 0 & 1 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ldots & \vdots \\ -\bar{\theta}_m & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{array}\right]$ and $\bar{\theta}_i$, $1 \leq i \leq m$, are positive reals satisfying $\prod_{i=1}^m (s^2 + \bar{\omega}_i^2) = s^{2m} + \sum_{i=0}^{m-1} s^{2i} \bar{\theta}_{m-i}$ with $\bar{\omega}_i$, $1 \leq i \leq m$ distinct positive reals such that $\bar{\omega}_i \leq \omega_M$, $1 \leq i \leq m$, ω_M being a known positive real. Define

$$\begin{array}{rcl} \epsilon_M & = & \min\limits_{\substack{0 \leq \bar{\omega}_i \leq \omega_M \\ 1 \leq i \leq m \\ \bar{\eta}_0 \in \mathbb{R}^{2m+1}}} \left\{ \sup\limits_{t \geq 0} |u_r(t) - \bar{u}_r(t)| \right\} \\ & \triangleq & \sup\limits_{t \geq 0} |u_r(t) - \hat{u}_r(t)| \end{array} \qquad (6)$$

with $\hat{u}_r(t)$ generated by

$$\begin{array}{rcl} \dot{\eta} & = & R_c \eta, \ \eta(0) = \eta_0 \\ \hat{u}_r & = & \eta_1 \end{array} \qquad (7)$$

in which $R_c = \left[\begin{array}{ccccc} 0 & 1 & 0 & \cdots & 0 \\ -\theta_1 & 0 & 1 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ldots & \vdots \\ -\theta_m & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{array}\right]$ with $\prod_{i=1}^m (s^2 + \omega_i^2) = s^{2m} + \sum_{i=0}^{m-1} s^{2i} \theta_{m-i}$. Let $\prod_{i=1}^m (s^2 + \omega_M^2) = s^{2m} + \sum_{i=0}^{m-1} s^{2i} \theta^*_{m-i}$ and denote by $r_\theta = \sqrt{m} \max_{1 \leq i \leq m} \{\theta_i^*\}$, so that $\theta = [\theta_1, \ldots, \theta_m]^T \in \Omega_\theta$, a closed ball centered at the origin with known radius r_θ. Find an adaptive control algorithm

$$\begin{array}{rcl} \dot{\sigma} & = & f_\sigma(\sigma, e, \hat{\theta}), \ \sigma \in \mathbb{R}^r \\ \dot{\hat{\theta}} & = & f_\theta(\sigma, \hat{\theta}, e), \ \hat{\theta} \in \mathbb{R}^m \\ u & = & f_u(e, \hat{\theta}, \sigma) \end{array} \qquad (8)$$

such that all signals in the closed loop system (4), (8) are bounded, $\|\hat{\theta}(t)\| \leq r_\theta + \epsilon_r$, with ϵ_r an arbitrary positive real, and: (i) if $u_r(t)$ is a sufficiently rich signal of order $2m$ (see [13]), then $|e(t)| \leq a_1 e^{-a_2 t} + a_3 \epsilon_M, \ \forall t \geq 0$ with a_i, $1 \leq i \leq 3$ suitable positive reals; (ii) if $u_r(t)$ is not a sufficiently rich signal of order $2m$, then for every $t \geq 0$ $|e(t)| \leq a_4 e^{-a_5 t} + a_6 \epsilon_M + a_7 \sup_{\tau \in [0,t)} \|\tilde{\theta}(\tau)\|$ with a_i, $4 \leq i \leq 7$ suitable positive reals. Moreover if $\epsilon_M = 0$, then $\lim_{t \to \infty} e(t) = 0$.

Theorem 2.1: There exists an adaptive control law which globally solves Problem 2.1 for system (4).

Proof. Define $\epsilon(t) = u_r(t) - \hat{u}_r(t) = u_r(t) - \eta_1(t)$. There exists a change of coordinates $\zeta \in \mathbb{R}^{n+2m+1}$

$$\zeta = T(\theta) \left[\begin{array}{c} \tilde{x} \\ \eta \end{array}\right] = T_1(\theta)\tilde{x} + T_2(\theta)\eta \qquad (9)$$

which maps (4), (7) into

$$\begin{array}{rcl} \dot{\zeta} & = & A_c \zeta + b[0](u - \epsilon) + \sum\limits_{i=1}^m \theta_i b[i](u - \epsilon) \\ & & + T_1(\theta)(\phi(y, w) - \phi(y_r, w)) \\ e & = & C_c \zeta \end{array} \qquad (10)$$

with ($b[i] \in \mathbb{R}^{n+2m+1}$, $0 \leq i \leq m$)

$$\begin{array}{rcl} b[0] & = & \left[\begin{array}{cccc} b^T & 0 & \cdots & 0 \end{array}\right]^T \\ b[1] & = & \left[\begin{array}{cccccc} 0 & 0 & b^T & 0 & \cdots & 0 \end{array}\right]^T \\ & \vdots & \\ b[m] & = & \left[\begin{array}{ccccc} 0 & \cdots & 0 & b^T & 0 \end{array}\right]^T . \end{array}$$

