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H_2 gain scheduling control for rational LPV systems using the descriptor framework

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Abstract— **This paper provides new LMI-based conditions for the design of H_2 gain scheduled controllers for rational linear parameter varying systems (LPV). Such systems are equivalently recast as affine descriptor LPV systems. Based on this, new sufficient LMI-based conditions for H_2 performance analysis are proposed. These conditions can be turned to a finite set of LMIs and allow the use of parameter-dependent Lyapunov functions. Accordingly, new LMI conditions for the H_2 gain scheduled controller synthesis problem are given. A numerical example highlights the effectiveness of the proposed conditions.**

I. Introduction

The interest for analysis and control of linear parameter varying models has been growing during the last two decades (see e.g. in [2], [16]). Such models potentially allow the description of a wide set of practical systems including some non-linear ones. Several approaches concerning gain scheduling control of LPV systems can be found in the literature. One of them consists into interpolating several invariant controllers tuned for different operating points. This classical gain scheduling design is obtained in three steps: define the operating points for the LPV system, design a linear time invariant controller for each of them and finally build the gain scheduled controller via interpolation techniques. Numerous schemes can be used, as for instance state-space matrices or poles-zeros-gains interpolations of the controllers (see e.g. [10], and [14]). This simple scheme often suffers from the drawback that the closed-loop stability is not guaranteed *a priori*. One exception can be found in [13], where the stability is ensured thanks to some additional constraints.

An alternative method to design gain scheduled controllers is the extension of the Lyapunov based LMI conditions known in the LTI case. The main advantage of this approach is that the closed-loop system stability is ensured. Recently, significant progress has been made in this area for some special class of LPV systems: e.g. LFT (linear fractional transformation), affine, polytopic, representations. In the case of polytopic systems, the design of a gain scheduled controller was considered in [1], [2], [6]. The particular case of the H_2 state-feedback control problem for polytopic systems is considered in [17]. Even if conditions proposed in [17] are less conservative than usual ones, they can hardly be exploited when considering a larger class of systems, as for instance rational LPV systems. Indeed, these conditions induce an infinite set of LMIs which makes it difficult to synthesize gain-scheduled state-feedback controllers for rational systems. To overcome these difficulties, parameterized LMIs can be solved as done in [12] thanks to full block S-procedure.

In this note, the design of H_2 gain scheduled controller for rational LPV systems is considered. Descriptor realizations are interesting tools for analysis and control of rational LPV systems (see e.g. [9], and [4]). This comes from the fact that state space equations with rational dependency on the parameter can be transformed into a polytopic descriptor form as pointed out in [8] , [9] and proved in [3]. Starting from this transformation, the present paper proposes new sufficient conditions for the design of H_2 gain scheduled controllers for rational LPV systems.

This paper is organized as follows: in section II, the class of systems considered is defined. The transformation of rational LPV systems into an equivalent descriptor realization is recalled, just as the H_2 norm evaluation for both classical state-space and descriptor LPV cases. In section III, a new strict LMI formulation is proposed for the characterization of H_2 performance analysis of rational LPV systems. Based on this, a gain scheduled controller achieving a given H_2 performance is proposed in section IV. Finally, a numerical example is presented in section V before concluding.

Notation: The notation $A > 0$ stands for A definite positive. The notation $He\{A\}$ stands for $A^T + A$ and $\bullet$ for terms that are induced by symmetry. $ker\{A\}$ (respectively $\mathrm{Img}\{A\}$) denotes the kernel (respectively the image) of matrix A.

II. Problem formulation

A. Class of systems considered

The LPV systems considered have the following form:

$$(\Sigma_r): \begin{cases} \dot{x}_1 = A_r(\theta)x_1 + B_{r_1}(\theta)w + B_{r_2}(\theta)u \\ z = C_{r_1}(\theta)x_1 \\ y = C_{r_2}(\theta)x_1 \end{cases} \tag{1}$$

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where the state matrices are assumed to be rational functions of the time varying vector $\theta(t)$.

$x_1(t) \in \mathbb{R}^{n_1}$ is the state, $w(t) \in \mathbb{R}^{n_w}$ the exogenous disturbance, $u(t) \in \mathbb{R}^{n_u}$ the input, $z(t) \in \mathbb{R}^{n_z}$ the controlled output and $y(t) \in \mathbb{R}^{n_y}$ is the measured output. The vector $\theta(t) = \begin{bmatrix}\theta_1(t) & \theta_2(t) & \ldots & \theta_q(t)\end{bmatrix}^T$ is assumed to be real, continuously time varying and satisfying the following constrains

i) Each parameter $\theta_i(t)$ is measured and ranges between *a priori* known extremal values $\theta_i(t) \in \begin{bmatrix}\underline{\theta}_i & \overline{\theta}_i\end{bmatrix}$.

ii) The variation rate of each parameter $\dot{\theta}_i(t)$ is limited as $\dot{\theta}_i(t) \in \begin{bmatrix}\underline{\tau}_i & \overline{\tau}_i\end{bmatrix}$.

