Proceedings of the
47th IEEE Conference on Decision and Control
Cancun, Mexico, Dec. 9-11, 2008

On the equivalence of closed-loop subspace predictive control with LQG

Jianfei Dong and Michel Verhaegen

***Abstract*— Subspace predictive control (SPC) is recently seen in the literature for joint system identification and control design. The existing SPCs parameterize $\mathcal{H}_2$ optimal control laws by the identified Markov parameters from data. It has been proved that the SPCs based on open-loop subspace identification are equivalent to the classical LQG design, when the data horizon goes to infinity. It is the purpose of this paper to establish this equivalence for the closed-loop SPC algorithm we have recently developed. When the data horizon is finite, we also present in this paper a state-feedback LQG control law based on the identified Markov paremters, where the states are estimated in an optimal sense from the past I/O samples of a plant.**

I. INTRODUCTION

Conventional LQG optimal control is based on system models either built from physical principles or identified from data, [1]. The subspace predictive control (SPC) approaches, as recently seen in [2], [3], [4], [5], circumvent the modeling step, and directly seek the predictors of future outputs from data. The combination of the subspace identification algorithm, N4SID in [6], with LQG design is first seen in [2]. The SPC does not proceed further to recover the state-space matrices from the estimated state sequences as does in N4SID, which is again an estimation problem and may need to choose model order (as also seen in [7], a state-space model is realized from the predictor to integrate the identified output predictor with the conventional MPC). The approaches in [2], [4] focus on open-loop data; while those in [3], [5] extend the applicability of the SPCs to closed loops. The closed-loop SPC algorithm in [5] is developed based on the VARX (vector autoregressive with exogenous inputs) algorithm, developed in [8], [9] for closed-loop identification. It is also shown in [5] the advantage of the VARX-based approach over that proposed in [3]; in the sense that the former excludes the need of any information about the controller in the loop; while the latter, based on the closed-loop identification algorithm of [10], leads to a biased predictor when the controller is not LTI.

The SPCs design control inputs by minimizing the summed squares of the predicted future tracking errors; and are hence closely related to LQG design. It is shown in [2] that the open-loop SPC is indeed equivalent to the classical LQG control, when the data horizon goes to infinity. When the horizon is finite, however, this equivalence is not direct, since the output predictor parameterized by the identified Markov parameters does not ensure the optimality of state estimation, hidden behind the input and output relation. This motivates us to address two problems in this paper.

1) Whether the closed-loop SPC algorithm of [5] is equivalent to the classical LQG?
2) How to establish the optimality of the state estimation step hidden behind the SPC design, when the horizon is finite?

We shall answer Problem 1 in Section III.

In SPCs, Markov parameters are needed to parameterize the control law, instead of a dynamic model. LQG design from Markov parameters is also seen in [11], [12]. The state at time instance t is optimally estimated as a linear combination of the past inputs and outputs, s sampling instances prior to t, [12]. A finite horizon LQG design is derived in a closed form based on the plant Markov parameters. This solution provides an answer to Problem 2. We shall therefore extends [12] to the identified Markov parameters.

The rest of this paper is organized as follows. We review the closed-loop SPC design of [5] in Section II. The infinite-horizon case is discussed in Section III. The finite-horizon LQG solution is derived in Section IV, with a closed-form. Section IV presents a simulation example. The paper concludes in Section VI with the direction for future research.

II. CLOSED-LOOP SUBSPACE PREDICTIVE CONTROL

A. Closed-loop Subspace Identification

Consider the innovation type state space model:

$$x(k+1) = Ax(k) + Bu(k) + Ke(k) \quad (1)$$
$$y(k) = Cx(k) + e(k), \quad (2)$$

where $e(k)$ is assumed to be a zero-mean white noise with a nonsingular covariance of $\mathcal{E}\mathcal{E}^T$. The dimensions are assumed to be $x(k) \in \mathbb{R}^n$, $y(k) \in \mathbb{R}^\ell$, and $u(k) \in \mathbb{R}^m$. We make the following assumption on the plant, which is commonly assumed in subspace identification.

Assumption 1: $D = 0$; i.e. no direct feedthrough.

Assumption 2: $\Phi \triangleq A - KC$ is stable, and the system is minimal.

Assumption 1 is to ensure a one-step delay from the inputs to the outputs, and hence the well-posedness of the closed-loop identification problem. Assumption 2 is actually not restrictive, since any LTI state-space model has such an observer form; where K is the steady-state Kalman gain, and yields a stable Φ.

This research is jointly funded by the grant of the European AI4IA project (Contract No. 514510) under the Marie Curie FP6 Framework, and *SKF ERC* in the Netherlands. The authors would also like to thank Dr. B. Kulcsar for the valuable discussions on this paper.

The authors are with Delft Center for Systems and Control, Delft University of Technology, 2628 CD Delft, The Netherlands. j.dong@tudelft.nl & m.verhaegen@moesp.org

In the sequel, we denote by s, f respectively the past and future horizon ($s \geq f$), in both identification and control. N represents the number of columns in the identification data matrices. Let t be the current time instance in the formulation of the identification problem. We shall denote by k the current time instance in the control formulation.