Define the filtered transformation (I_j denotes the $(j \times j)$ identity matrix)

$$\begin{array}{rcl} \dot{\xi}_i & = & D\xi_i + \left[\begin{array}{cc} 0 & I_{n+2m} \end{array}\right] b[i]u, \quad \xi_i \in \mathbb{R}^{n+2m} \\ \mu_i & = & \left[\begin{array}{cccc} 1 & 0 & \cdots & 0 \end{array}\right] \xi_i, \quad 1 \leq i \leq m \\ z & = & \zeta - \left[\begin{array}{c} 0 \\ \sum_{i=1}^m \xi_i \theta_i \end{array}\right], \quad z \in \mathbb{R}^{n+2m+1} \end{array} \qquad (11)$$

in which $D = \left[\begin{array}{cccc} -d_2 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -d_{n+2m} & 0 & \cdots & 1 \\ -d_{n+2m+1} & 0 & \cdots & 0 \end{array}\right]$ is a Hurwitz matrix. From (10) and (11), we obtain

$$\begin{array}{rcl} \dot{z} & = & A_c z + b[0]u + d \sum\limits_{i=1}^m \theta_i \mu_i \\ & & + T_1(\theta)(\phi(y, w) - \phi(y_r, w)) + \epsilon \bar{b}(\theta) \\ e & = & C_c z \end{array} \qquad (12)$$

with $d = [1, d_2, \ldots, d_{n+2m+1}]^T$ and $\bar{b}(\theta) = b[0] + \sum_{i=1}^{m} \theta_i b[i]$. First, we consider the case $\rho = 1$. Introduce the observer

$$\dot{\hat{z}} = A_c \hat{z} + b[0]u + d \sum_{i=1}^{m} \hat{\theta}_i \mu_i - k_o(e - C_c \hat{z}) \quad (13)$$

with error dynamics

$$\begin{aligned} \dot{\tilde{z}} &= A\tilde{z} + d\mu^T \tilde{\theta} \\ &\quad + T_1(\theta)[\phi(y,w) - \phi(y_r,w)] + \epsilon \bar{b}(\theta) \end{aligned} \quad (14)$$

with $A = A_c + k_o C_c$, $\tilde{\theta} = \theta - \hat{\theta}$ and $\mu = [\mu_1, \ldots, \mu_m]^T$. The vector k_o is chosen so that the triple (A, d, C_c) is strictly positive real, i.e. such that the matrix A is Hurwitz and $n + 2m$ of its $n + 2m + 1$ eigenvalues coincide with the eigenvalues of the matrix D. Define the linear change of coordinates

$$\begin{aligned} z_1 &= z_1 \\ \bar{z}_i &= z_i - b_i z_1, \quad 2 \le i \le n \\ z_i &= z_i, \quad n+1 \le i \le n+2m+1 \ . \end{aligned} \quad (15)$$

From (12) and (15), we have

$$\begin{aligned} \dot{z}_1 &= \bar{z}_2 + b_2 z_1 + u + \mu^T \theta + \epsilon \bar{b}_1 \\ &\quad + T_{11}[\phi(y,w) - \phi(y_r,w)] \end{aligned} \quad (16)$$