As a consequence, the varying parameter is such that θ and $\dot{\theta}$ evolve respectively in the hyper-rectangles P and Ω:

$$\mathrm{P} = \left\{(\omega_1,...,\omega_{2^q}) \setminus \omega_i \in \left\{\underline{\theta}_i \quad \overline{\theta}_i\right\}\right\}$$

$$\Omega = \left\{(\tau_1,...,\tau_{2^q}) \setminus \tau_i \in \left\{\underline{\tau}_i \quad \overline{\tau}_i\right\}\right\}.$$

The set of the admissible trajectories of $\theta(t)$ is noted Θ.

(Σ_r) is supposed to admit a well-posed LFT representation. We assume, without loss of generality, that matrices $B_{r_i}(\theta)$, $C_{r_i}(\theta), i \in \{1,2\}$ are constant matrices. Otherwise this can be obtained using a pre-filtering technique as proposed in [1].

It has been shown in [3] that this class of systems can be equivalently recast into a descriptor realization:

$$(\Sigma_d): \begin{cases} E\dot{x} = A(\theta)x + B_1 w + B_2 u \\ z(t) = C_1 x \\ y(t) = C_2 x \end{cases} \tag{2}$$

where the descriptor state vector is partitioned as: $x = \begin{bmatrix}x_1^T & x_2^T\end{bmatrix}^T \in \mathbb{R}^n$ with $x_2 \in \mathbb{R}^{n_2}$, $n = n_1 + n_2$ while the state matrices are given by

$$E = \begin{pmatrix} I_{n_1} & 0 \\ 0 & 0 \end{pmatrix},\ A(\theta) = \begin{pmatrix} A_1(\theta) & A_2(\theta) \\ A_3(\theta) & A_4(\theta) \end{pmatrix},$$

$$\forall i \in \{1,2\}:\ B_i = \begin{pmatrix} \tilde{B}_i \\ 0 \end{pmatrix},\ C_i = \begin{pmatrix} \tilde{C}_i & 0 \end{pmatrix},$$

with $(A_i(\theta))_{i\in\{1,2,3,4\}}$ affine functions of $\theta(t)$ and $B_i, C_i, i \in \{1,2\}$ constant matrices. Moreover, the matrix $A_4(\theta) \in \mathbb{R}^{n_2 \times n_2}$ is non singular for all admissible trajectories $\theta(.) \in \Theta$. The following equations describe the relation between (Σ_r) and (Σ_d):

$$A_r(\theta) = A_1(\theta) - A_2(\theta)A_4(\theta)^{-1}A_3(\theta) \tag{3}$$

$$\forall i \in \{1,2\}:\ B_{r_i} = \tilde{B}_i \tag{4}$$

$$\forall i \in \{1,2\}:\ C_{r_i} = \tilde{C}_i \tag{5}$$

Remark 1: The non singularity of matrix $A_4(\theta)$ is due to the existence of a well-posed LFT representation of the rational system (Σ_r). Moreover, it can be shown that $A_4(\theta)^{-1}$ is continuously differentiable (see [3] for details).

B. H_2 cost function

1) H_2 performance index for LPV systems

Let us recall the notion of H_2 norm for the LPV system (Σ_r). Generally, the definition of the H_2 performance index has been extended to linear time-varying (LTV) systems as the L_2-norm of the output signal when the system input considered is either a unit impulse or a stationary white noise (see [11]). This second interpretation is considered from now on.

Definition 1: Let (Σ_r) be exponentially stable. The H_2 norm can be defined as

$$\|\Sigma_r\|_2^2 = \lim_{h\to\infty} \Xi\left\{\frac{1}{h}\int_0^h z^T(t)z(t)dt\right\}$$

when $x_1(0) = 0$ and $w(t)$ is a non zero-mean white process with an identity power spectrum density matrix. The symbol Ξ above denotes the mathematical expectation.

Following the lines of [17], (Σ_r) will be said H_2 γ-suboptimal if the H_2 performance index is such that $\|\Sigma_r\|_2 < \gamma$. A characterization of the H_2 performance index for a LPV system is recalled next.

Lemma 1: For a given $\gamma > 0$, the LPV system (Σ_r) is said exponentially stable and $\|\Sigma_r\|_2 < \gamma$ if there exists a continuously differentiable function $Q : \theta \to Q(\theta)$ of appropriate dimensions such that for all $\theta(\cdot)$:

$$Q(\theta) = Q(\theta)^T > 0 \tag{6}$$

$$He\{A_r(\theta)Q(\theta)\} + \frac{dQ(\theta)}{dt} + B_{r_1}B_{r_1}^T < 0 \tag{7}$$

$$tr\left\{C_{r_1}Q(\theta)C_{r_1}^T\right\} < \gamma^2 \tag{8}$$

2) H_2 performance index for LPV descriptor systems

Considering the class of descriptor systems described by (2), the H_2 index is characterized next.

Let us first remark that the following condition:

$$ker\{C_1\} \supseteq ker\{E\} \tag{9}$$

holds for system (2), since:

$$E = \begin{pmatrix} I_{n_1} & 0 \\ 0 & 0 \end{pmatrix} \text{ and } C_1 = \begin{pmatrix} \tilde{C}_1 & 0 \end{pmatrix}$$

This condition ensures the finiteness of the H_2 index (see [7] and [15]).