The first step in identifying the state space model (1) and (2) is to solve the following least square problem,

$$Y_t = C\Phi^s X_{t-s} + \Xi_0 \mathcal{Z}_{[t-s,t)} + E_t. \tag{3}$$

$\Xi_0 \triangleq \left[C\Phi^{s-1}B \; C\Phi^{s-1}K \; \cdots \; CB \; CK\right]$ contains the Markov parameters of the system, in terms of Φ. $X_{t-s} = \left[x(t-s) \; x(t-s+1) \; \cdots \; x(t-s+N-1)\right]$ is the sequence of the unknown initial states. $Y_t = \left[y(t) \; y(t+1) \; \cdots \; y(t+N-1)\right]$ and $E_t = \left[e(t) \; e(t+1) \; \cdots \; e(t+N-1)\right]$ are respectively the future output and innovation sequence. The past I/O data are collected in the matrix

$$\mathcal{Z}_{[t-s,t)} = \begin{bmatrix} u(t-s) & u(t-s+1) & \cdots & u(t-s+N-1) \\ y(t-s) & y(t-s+1) & \cdots & y(t-s+N-1) \\ u(t-s+1) & u(t-s+2) & \cdots & u(t-s+N) \\ y(t-s+1) & y(t-s+2) & \cdots & y(t-s+N) \\ \vdots & \vdots & & \vdots \\ u(t-1) & u(t) & \cdots & u(t+N-2) \\ y(t-1) & y(t) & \cdots & y(t+N-2) \end{bmatrix}.$$

The subscript "$[t-s,t)$" stands for the range of the time index along the first column of the $\mathcal{Z}$ matrix; while the number of columns, N, is omitted to simplify the notations. To ensure $\mathcal{Z}_{[t-s,t)}$ has full row rank, we need to make the following assumption, which is the necessary and sufficient condition for persistently exciting the system of any order.

Assumption 3: [9], the spectrum of the joint input and output signals $\mathbf{z}(k) = \left[u^T(k) \; y^T(k)\right]^T$ (denoted by $\mathbf{\Phi_z}$) is bounded and bounded away from zero on the unit circle, i.e. $\exists 0 < c \leq M < \infty$, s.t. $cI \leq \mathbf{\Phi_z}(e^{j\omega}) \leq MI$, $\forall \omega \in [0, 2\pi)$.

In the sequel, we shall denote the estimated Ξ_0 by $\hat{\Xi}_0 = \left[\overline{C\Phi^{s-1}B} \; \overline{C\Phi^{s-1}K} \; \cdots \; \overline{CB} \; \overline{CK}\right]$. With $\hat{\Xi}_0$, we are able to predict the future output trajectory.

B. The Closed-loop SPC

Let the prediction horizon be f. To distinguish with $\mathcal{Z}_{[t-s,t)}$ in the identification problem, we use $\bar{Z}_{[k-s,k)} = \left[u(k-s)^T \; y(k-s)^T \; \cdots \; u(k-1)^T \; y(k-1)^T\right]^T$ to represent the past I/Os in the control problem. It is shown in [5] that the future f step ahead output predictor, with true plant Markov parameters, takes the following form,

$$\hat{\mathbf{y}}_{[k,k+f)} = \underbrace{\begin{bmatrix} C\Phi^s x(k-s) \\ C\Phi^s x(k-s+1) \\ \vdots \\ C\Phi^s x(k-s+f-1) \end{bmatrix}}_{\mathbf{b}_x} + \begin{bmatrix} \Xi_0 \\ \Xi_1 \\ \vdots \\ \Xi_{f-1} \end{bmatrix} \bar{Z}_{[k-s,k)}$$
$$+ \begin{bmatrix} 0 & & & \\ \Psi_1 & 0 & & \\ \vdots & \ddots & \ddots & \\ \Psi_{f-1} & \cdots & \Psi_1 & 0 \end{bmatrix} \cdot \begin{bmatrix} u(k) \\ y(k) \\ \vdots \\ u(k+f-1) \\ y(k+f-1) \end{bmatrix}, \tag{4}$$

where $\Psi_\tau \triangleq C\Phi^{\tau-1}\left[B \; K\right]$, $\tau = 1, \cdots, f-1$; and $\Xi_i = \left[0_{l\times i(m+\ell)} \; C\Phi^{s-1}B \; C\Phi^{s-1}K \; \cdots C\Phi^i B \; C\Phi^i K\right]$ is simply a right-shifted and zero-padded version of Ξ_0. Here we use $0_{m\times n}$ to represent an m-by-n zero matrix; and I_m an m-dimensional identity matrix. $\mathbf{b}_x$ is not known. We can either simply ignore it by choosing a large s as in all the existing SPCs; or treat it as a deterministic disturbance as in [13]. If both $\mathbf{b}_x$ and the estimation errors in the Markov parameters are ignored, we shall refer to (4), as the "nominal deterministic predictor", denoted by $\hat{\mathbf{y}}^d_{[k,k+f)} \triangleq \left[(\hat{y}^d(k))^T \; (\hat{y}^d(k+1))^T \; \cdots \; (\hat{y}^d(k+f-1))^T\right]^T$. Let $\mathbf{u}_{[k,k+f-1)} \triangleq \left[u^T(k) \; u^T(k+1) \; \cdots \; u^T(k+f-2)\right]^T$ be the future control inputs. The following lemma computes $\hat{\mathbf{y}}^d_{[k,k+f)}$ from the identified $\hat{\Xi}_0$.