$$\begin{aligned} \begin{bmatrix} \dot{\bar{z}}_2 \\ \vdots \\ \dot{\bar{z}}_n \end{bmatrix} &= \begin{bmatrix} -b_2 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ -b_{n-1} & 0 & 0 & \cdots & 1 \\ -b_n & 0 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} \bar{z}_2 \\ \vdots \\ \bar{z}_n \end{bmatrix} \\ &\quad + z_1 \begin{bmatrix} b_3 - b_2^2 \\ \vdots \\ b_n - b_2 b_{n-1} \\ -b_n b_2 \end{bmatrix} + \mu^T \theta \begin{bmatrix} d_2 - b_2 \\ \vdots \\ d_n - b_n \end{bmatrix} \\ &\quad + \begin{bmatrix} T_{12} - b_2 T_{11} \\ \vdots \\ T_{1n} - b_n T_{11} \end{bmatrix} [\phi(y,w) - \phi(y_r,w)] \\ &\quad + \epsilon \begin{bmatrix} \bar{b}_2 - b_2 \bar{b}_1 \\ \vdots \\ \bar{b}_n - b_n \bar{b}_1 \end{bmatrix} \\ &\triangleq F \begin{bmatrix} \bar{z}_2 \\ \vdots \\ \bar{z}_n \end{bmatrix} + f_1 z_1 + \mu^T \theta f_2 \\ &\quad + \bar{T}(\theta)[\phi(y,w) - \phi(y_r,w)] + \bar{\bar{b}}(\theta) \end{aligned} \quad (17)$$

in which T_{1j} denotes the j-th row of matrix T_1. With reference to (11), we note that $\mu_i(t)$ may be equivalently generated by the following filters with proper initial conditions

$$\begin{aligned} \dot{\bar{\xi}}_i[1] &= D\bar{\xi}_i[1] - M_i[\phi(y,w) - \phi(y_r,w)] \\ \mu_i[1] &= \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix} \bar{\xi}_i[1] \\ \dot{\bar{\xi}}_i[2] &= D\bar{\xi}_i[2] + \begin{bmatrix} 0 & I_{n+2m} \end{bmatrix} b[i](\eta_1 + \epsilon) \\ \mu_i[2] &= \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix} \bar{\xi}_i[2], \ 1 \le i \le m \end{aligned} \quad (18)$$

by means of the relations

$$\bar{\xi}_i[1] + \bar{\xi}_i[2] = \xi_i - M_i \tilde{x}, \ 1 \le i \le m \ . \quad (19)$$

If $u_r(t) = \eta_1(t) + \epsilon(t)$ is a sufficiently rich signal of order $2m$, $\mu[2] = [\mu_1[2], \ldots, \mu_m[2]]^T$ is a persistently exciting vector (see [13]). This fact implies that (see [14]) the solution of the matrix differential equation

$$\dot{Q} = -Q + \mu[2]\mu^T[2], \ Q(0) = e^{-T_p} k_p I \quad (20)$$

with T_p and k_p positive reals satisfying

$$\int_t^{t+T_p} \mu[2](\tau)\mu^T[2](\tau) d\tau \ \ge \ k_p I, \ \forall t \ge 0 \quad (21)$$

is such that

$$\sup_{t \ge 0} \|\mu[2](t)\|^2 I \ge Q(t) \ge k_p e^{-2T_p} I, \ \forall t \ge 0 \ .$$

Moreover, recalling (2), we can write ($\forall e \in \mathbb{R}$, $\forall w \in \Omega_w$)

$$\begin{aligned} \|\phi(y,w) - \phi(y_r,w)\| &\le \bar{\phi}(y, y_r, w)|e| \\ &\le \phi_M(e)|e| \end{aligned} \quad (22)$$

in which $\phi_M(e)$ is a known smooth function. Let

$$\begin{aligned} u &= -\hat{z}_2 - ke - \mu^T \hat{\theta} - h_1 \phi_M(e) e - h_2 \phi_M^2(e) e \\ \dot{\hat{\theta}} &= \text{Proj}(\mu(\tilde{z}_1 + z_1), \hat{\theta}) \end{aligned} \quad (23)$$

in which $\text{Proj}(\cdot,\cdot)$ is the smooth projection operator defined as (see [15])

$$\begin{aligned} \text{Proj}(\varphi, \hat{\theta}) &= \varphi, \text{ if } p_r(\hat{\theta}) \le 0 \\ \text{Proj}(\varphi, \hat{\theta}) &= \varphi, \text{ if } p_r(\hat{\theta}) \ge 0 \text{ and } \langle \text{grad } p_r(\hat{\theta}), \varphi \rangle \le 0 \\ \text{Proj}(\varphi, \hat{\theta}) &= \left[I - \frac{p_r(\hat{\theta}) \text{grad } p_r(\hat{\theta}) \text{grad } p_r(\hat{\theta})^T}{\|\text{grad } p_r(\hat{\theta})\|^2} \right] \varphi, \\ &\quad \text{if } p_r(\hat{\theta}) > 0 \text{ and } \langle \text{grad } p_r(\hat{\theta}), \varphi \rangle > 0 \end{aligned}$$