Lemma 1: The nominal deterministic predictor for the future f outputs is

$$\hat{\mathbf{y}}^d_{[k,k+f)} = \Gamma \bar{Z}_{[k-s,k)} + \Lambda \mathbf{u}_{[k,k+f)},$$
$$\Gamma = \begin{bmatrix} \Gamma_0 \\ \Gamma_1 \\ \Gamma_2 \\ \vdots \\ \Gamma_{f-1} \end{bmatrix}, \Lambda = \begin{bmatrix} 0 & & & & \\ \Lambda_1 & 0 & & & \\ \Lambda_2 & \Lambda_1 & 0 & & \\ \vdots & \vdots & \ddots & \ddots & \\ \Lambda_{f-1} & \Lambda_{f-2} & \cdots & \Lambda_1 & 0 \end{bmatrix} \tag{5}$$

where the parameters, $\{\Gamma_i, \; \Lambda_j | i,j = 1, \cdots, f-1\}$, are derived from

$$\begin{array}{l} \Gamma_i = \hat{\Xi}_i + \sum_{\tau=0}^{i-1} \overline{C\Phi^{i-\tau-1}K} \cdot \Gamma_\tau, \; \Gamma_0 = \hat{\Xi}_0, \\ \Lambda_j = \overline{C\Phi^{j-1}B} + \sum_{\tau=1}^{j-1} \overline{C\Phi^{j-\tau-1}K} \cdot \Lambda_\tau, \; \Lambda_1 = \overline{CB}, \end{array} \tag{6}$$

$$\hat{\Xi}_i = \left[0_{l\times i(m+\ell)}, \overline{C\Phi^{s-1}B}, \overline{C\Phi^{s-1}K}, \cdots, \overline{C\Phi^i B}, \overline{C\Phi^i K}\right].$$

Proof: See [5]. ■

Remark 1: Note that in (4), the true $\mathbf{y}_{[k,k+f-1)}$ is used on the right hand side. If we substitute $\mathbf{y}_{[k,k+f-1)}$ with $\hat{\mathbf{y}}_{[k,k+f-1)}$, then we have to modify the bias term $\mathbf{b}_x$ by

$$\widetilde{\mathbf{b}}_x = \underbrace{\begin{bmatrix} L_1 & & & \\ L_2 & L_1 & & \\ \vdots & \vdots & \ddots & \\ L_f & L_{f-1} & \cdots & L_1 \end{bmatrix}}_{\mathcal{L}} \mathbf{b}_x, \tag{7}$$

where $L_j = \sum_{\tau=1}^{j-1} \overline{C\Phi^{j-1-\tau}K} \cdot L_\tau$, $j = 2, \cdots, f$ with $L_1 = I_l$. This can be verified by propagating $\{C\Phi^s x(k-s), \cdots, C\Phi^s x(k-s+i-1)\}$ to $\hat{y}(k+i)$, $1 \leq i \leq f-1$. In fact, the bias $\mathbf{b}_x$ contains the unknown states, which are however negligible under the conditions stated in the next two sections. The expression, (7), is particularly interesting in establishing the equivalence of the closed-loop SPC with the classical LQG.

The closed-loop SPC in [5] is based on this nominal deterministic predictor. Consider the following quadratic cost function,

$$J(k) = \|\hat{\mathbf{y}}^d_{[k,k+f)}\|_Q^2 + \|\mathbf{u}_{[k,k+f)}\|_R^2, \tag{8}$$

where $Q, R \succ 0$ are weighting matrices. $\|v\|_Q^2 \triangleq v^T Q v$ defines a weighted 2-norm. Then the solution to the unconstrained closed-loop SPC design problem,

$$\mathbf{u}_{[k,k+f)} = \arg \min_{\mathbf{u}_{[k,k+f)}} J(k), \tag{9}$$

has the following closed form

$$\mathbf{u}^*_{[k,k+f)} = -(R + \Lambda^T Q \Lambda)^{-1} \Lambda^T Q \Gamma \bar{Z}_{[k-s,k)}. \tag{10}$$

We shall refer to the constrained SPC problem to [5]; while only focus on the unconstrained LQG solution, (10), in this paper. The goal is to establish the equivalence of (10) with the classical LQG design.

III. The equivalence of the infinite-horizon case with the classical LQG

A. The explicit form of Λ_j and L_j

Before establishing the equivalence, we need to explore the information hidden behind the implicit form of Λ_j in (6) and L_j in (7). We emphasize that the results in this subsection hold no matter how long the data horizon is. It is straightforward to prove the following statement by induction.

Lemma 2: Λ_j and L_j respectively estimate the Markov parameters from the input and the noise channels to the output channels; i.e.

$$\Lambda_j = \overline{CA^{j-1}B}, \tag{11}$$
$$L_j = \overline{CA^{j-2}K}. \tag{12}$$

B. The closed-loop identification with infinite data horizon

The following assumption explains rigorously what we mean by "infinite data horizon".

Assumption 4: [9], the past horizon s goes to infinity with the length N while satisfying, $s \geq \frac{\log N^{-d/2}}{\log|\rho|}$ with $1 < d < \infty$ and $s = o(\log(N)^\alpha)$ with $\alpha < 1$.

Lemma 3: Under Assumption 1, 2, 3, and 4,

$$\hat{\Xi}_0 = Y_t \cdot \mathcal{Z}^\dagger_{[t-s,t)} \tag{13}$$

is an unbiased estimate of Ξ_0. "$\dagger$" stands for pseudo inverse.

Proof: Since by Assumption 2, $\lim_{s\to\infty} \|\Phi^s\|_2 = 0$; the bias term, $\lim_{s\to\infty} C\Phi^s X_{t-s} = 0$, where $\|X_{t-s}\|_2 < \infty$ under Assumption 3. On the other hand, note that in (3), each column of E_t is at least one step ahead in the future of the I/Os along the corresponding column of $\mathcal{Z}_{[t-s,t)}$. Due to the causality of the system, $\lim_{N\to\infty} \frac{1}{N} E_t \cdot \mathcal{Z}^T_{[t-s,t)} = 0$. Then $\hat{\Xi}_0$ is unbiased. ■

Remark 2: Note that $\hat{\Xi}_0$ is a random variable, although it is unbiased. The covariance matrix of the vectorized estimation error, $vec(\hat{\Xi}_0 - \Xi_0)$, depends on the signal to noise ratio (SNR) of the identification signals. The higher the ratio, the less uncertain the estimate $\hat{\Xi}_0$ is. See [13] for details.