with $p_r(\hat{\theta}) = \frac{\|\hat{\theta}\|^2 - r_\theta^2}{\epsilon_r^2 + 2\epsilon_r r_\theta}$, and ϵ_r an arbitrary positive real. If $\|\hat{\theta}(0)\| \le r_\theta$ then, $\forall t \ge 0$: (a1) $\|\hat{\theta}(t)\| \le r_\theta + \epsilon_r$; (a2) $\text{Proj}(\varphi, \hat{\theta})$ is Lipschitz continuous; (a3) $\|\text{Proj}(\varphi, \hat{\theta})\| \le \|\varphi\|$; (a4) $\tilde{\theta}^T \text{Proj}(\varphi, \hat{\theta}) \ge \tilde{\theta}^T \varphi$. Consider the function

$$V = \frac{1}{2}\tilde{z}^T P \tilde{z} + \frac{1}{2}e^2 + \frac{1}{2}\tilde{\theta}^T \tilde{\theta} \ . \quad (24)$$

The symmetric positive definite matrix P is obtained by solving the matrix equations

$$\begin{aligned} A^T P + PA &= -qq^T - 6\alpha I \\ Pd &= C_c^T \end{aligned} \quad (25)$$

with respect to P, q and the positive real α. Such a solution exists since the triple (A, d, C_c) is real strictly positive (see [16]). If h_1 and h_2 are chosen so that

$$\begin{aligned} h_1 &\ge \|T_{11}(\theta)\|, \ \forall \theta \in \Omega_\theta \\ h_2 &\ge \frac{1}{4\alpha}\|P\|^2 \|T_1(\theta)\|^2, \ \forall \theta \in \Omega_\theta \end{aligned} \quad (26)$$

it follows that

$$\dot{V} \le -c_{v_a} V + c_{v_b} \left\| \begin{bmatrix} \epsilon \\ \tilde{\theta} \end{bmatrix} \right\|^2 \quad (27)$$

i.e.

$$V(t) \;\leq\; V(0)\exp(-c_{v_a}t) + \frac{c_{v_b}}{c_{v_a}} \sup_{\tau\in[0,t)} \left\| \begin{bmatrix} \epsilon(\tau) \\ \tilde{\theta}(\tau) \end{bmatrix} \right\|^2$$

so that $\tilde{z}(t)$ and $e(t)$ are bounded, since $\epsilon(t)$ and $\tilde{\theta}(t)$ are bounded. From (18) it follows that $\bar{\xi}_i[1]$ are bounded and, consequently, $\mu_i[1]$, $\mu_i[2]$, $\dot{\mu}_i[1]$ and $\dot{\mu}_i[2]$ are bounded and property (ii) of Problem 2.1 is proved. From (17) it follows that $\bar{z}_i$, $2 \leq i \leq n$, are bounded, so that (23) implies that $u(t)$ is bounded. Now, consider system (4) and apply the change of coordinates

$$\chi_i \;=\; \tilde{x}_{i+1} - b_{i+1}\tilde{x}_1,\ 1 \leq i \leq n-1 \qquad (28)$$

which maps (4) into ($\chi = [\chi_1, \ldots, \chi_{n-1}]^T$)

$$\begin{aligned} \dot{e} &= \chi_1 + b_2 e + \phi_1(y,w) - \phi_1(y_r,w) + u - u_r \\ \dot{\chi} &= F\chi + ef_1 + \begin{bmatrix} \phi_2(y,w) - \phi_2(y_r,w) \\ \vdots \\ \phi_n(y,w) - \phi_n(y_r,w) \end{bmatrix} \\ &\quad - \begin{bmatrix} b_2(\phi_1(y,w) - \phi_1(y_r,w)) \\ \vdots \\ -b_n(\phi_1(y,w) - \phi_1(y_r,w)) \end{bmatrix} \end{aligned} \qquad (29)$$

so that since F is Hurwitz and $e(t)$, $y_r(t)$, $w(t)$ are bounded, it follows that $\chi(t)$ and, consequently from (28), $\tilde{x}(t)$ are bounded. Now, consider the function

$$W \;=\; V + p\|Q\tilde{\theta} - \mu[2]e\|^2 + p_1 \sum_{i=1}^{m} \bar{\xi}_i^T[1]P_2\bar{\xi}_i[1] \qquad (30)$$

with P_2 solution of

$$D^T P_2 + P_2 D \;=\; -I \qquad (31)$$

and p, p_1 suitable positive reals yet to be defined. If $k > 0$ and

$$\begin{aligned} c_1 &> 2 \\ p_1 &= \frac{2p}{c_1 - 1}\sup_{t\geq 0}\|\mu[2](t)\|^2\|\tilde{\theta}(t)\|^2 \\ p &< \min_{t\geq 0}\{a_1(t), a_2(t)\} \end{aligned} \qquad (32)$$