C. Infinite-horizon closed-loop SPC

Define the state-space model hidden behind the unbiased estimates of the Markov parameters in (11) and (12) as

$$x(k+1) = \bar{A}x(k) + \bar{B}u(k) + \bar{K}e(k) \tag{14}$$
$$y(k) = \bar{C}x(k) + e(k), \tag{15}$$

The "bar" over A, B, C, K emphasizes the randomness of the identified Markov parameters. When the SNR of the identification data is high enough, the randomness of the identified A, B, C, K can be neglected.

Define the extended observability matrix and the Toeplitz Markov parameter matrix of this model as,

$$\mathcal{O}_f = \begin{bmatrix} \overline{C} \\ \overline{CA} \\ \vdots \\ \overline{CA^{f-1}} \end{bmatrix}, \ \mathcal{H}_f = \begin{bmatrix} 0 & & & \\ \overline{CB} & 0 & & \\ \vdots & \vdots & \ddots & \\ \overline{CA^{f-2}B} & \overline{CA^{f-3}B} & \cdots & 0 \end{bmatrix}.$$

The equivalence of the SPC design in [2] with the classical LQG is briefly reviewed in the appendix. We establish the same equivalence for the closed-loop SPC design (10) in the following theorem.

Theorem 1: With $N, s, f \to \infty$, the closed-loop SPC (10), is equivalent to (26). If in addition, the closed-loop SPC is started from a stable equilibrium of the system ((14) and (15)), then it produces the same stabilizing control input $u^*(k)$ as resulted from the steady-state solution of the classical LQG-controller (28).

Proof: Obviously, under Assumption 4, $\lim_{s\to\infty} \|\widetilde{\mathbf{b}}_x\|_2 = 0$, provided the closed-loop plant is internally stable prior to the time instance $k - s + f - 1$. This is clearly true, if the system stays at a stable equilibrium before starting the closed-loop SPC. If $u^*(k)$ is indeed equivalent to the steady-state solution of the classical LQG-controller (28), then it stabilizes the closed-loop system, and ensures $\|\widetilde{\mathbf{b}}_x\|_2 < \infty$ for ever. In this case, we do not have to estimate the states in $\widetilde{\mathbf{b}}_x$; because no matter being estimated or true, they simply vanish due to the zero weighting $C\Phi^s$. We will only use $\widetilde{\mathbf{b}}_x$ as a dummy variable in this proof. And we shall use $\overline{C\Phi^s}$ to replace $C\Phi^s$.

We only need to establish the equivalence between (10) and (26). Then what remains is completely the same with the proof in [2], and shall be omitted. Since from Lemma 2, $\Lambda = \mathcal{H}_f$; we are only left with showing $\Gamma\bar{Z}_{[k-s,k)} = \mathcal{O}_f \hat{x}(k|k-1)$. $\hat{x}(k|k-1)$ is the state estimated from the steady-state Kalman filter ($q = k-s, \cdots, k-1$),

$$\hat{x}(q+1|q) = \bar{A}\hat{x}(q|q-1) + \bar{B}u(q) + \bar{K}\big[y(q) - \bar{C}\hat{x}(q|q-1)\big], \tag{16}$$

which is stable from Assumption 2. It is in fact a common practice to treat $\bar{K}$ as a steady-state Kalman gain in the subspace identification literature, [2], [14].

It is straightforward to see that

$$\begin{aligned} &\Gamma_0 \bar{Z}_{[k-s,k)} + \overline{C\Phi^s} x(k-s) = \bar{C}\hat{x}(k|k-1), \\ &\Gamma_1 \bar{Z}_{[k-s,k)} + \overline{CK} \cdot \overline{C\Phi^s} x(k-s) + \overline{C\Phi^s} x(k-s+1) \\ = \ & \overline{CA} \cdot \hat{x}(k|k-1). \end{aligned}$$

Now, suppose $\forall 0 \leq p \leq i$,

$$\Gamma_p \cdot \bar{Z}_{[k-s,k)} + \begin{bmatrix} L_{p+1} & L_p & \cdots & L_1 \end{bmatrix} \cdot \begin{bmatrix} \overline{C\Phi^s}x(k-s) \\ \overline{C\Phi^s}x(k-s+1) \\ \vdots \\ \overline{C\Phi^s}x(k-s+p) \end{bmatrix} = \overline{CA^p} \cdot \hat{x}(k|k-1).$$