with

$$\begin{aligned} a_1(t) &= \frac{\alpha}{c_1(\|Q\|\|\mu\| + \|\mu[2]\|)^2}, \\ a_2(t) &= (c_1-1)k[\frac{1}{a_3(t)} + \frac{1}{2c_1\|P_2\|^2\phi_M^2\|\mu[2]\|^2\|\tilde{\theta}\|^2}] \\ a_3(t) &= c_1(c_1-1)[\|Q\|\|\mu\| + \|\dot{\mu}[2]\| + \|\mu[2]\| \\ &\quad \cdot(k + 1 + h_1\phi_M + h_2\phi_M^2 + \|T_{11}\|\phi_M)]^2 \end{aligned}$$

we have

$$\dot{W}(t) \;\leq\; -c_{w_a}W(t) + c_{w_b}\epsilon^2 \qquad (33)$$

for suitable positive reals c_{w_a} and c_{w_b}. By integrating (33), we obtain

$$W(t) \;\leq\; W(0)\exp(-c_{w_a}t) + \frac{c_{w_b}}{c_{w_a}}\sup_{\tau\in[0,t)}\{\epsilon^2(\tau)\} \qquad (34)$$

which proves property (i) of Problem 2.1. Now, consider the case $\rho > 1$ and let system (1) be written as ($\bar{b} = [0, \ldots, 1, \bar{b}_{\rho+1}, \ldots, \bar{b}_n]^T$)

$$\begin{aligned} \dot{\bar{x}} &= A_c\bar{x} + \phi(y,w) + \bar{b}u \\ \dot{w} &= R(w) \\ e &= C_c\bar{x} - q(w),\ w \in \Omega_w \subset \mathbb{R}^r \end{aligned} \qquad (35)$$

Assume that there exists a solution ($\bar{x}_r = \bar{\Gamma}(w)$, $u_r = \bar{\gamma}(w)$) to the regulator equations

$$\begin{aligned} \dot{\bar{x}}_r &= A_c\bar{x}_r + \phi(y_r,w) + \bar{b}u_r \\ y_r &= C_c\bar{x}_r \\ C_c\bar{x}_r - q(w) &= 0\ . \end{aligned} \qquad (36)$$

Consider the filtered transformation

$$\begin{aligned} \dot{\varphi} &= \begin{bmatrix} -\lambda_1 & 1 & 0 & \cdots & 0 \\ 0 & -\lambda_2 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ldots & \vdots \\ 0 & 0 & 0 & \cdots & -\lambda_{\rho-1} \end{bmatrix}\varphi + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}u \\ x_1 &= \bar{x}_1,\ x_i = \bar{x}_i - \sum_{j=2}^{\rho} b_i[j]\varphi_{j-1},\ 2 \leq i \leq n \end{aligned} \qquad (37)$$

where $\lambda_i > 0$, $1 \leq i \leq \rho - 1$ and the vectors $b[j]$ are recursively obtained by

$$\begin{aligned} b[\rho] &= \bar{b} \\ b[j-1] &= A_c b[j] + \lambda_{j-1}b[j], \rho \geq j \geq 2 \\ b[1] &\triangleq b = [1, b_2, \ldots, b_n]^T \end{aligned} \qquad (38)$$

with b_i, $2 \leq i \leq n$, solutions of

$$\begin{aligned} s^{n-1} + b_2 s^{n-2} + \cdots + b_n = (s^{n-\rho} + \bar{b}_{\rho+1}s^{n-\rho-1} \\ + \cdots + \bar{b}_n)\prod_{i=1}^{\rho-1}(s + \lambda_i)\ . \end{aligned} \qquad (39)$$

From (35) and (37), we have

$$\begin{aligned} \dot{x} &= A_c x + \phi(y,w) + b\varphi_1 \\ \dot{w} &= R(w) \\ e &= C_c x - q(w)\ . \end{aligned} \qquad (40)$$

By virtue of (36) there exists a solution ($x_r = \Gamma(w), \varphi_{1r} = \gamma(w)$) to the regulator equations

$$\begin{aligned} \dot{x}_r &= A_c x_r + \phi(y_r,w) + b\varphi_{1r} \\ y_r &= C_c x_r \\ C_c x_r - q(w) &= 0\ . \end{aligned} \qquad (41)$$