Then for $p = i+1$,

$$
\begin{aligned}
& \Gamma_{i+1} \cdot \bar{Z}_{[k-s,k)} + \left[\begin{array}{cccc} L_{i+2} & L_{i+1} & \cdots & L_1 \end{array}\right] \cdot \\
& \left[\begin{array}{c} \overline{C\Phi^s} x(k-s) \\ \overline{C\Phi^s} x(k-s+1) \\ \vdots \\ \overline{C\Phi^s} x(k-s+i+1) \end{array}\right] \\
= \; & (\hat{\Xi}_{i+1} + \sum_{j=0}^{i} \overline{C\Phi^{i-j}K} \cdot \Gamma_j) \cdot \bar{Z}_{[k-s,k)} + \\
& \overline{C\Phi^s} x(k-s+i+1) + \left[\begin{array}{ccc} \overline{C\Phi^i K} & \cdots & \overline{CK} \end{array}\right] \cdot \\
& \left[\begin{array}{cccc} L_1 & & & \\ L_2 & L_1 & & \\ \vdots & \vdots & \ddots & \\ L_{i+1} & L_i & \cdots & L_1 \end{array}\right] \cdot \left[\begin{array}{c} \overline{C\Phi^s} x(k-s) \\ \overline{C\Phi^s} x(k-s+1) \\ \vdots \\ \overline{C\Phi^s} x(k-s+i) \end{array}\right] \\
= \; & \overline{C\Phi^s} x(k-s+i+1) + \hat{\Xi}_{i+1} \bar{Z}_{[k-s,k)} + \\
& \left[\begin{array}{ccc} \overline{C\Phi^i K} & \cdots & \overline{CK} \end{array}\right] \cdot \left(\left[\begin{array}{c} \Gamma_0 \\ \vdots \\ \Gamma_i \end{array}\right] \cdot \bar{Z}_{[k-s,k)} + \right. \\
& \left. \left[\begin{array}{ccc} L_1 & & \\ \vdots & \ddots & \\ L_{i+1} & \cdots & L_1 \end{array}\right] \cdot \left[\begin{array}{c} \overline{C\Phi^s} x(k-s) \\ \vdots \\ \overline{C\Phi^s} x(k-s+i) \end{array}\right] \right) \\
= \; & \overline{C\Phi^{i+1}} \cdot \hat{x}(k|k-1) + \left[\begin{array}{ccc} \overline{C\Phi^i K} & \cdots & \overline{CK} \end{array}\right] \cdot \\
& \left[\begin{array}{c} \bar{C} \\ \overline{CA} \\ \vdots \\ \overline{CA^i} \end{array}\right] \cdot \hat{x}(k|k-1) \\
= \; & \left(\overline{C\Phi^{i+1}} + \left[\begin{array}{ccc} \overline{C\Phi^i K} & \cdots & \overline{CK} \end{array}\right] \cdot \left[\begin{array}{c} \bar{C} \\ \vdots \\ \overline{CA^i} \end{array}\right] \right) \cdot \\
& \hat{x}(k|k-1) \\
= \; & \overline{CA^{i+1}} \cdot \hat{x}(k|k-1),
\end{aligned}
$$

which indicates that indeed $\Gamma \bar{Z}_{[k-s,k)} = \mathcal{O}_f \hat{x}(k|k-1)$. This completes the proof. ∎

IV. The finite-horizon closed-loop SPC and LQG

Since the closed-loop SPC avoids the intermediate steps of identifying the parametric matrices in a state-space form, they are particularly attractive for adaptively tuning predictive controllers online, because the estimation of the Markov parameters is simply a least square problem, [5]. This, however, limits the data horizon to finitely long. In this case, the estimate, (3), is no longer unbiased. Besides, the term, $\widetilde{\mathbf{b}}_x$, is not zero any more. The rigorous treatment of these biases is more involved than the scope of this paper, and is presented in [13]. Due to the scaling by $C\Phi^s$, the biases can in fact be made arbitrarily small by choosing a sufficiently large s as long as allowed by the hardware capacity. Specifically, we shall make the following assumption to simplify the analysis in this paper.

Assumption 5: The SNR of the identification signals is so large that the identified Markov parameters is almost certain. In addition, the past horizon s is large enough to neglect the biases, $C\Phi^s X_{t-s}$ and $\widetilde{\mathbf{b}}_x$, respectively in the identified Markov parameters and the future output predictor.

Under this assumption, we can ignore the model-plant mismatch, in terms of the Markov parameters. Now, we are ready to propose a finite-horizon LQG solution for the closed-loop SPC design, based on the nominal deterministic estimates in (13). The derivation is mainly based on [12].

A. Optimal state estimation with finite data horizon

As argued in the introduction, although (10) minimizes the quadratic cost of (8), it is not directly an LQG design, in the sense that no optimality can be claimed for estimating the states hidden behind the input and output relation.

An optimal state estimator is developed in [12] by minimizing a mean squared estimation error,

$$\mathcal{J} = \mathbb{E}\big[\|\xi^T x(k) - \hat{x}(k)\|_2^2\big], \tag{17}$$

for some scaling factor ξ; where $\hat{x}(k)$ is the linear combination of the past inputs and outputs,

$$\hat{x}(k) = \underbrace{\left[\begin{array}{ccccc} c_{k-s}^T & d_{k-s}^T & \cdots & c_{k-1}^T & d_{k-1}^T \end{array}\right]}_{\mathcal{K}} \cdot \bar{Z}_{[k-s,k)}. \tag{18}$$

A closed-form optimal solution is derived in [15], [12], which is equivalent to a classical Kalman filter, derived from Riccati recursions, [14].

B. Finite horizon LQG solution

The classical LQG is a state feedback design, i.e.

$$u^*(k) = \mathcal{T}_k \cdot \hat{x}(k),$$

with $\hat{x}(k)$ an estimate of $x(k)$ from a Kalman filter. Substitute (18) into the above equation,

$$u^*(k) = \mathcal{T}_k \cdot \mathcal{K} \cdot \bar{Z}_{[k-s,k)}.$$

It is shown in [12] that the control gain $\mathcal{T}_k \cdot \mathcal{K}$ can be completely determined by the plant Markov parameters. In what follows, we shall develop such a solution based on the identified Markov parameters.

Define the controlled outputs of the system, (14) and (15), as,

$$z(k) = \underbrace{\left[\begin{array}{c} q \cdot \bar{C} \\ 0 \end{array}\right]}_{C_z} x(k) + \underbrace{\left[\begin{array}{c} 0 \\ \mathcal{R} \end{array}\right]}_{D_z} u(k), \tag{19}$$

where $q > 0$ is a tuning factor; and $\mathcal{R}$ is the Cholesky factorization of the weight on the input at each time instance; i.e. taking the weighting matrix R in (8) as $diag(\mathcal{R}^T\mathcal{R}, \cdots, \mathcal{R}^T\mathcal{R}) \succ 0$. Consider now the cost function

$$\mathcal{J}(k) = \sum_{\tau=0}^{f-1} z^T(k+\tau) z(k+\tau) = \|\mathbf{x}_{[k,k+f)}\|_{\bar{Q}}^2 + \|\mathbf{u}_{[k,k+f)}\|_R^2, \tag{20}$$

where $\bar{Q} = diag(q^2 \cdot \bar{C}^T\bar{C}, \cdots, q^2 \cdot \bar{C}^T\bar{C})$.

Let the past and future horizon be $s = f = 2h$. Collect the h I/Os prior to the current time k in the vector forms as

$$\tilde{\mathbf{u}}_{[k-h,k)} = \left[\begin{array}{c} u(k-1) \\ \vdots \\ u(k-h) \end{array}\right] \text{and}\ \tilde{\mathbf{y}}_{[k-h,k)} = \left[\begin{array}{c} y(k-1) \\ \vdots \\ y(k-h) \end{array}\right].$$

We summarize the closed-form solution to (20) in the following theorem.

Theorem 2: If Assumption 5 holds, then the finite horizon LQG design can be completely determined by the Markov parameters identified from closed-loop I/O signals. The optimal control input at the current time instance k takes the following closed form,

$$\begin{aligned} &u^*(k) = -\left(\mathcal{R}^T\mathcal{R} + \mathcal{N}^T\mathcal{P}^T\Pi\mathcal{P}\mathcal{N}\right)^{-1}\cdot \\ &\left[\mathcal{N}^T\mathcal{P}^T\Pi\mathcal{P}\mathcal{F}\cdot\tilde{\mathbf{u}}_{[k-h,k)} - \mathcal{N}^T\mathcal{P}^T\Pi\mathcal{P}\mathcal{G}\right. \\ &\left.\cdot\mathcal{W}^T(\mathcal{W}\mathcal{W}^T)^\dagger\cdot(\mathcal{V}\tilde{\mathbf{u}}_{[k-h,k)} - \tilde{\mathbf{y}}_{[k-h,k)})\right], \end{aligned} \tag{21}$$

where the parametric matrices are defined in (22); and

$$\Pi = I - \mathcal{D}(\mathcal{D}^T\mathcal{D})^\dagger\mathcal{D}^T.$$

Remark 3: The stabilizing finite-horizon SPC control law can be found if the horizon $h \geq n$, [12]. This condition, as well as Assumption 5, requires to set h to a large integer. Note that the other three conditions mentioned in [12] are automatically satisfied in the design; i.e. the open-loop system is assumed to be minimal and therefore contains no unstable hidden modes; D_z has full column rank (since $\mathcal{R}^T\mathcal{R} \succ 0$); and the feed-through matrix I from the noise to the output, in the model (14) and (15), has full row rank.

Three steps are needed to implement the control law.

1) solve (13) for $\hat{\Xi}_0$;
2) compute the Markov parameters $\{\overline{C\Phi^\tau B},\ \overline{C\Phi^\tau K}|\tau = 0, \cdots, 2h-1\}$ according to (6) and (7);
3) compute the input at the current instance k by (21).

V. A SIMULATION EXAMPLE

As a case study, we consider a reference-tracking problem of a linearized VTOL (vertical takeoff and landing) aircraft model, originally appeared in [16],

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{23}$$

$$y(t) = Cx(t) + e(t) \tag{24}$$

$$A = \begin{bmatrix} -0.0366 & 0.0271 & 0.0188 & -0.4555 \\ 0.0482 & -1.01 & 0.0024 & -4.0208 \\ 0.1002 & 0.3681 & -0.707 & 1.42 \\ 0 & 0 & 1 & 0 \end{bmatrix},$$

$$B = \begin{bmatrix} 0.4422 & 0.1761 \\ 3.5446 & -7.5922 \\ -5.52 & 4.49 \\ 0 & 0 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix},$$

where $e(t)$ is a 4-dimensional zero-mean white noise, with a covariance of $\mathcal{S}\mathcal{S}^T = 10^{-12}\cdot I_4$, added as the measurement noise. The model is discretized with a sampling period of 0.1 second. No stochastic disturbance term, "$Ke(t)$", appears in (23), as assumed in (1). We shall use the VTOL model to demonstrate the generality of the proposed approaches.

$N = 10000$ I/O samples are generated to identify the output predictor, from a closed-loop experiment with an initial stabilizing controller, $u(k) = Lx(k) + d(k)$, where the dither signals $d(k)$ are a 2-dimensional zero-mean white noise with a unity covariance matrix. The state feedback gain, L, is computed from the LQR control law.