Indeed, let φ_{1r} be obtained by ($\varphi_r = [\varphi_{r1}, \ldots, \varphi_{r,\rho-1}]^T$)

$$\begin{aligned} \dot{\varphi}_r &= \begin{bmatrix} -\lambda_1 & 1 & 0 & \cdots & 0 \\ 0 & -\lambda_2 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ldots & \vdots \\ 0 & 0 & 0 & \cdots & -\lambda_{\rho-1} \end{bmatrix}\varphi_r + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ u_r \end{bmatrix} \\ \varphi_{1r} &= \varphi_{r1}\ . \end{aligned} \qquad (42)$$

Defining $x_r = [x_{r1}, \ldots, x_{rn}]^T$ with

$$x_{r1} = \bar{x}_{r1},\ x_{ri} = \bar{x}_{ri} - \sum_{j=2}^{\rho} b_i[j]\varphi_{r,j-1},\ 2 \le j \le n$$

and recalling (36) and (42), we obtain (41). The regulator error equations become

$$\begin{aligned} \dot{\tilde{x}} &= A_c\tilde{x} + \phi(y,w) - \phi(y_r,w) + b(\varphi_1 - \varphi_{1r}) \\ e &= C_c\tilde{x} \\ \dot{w} &= R(w)\ . \end{aligned} \tag{43}$$

If we consider φ_1 as the control input, system (43) is in the form (4) and, therefore, we can follow the same steps of the relative-degree-one case (using φ_1 in place of u in the filters (11)) to obtain the ideal control

$$\begin{aligned} \varphi_1^* &= -\hat{z}_2 - ke - \mu^T\hat{\theta} - h_1\phi_M e - h_2\phi_M^2 e \\ &\triangleq \varphi_1^*(e, \hat{z}_2, \mu, \hat{\theta}) \\ \dot{\hat{z}} &= A_c\hat{z} + b[0]\varphi_1 + d\sum_{i=1}^{m}\mu_i\hat{\theta}_i - k_o(e - C_c\hat{z}) \\ \dot{\xi}_i &= D\xi_i + \left[\begin{array}{cc} 0 & I_{n+2m} \end{array}\right] b[i]\varphi_1 \\ \mu_i &= \left[\begin{array}{cccc} 1 & 0 & \cdots & 0 \end{array}\right]\xi_i \end{aligned} \tag{44}$$

in which $\hat{\theta}$ is yet to be defined. Defining $\tilde{\varphi}_1 = \varphi_1 - \varphi_1^*$, we obtain for the error equations ($\varphi_{1r} = \eta_1 + \epsilon$, with η_1 generated by the exosystem (7))

$$\begin{aligned} \dot{z}_1 &= \tilde{z}_2 + \mu^T\tilde{\theta} + T_{11}(\theta)[\phi(y,w) - \phi(y_r,w)] \\ &\quad + \epsilon\bar{b}_1(\theta) + \tilde{\varphi}_1 - ke - h_1\phi_M e - h_2\phi_M^2 e \\ \left[\begin{array}{c} \dot{\bar{z}}_2 \\ \vdots \\ \dot{\bar{z}}_n \end{array}\right] &= F\left[\begin{array}{c} \bar{z}_2 \\ \vdots \\ \bar{z}_n \end{array}\right] + f_1 z_1 + \mu^T\theta f_2 \\ &\quad + \bar{T}(\theta)[\phi(y,w) - \phi(y_r,w)] + \epsilon\bar{\bar{b}}(\theta) \end{aligned} \tag{45}$$

$$\begin{aligned} \dot{\tilde{\varphi}}_1 &= -\lambda_1\varphi_1 + \varphi_2 - \frac{\partial\varphi_1^*}{\partial e}\dot{e} - \frac{\partial\varphi_1^*}{\partial\hat{z}_2}\dot{\hat{z}}_2 \\ &\quad - \frac{\partial\varphi_1^*}{\partial\mu}\dot{\mu} - \frac{\partial\varphi_1^*}{\partial\hat{\theta}}\dot{\hat{\theta}}\ . \end{aligned} \tag{46}$$