The past and future horizon are respectively set as $s = 30$ and $f = 30$; and hence $h = 15$. In this case, $\|Cov(vec(\hat{\Xi}_0 - \Xi_0))\|_2 = 4.2\times10^{-3}$ and $\|\overline{C\Phi^{s-1}B}\|_2 = 4.4\times10^{-2}$, $\|\overline{C\Phi^{s-1}K}\|_2 = 7.4\times10^{-2}$. Assumption 5 is clearly satisfied. The biases and stochastic errors in the identification and control can therefore be safely ignored. Choose $q = 10^6, Q = qI$, and $\mathcal{R} = I$. Three schemes are tested; namely (10), (21), and the classical LQG law of $u(k) = \bar{L}\hat{x}(k)$, where $\hat{x}(k)$ is estimated by the steady-state Kalman filter with gain $\bar{K}$. The simulation results are shown in figure 1. Clearly, the closed-loop SPC leads to similar closed-loop response with the classical LQG, which verifies their equivalence. Starting from an earlier instance, the control law, (21), also stabilizes the system.

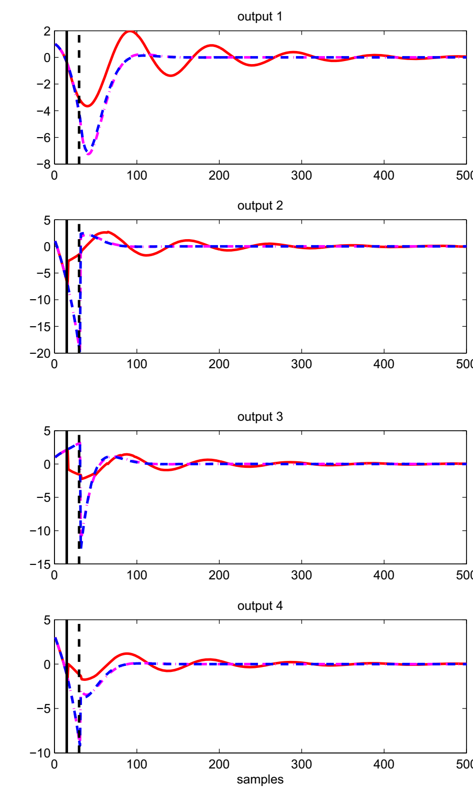

Fig. 1. The outputs of the VTOL control system: red solid, (21); blue dashed, the classical LQG; purple dash-dotted, (10); black bold solid, the starting instance of (21) after the first $h = 15$ samples without control efforts; black bold dashed, the starting instance of (10) and the LQG after the first $s = 30$ samples without control efforts.

VI. CONCLUSION

We have established in this paper the equivalence of the closed-loop SPC in [5] with the classical LQG when the data horizon is infinite. When this horizon is finite, the solution of the closed-loop SPC problem by the LQG design in [12] ensures the optimality both in the state estimation step hidden behind the input and output relation and in the control gain, just based on the identified Markov parameters. The future research shall be focused on ensuring the closed-loop stability in the finite-horizon design.

$$\mathcal{N} = \begin{bmatrix} q\cdot\overline{CB} \\ q\cdot\overline{CAB} \\ \vdots \\ q\cdot\overline{CA^{h-2}B} \end{bmatrix}, \mathcal{F} = \begin{bmatrix} q\cdot\overline{CAB} & \cdots & q\cdot\overline{CA^{h}B} \\ \vdots & & \vdots \\ q\cdot\overline{CA^{h-1}B} & \cdots & q\cdot\overline{CA^{s-2}B} \end{bmatrix}, \mathcal{G} = \begin{bmatrix} q\cdot\overline{CAK} & \cdots & q\cdot\overline{CA^{h}K} \\ \vdots & & \vdots \\ q\cdot\overline{CA^{h-1}K} & \cdots & q\cdot\overline{CA^{s-2}K} \end{bmatrix},$$
$$\mathcal{P} = \begin{bmatrix} I_l & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ 0 & I_l & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & I_l \\ 0 & 0 & \cdots & 0 \end{bmatrix}, \mathcal{D} = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ \mathcal{R} & 0 & \cdots & 0 & 0 \\ q\cdot\overline{CB} & 0 & \cdots & 0 & 0 \\ 0 & \mathcal{R} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ q\cdot\overline{CA^{h-3}B} & q\cdot\overline{CA^{h-4}B} & \cdots & q\cdot\overline{CB} & 0 \\ 0 & 0 & \cdots & 0 & \mathcal{R} \end{bmatrix}, \tag{22}$$
$$\mathcal{W} = \begin{bmatrix} I & \overline{CK} & \cdots & \overline{CA^{h-2}K} \\ & \ddots & \ddots & \vdots \\ & & \ddots & \overline{CK} \\ & & & I \end{bmatrix}, \mathcal{V} = \begin{bmatrix} 0 & \overline{CB} & \cdots & \overline{CA^{h-2}B} \\ & \ddots & \ddots & \vdots \\ & & \ddots & \overline{CB} \\ & & & 0 \end{bmatrix}$$

APPENDIX
THE SPC DESIGN IN [2]

In [2], the following SPC control design minimizing the cost (8) is proposed

$$\mathbf{u}^*_{[k,k+f)} = -(R + L_u^T Q L_u)^{-1} L_u^T Q L_w \bar{Z}_{[k-s,k)}. \tag{25}$$

L_u, L_w are estimated from N4SID subspace identification, which define the future output predictor, i.e.

$$\hat{\mathbf{y}}^d_{[k,k+f)} = L_u \mathbf{u}_{[k,k+f)} + L_w \bar{Z}_{[k-s,k)}.$$