If $\rho = 2$, then we define $\varphi_2 = u$ as

$$\begin{aligned} u &= \lambda_1\varphi_1 + \frac{\partial\varphi_1^*}{\partial\hat{z}_2}\dot{\hat{z}}_2 + \frac{\partial\varphi_1^*}{\partial\mu}\dot{\mu} + \frac{\partial\varphi_1^*}{\partial\hat{\theta}}\dot{\hat{\theta}} + \frac{\partial\varphi_1^*}{\partial e}[\hat{z}_2 \\ &\quad + \varphi_1 + \mu^T\hat{\theta} - h_3\frac{\partial\varphi_1^*}{\partial e}\tilde{\varphi}_1 - h_4\frac{\partial\varphi_1^*}{\partial e}\phi_M^2\tilde{\varphi}_1] \\ \dot{\hat{\theta}} &= \text{Proj}(\mu(e + \tilde{z}_1) - \mu\frac{\partial\varphi_1^*}{\partial e}\tilde{\varphi}_1, \hat{\theta}) \end{aligned} \tag{47}$$

and consider the function

$$\begin{aligned} V_1 &= \frac{1}{2}\tilde{z}^T P\tilde{z} + \frac{1}{2}e^2 + \frac{1}{2}\tilde{\varphi}_1^2 + \frac{1}{2}\tilde{\theta}^T\tilde{\theta} \\ &= V + \frac{1}{2}\tilde{\varphi}_1^2 \end{aligned} \tag{48}$$

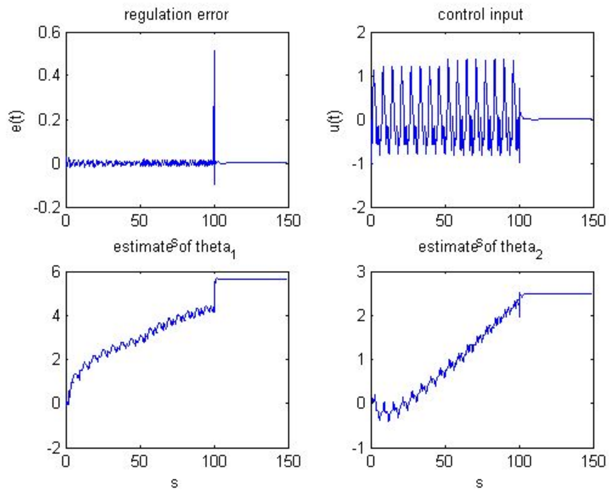

Fig. 1. Error, input and parameters estimates

in which P satisfies (25). The time derivative of (48), recalling (26) and property (a4) of Proj, is such that

$$\begin{aligned} \dot{V}_1 &\le -\alpha\tilde{z}^T\tilde{z} - ke^2 + \epsilon[\tilde{z}^T P\bar{b}(\theta) + e\bar{b}_1(\theta)] \\ &\quad + e\tilde{\varphi}_1 - \lambda_1\tilde{\varphi}_1^2 - \tilde{\varphi}_1\frac{\partial\varphi_1^*}{\partial e}[\tilde{z}_2 \\ &\quad + T_{11}(\theta)(\phi(y,w) - \phi(y_r,w)) + \epsilon\bar{b}_1(\theta)] \\ &\quad - h_3(\frac{\partial\varphi_1^*}{\partial e})^2\tilde{\varphi}_1^2 - h_4(\frac{\partial\varphi_1^*}{\partial e})^2\phi_M^2\tilde{\varphi}_1^2 \\ &\quad + \|\tilde{\theta}\|^2 - \|\tilde{\theta}\|^2 \end{aligned} \tag{49}$$

from which, since $\tilde{\theta}(t)$ is bounded, for sufficiently large h_3 and h_4 and suitable k, λ_1 it follows that $\tilde{z}$, e, $\tilde{\varphi}_1$ are bounded, so that $\mu(t)$ is bounded from (18), while $\bar{z}_2, \ldots, \bar{z}_n$ are bounded from (45). Consequently, from (44), φ_1^* is bounded and, in turn, φ_1 is bounded. From (47), $u(t)$ is bounded. The boundedness of $\bar{x}(t)$ in (35) may be proved as in the case $\rho = 1$ by considering the transformation (28). Moreover, (48) and (49) imply property (ii) in Problem 2.1. By using the function

$$W_1 = V_1 + p\|Q\tilde{\theta} - \mu[2]e\|^2 + p_1\sum_{i=1}^{m}\bar{\xi}_i[1]^T P_2\bar{\xi}_i[1]$$

it is possible to prove that for a suitable choice of p and p_1

$$W_1(t) \le W_1(0)\exp(-c_{w_{1a}}) + \frac{c_{w_{1b}}}{c_{w_{1a}}}\sup_{\tau\in[0,t)}\epsilon^2(\tau)$$

for proper $c_{w_{1a}} > 0$ and $c_{w_{1b}} > 0$, which proves (i) in Problem 2.1. The previous arguments can be iterated to prove the same result for any relative degree. □

III. EXAMPLE

Consider the second order system

$$\begin{aligned}\dot{x}_1 &= x_2 + w_1 x_1^2 + u\\ \dot{x}_2 &= u\\ e &= x_1 + w_2 \end{aligned} \tag{50}$$

in which:

$$\begin{aligned} w_1(t) &= \begin{cases} \sin(2t) + 0.2\sin(4t), & t < 50 \text{ s}\\ \sin(2t), & t \geq 50 \text{ s}\end{cases}\\ w_2(t) &= \begin{cases} \sin(t), & t < 100 \text{ s}\\ 0, & t \geq 100 \text{ s}\end{cases}.\end{aligned}$$

The unknown input reference u_r is obtained by $x_{1r} = -w_2$, $\dot{z}_{2r} = -z_{2r} + w_2 - w_1 w_2^2$, $z_{2r}(0) = z_{2r0}$, $u_r = -\dot{w}_2 - z_{2r} + w_2 - w_1 w_2^2$. We assume that the estimate $\hat{u}_r$ of u_r is generated by an exosystem of dimension 5 (i.e. we assume that $m = 4$ in (7)). Following the design outlined in Section II, the resulting control is given by

$$\begin{aligned} \dot{\xi}_i &= D\xi_i + \begin{bmatrix} 0 & I_6 \end{bmatrix}^T (E_{2i+1} + E_{2i+2})u\\ \mu_i &= \xi_{i1},\ \xi_i \in \mathbb{R}^6,\ i = 1,2\\ \dot{\hat{z}} &= A_c \hat{z} + (E_1 + E_2)u + d\mu^T \hat{\theta}\\ &\quad -k_o(e - \hat{z}_1),\ \hat{z} \in \mathbb{R}^7\\ u &= -\hat{z}_2 - ke - \mu^T\hat{\theta} - h_1 \phi_M e - h_2 \phi_M^2 e\\ \dot{\hat{\theta}} &= g\text{Proj}(\mu(\tilde{z}_1 + e), \hat{\theta}) \end{aligned} \tag{51}$$

with E_i the i-th column of the (7×7) identity matrix, g a positive adaptation gain, $D = \begin{bmatrix} -d_2 & 1 & \cdots & 0\\ \vdots & \vdots & \vdots & \vdots\\ -d_7 & 0 & \cdots & 0\end{bmatrix}$, $d = \begin{bmatrix} 1\\ d_2\\ \vdots\\ d_7 \end{bmatrix}$, $\phi_M = w_{1M}(|e| + 2w_{2M})$ and w_{1M}, w_{2M} the known admissible largest values for $w_1(t)$ and $w_2(t)$. Even though $g = 1$ in the proof of Theorem 2.1, it is obvious the extension to any positive value of g. Some numerical simulations have been carried out for system (50) controlled by (51). The parameters of the controller have been chosen as: $k = 5$, $g = 1000$, $d = [1, 12, 58, 144, 193, 132, 36]^T$, $k_o = [13, 70, 202, 337, 325, 168, 36]^T$. All initial conditions of the system and of the controller have been set to zero. The results are illustrated in Figs. 1 and 2. In Fig. 1 are reported the time histories of the output regulation error $e(t)$, the control input $u(t)$ and the estimates of the parameters $\hat{\theta}_1(t)$ and $\hat{\theta}_2(t)$, while Fig. 2 represents the state variables $[x_1(t), x_2(t)]$, the disturbance $w_1(t)$ and the output reference $-w_2(t)$.

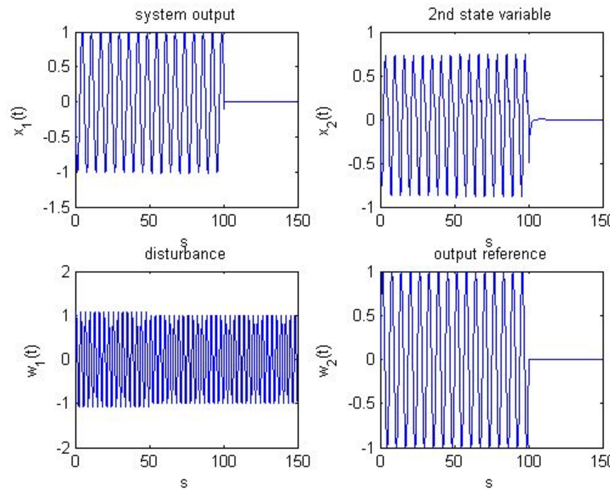

Fig. 2. State variables, disturbance and reference

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