It is shown in [2] that under the assumption that $s = f \to \infty$, the right hand side of (25) is equivalent to

$$\mathbf{u}^*_{[k,k+f)} = -(R + \mathcal{H}_f^T Q \mathcal{H}_f)^{-1} \mathcal{H}_f^T Q \mathcal{O}_f \hat{x}(k|k-1), \tag{26}$$

where $\hat{x}(k|k-1)$ is the state estimate by the Kalman filter,

$$\begin{aligned} &\hat{x}(q+1|q) = \bar{A}\hat{x}(q|q-1) + \bar{B}u(q) + \\ &\qquad K_q\big[y(q) - \bar{C}\hat{x}(q|q-1)\big], \\ &K_q = (\bar{K}\mathcal{S}\mathcal{S}^T + \bar{A}\Sigma_q\bar{C}^T)(\mathcal{S}\mathcal{S}^T + \bar{C}\Sigma_q\bar{C}^T)^{-1}, \\ &\Sigma_{q+1} = \bar{A}\Sigma_q\bar{A}^T + \bar{K}\mathcal{S}\mathcal{S}^T\bar{K}^T - (\bar{K}\mathcal{S}\mathcal{S}^T + \bar{A}\Sigma_q\bar{C}^T) \\ &\qquad \cdot(\mathcal{S}\mathcal{S}^T + \bar{C}\Sigma_q\bar{C}^T)^{-1}(\bar{K}\mathcal{S}\mathcal{S}^T + \bar{A}\Sigma_q\bar{C}^T)^T, \end{aligned} \tag{27}$$

where $q = k-s, \cdots, k-1$. Since $s \to \infty$, the Kalman filter is at the steady state; meaning that the Kalman gain is the steady-state gain; and the influence of the initial state $\hat{x}(k-s)$ dies out.

On the other hand, in the classical LQR design, the input $u(k)$ that minimizes the cost (8) is

$$u^*(k) = L_k \hat{x}(k|k-1), \tag{28}$$

where L_k is the solution to the following recursive equations,

$$\begin{aligned} &L_k^T = -(C^T Q_1 D + A^T P_f B)\cdot \\ &\quad (R_1 + D^T Q_1 D + B^T P_f B)^{-1}, \\ &P_{q+1} = A^T P_q A + C^T Q_{f-q} C - (C^T Q_{f-q} D + \\ &\quad A^T P_q B)\cdot(R_{f-q} + D^T Q_{f-q} D + B^T P_q B)^{-1}\cdot \\ &\quad (C^T Q_{f-q} D + A^T P_q B)^T, \end{aligned} \tag{29}$$

where $q = 1, \cdots, f-1$. $Q_i \in \mathbb{R}^{l\times l}$ is the i-th diagonal block of Q. When $f \to \infty$, L_k results from the steady-state Riccati equation, (29).

It is proved in [2] that when $s = f \to \infty$ the control input at the current time instance, $u^*(k)$, from the SPC design (25) is equivalent to the classical LQG design of (28).

REFERENCES

[1] K. Åström and B. Wittenmark, *Computer controlled systems: theory and design*. Prentice Hall, 1984.

[2] W. Favoreel, B. de Moor, M. Gevers, and P. van Overschee, "Model-free subspace-based LQG-design," *Tech. Rep. ESAT-SISTA/TR 1998-34, KU Leuven, Leuven, Belgium*, 1998.

[3] ——, "Closed-loop model-free subspace-based LQG-design," in *the Proceedings of the 7th Mediterranean Conference on Control and Automation*, Haifa, Israel, 1999.

[4] B. Woodley, *Model free subspace based $\mathcal{H}_\infty$ control, PhD Dissertation*. Stanford University, 2001.

[5] J. Dong, M. Verhaegen, and E. Holweg, "Closed-loop subspace predictive control for fault tolerant MPC design," in *the Proceedings of the 17th IFAC World Congress*, Seoul, South Korea, 2008, pp. 3216–3221.

[6] P. van Overschee and B. de Moor, "N4SID: Subspace algorithms for the identification of combined deterministic-stochastic systems," *Automatica*, vol. 36, pp. 75–93, 1994.

[7] E. Hale and S. Qin, "Subspace model predictive control and a case study," in *the Proceedings of American Control Conference*, Anchorage, US, 2002, pp. 4758–4763.

[8] A. Chiuso, "The role of vector autoregressive modeling in predictor based subspace identification," *Automatica*, vol. 43, pp. 1034–1048, 2007.

[9] ——, "On the relation between CCA and predictor-based subspace identification," *IEEE Transactions on Automatic Control*, vol. 52, pp. 1795–1812, 2007.

[10] P. van Overschee and B. de Moor, "Closed-loop subspace identification," *Tech. Rep. ESAT-SISTA/TR 1996-52I, KU Leuven, Leuven, Belgium*, 1996.

[11] K. Furuta and M. Wongsaisuwan, "Closed-form solutions to discrete-time LQ optimal control and disturbance attenuation," *Systems & Control Letters*, vol. 20, pp. 427–437, 1993.

[12] ——, "Discrete-time LQG dynamic controller design using plant markov parameters," *Automatica*, vol. 31, pp. 1317–1324, 1995.

[13] J. Dong and M. Verhaegen, "Model free probabilistic design with uncertain markov parameters identified in closed loop," in *accepted for publication in the Proceedings of the 47th IEEE Conference on Decision and Control*, Cancun, Mexico, 2008.

[14] M. Verhaegen and V. Verdult, *Filtering and System Identification: A Least Squares Approach*. Cambridge University Press, 2007.

[15] K. Furuta, "Alternative solution of discrete-time kalman filter," *Systems & Control Letters*, vol. 22, pp. 429–435, 1994.

[16] K. Narendra and S. Tripathi, "Identification and optimization of aircraft dynamics," *Journal of Aircraft*, vol. 10(4), pp. 193–199, 1973.