At second, let us consider the sufficient condition for the admissibility of (Σ_d), using the constant matrices

$U_0 = \begin{pmatrix} 0_{n_1 \times n_2} \\ I_{n_2} \end{pmatrix}$ and $U = \begin{pmatrix} I_{n_1} \\ 0_{n_2 \times n_1} \end{pmatrix}$.

Lemma 2: The descriptor LPV system (Σ_d) is admissible if there exist continuously differentiable functions $W : \theta \to W(\theta)$ and $S : \theta \to S(\theta)$ of appropriate dimension such that for all $\theta(.)$:

$$W(\theta) = W(\theta)^T > 0 \tag{10}$$

$$He\left\{ A(\theta)\left(UW(\theta)U^T + S(\theta)U_0^T\right)^T \right\} + U\left(\frac{d}{dt}\{W(\theta)\}\right)U^T < 0 \tag{11}$$

Proof: Is omitted for brevity reasons

The H_2 index for the descriptor system (2) is now characterized in next lemma.

Lemma 3: For a given scalar $\gamma > 0$, the descriptor LPV system (Σ_d) is admissible and $\|\Sigma_d\|_2 < \gamma$ if there exist continuously differentiable functions $W : \theta \to W(\theta)$ and $S : \theta \to S(\theta)$ of appropriate dimensions such that for all $\theta(.)$:

$$W(\theta) = W(\theta)^T > 0 \tag{12}$$

$$He\left\{ A(\theta)\left(UW(\theta)U^T + S(\theta)U_0^T\right)^T \right\} + U\left(\frac{d}{dt}\{W(\theta)\}\right)U^T + B_1 B_1^T < 0 \tag{13}$$

$$tr\left\{ C_1 U W(\theta) U^T C_1^T \right\} < \gamma^2 \tag{14}$$

Note that a dual version of **Lemma 3** can be given when considering the dual system of (2). If it is usual to assume the following condition

$$\mathrm{Img}\{E\} \supseteq \mathrm{Img}\{B_1\}$$

for the dual form of **Lemma 3**, in our case this condition naturally holds since we have:

$$E = \begin{pmatrix} I_{n_1} & 0 \\ 0 & 0 \end{pmatrix} \text{ and } B_1 = \begin{pmatrix} \tilde{B}_1 \\ 0 \end{pmatrix}$$

Remark 2: If $\left(UW(\theta)U^T + S(\theta)U_0^T\right)$ is non singular for all $\theta(.)$ with $W(\theta) > 0$ then there exist $\overline{W}(\theta)$ and $\overline{S}(\theta)$ such that

$$\left(UW(\theta)U^T + S(\theta)U_0^T\right)^{-1} = \left(U\overline{W}(\theta)U^T + \overline{S}(\theta)U_0^T\right)$$

with $\overline{W}(\theta) = W(\theta)^{-1} > 0$. This property is a direct consequence of the particular structure of matrix E (see [18] for more details) and will be useful when dealing with the H_2 gain scheduling control as it will be shown in the sequel.

Now that the H_2 performance index have been characterized for the descriptor LPV system (Σ_d) in **Lemma 3**, we shall examine the relations between the state-space realization (Σ_r) and the descriptor system (Σ_d) and their H_2 performance indexes.

C. *State-space rational LPV systems versus descriptor affine LPV systems*

In this paper, rational LPV systems are treated through a descriptor realization with an affine dependence on the varying parameter. In this subsection, we give some preliminary results concerning the relation between these two realizations. Let us consider two descriptor systems given by

$$\left(\overline{\Sigma}\right): \begin{cases} \overline{E}\dot{x} = \overline{A}(\theta)x + \overline{B}_1(\theta)w + \overline{B}_2(\theta)u \\ z(t) = \overline{C}_1(\theta)x \\ y(t) = \overline{C}_2(\theta)x \end{cases}$$

and

$$\left(\hat{\Sigma}\right): \begin{cases} \hat{E}\dot{\hat{x}} = \hat{A}(\theta)\hat{x} + \hat{B}_1(\theta)w + \hat{B}_2(\theta)u \\ z(t) = \hat{C}_1(\theta)\hat{x} \\ y(t) = \hat{C}_2(\theta)\hat{x} \end{cases}$$

Definition 2: "Strong Equivalence"

The descriptor realizations $\left(\overline{\Sigma}\right)$ and $\left(\hat{\Sigma}\right)$ are said *strongly equivalent* if and only if there exist two continuously differentiable functions $M : \theta \to M(\theta)$, $N : \theta \to N(\theta)$ such that:

i) $M(\theta)$ and $N(\theta)$ are non singular matrices

ii) $M(\theta)^{-1}$, $N(\theta)^{-1}$ are continuously differentiable

and the following equations hold

$$M(\theta)\overline{E}N(\theta) = \hat{E} \tag{15}$$

$$M(\theta)\overline{A}(\theta)N(\theta) = \hat{A}(\theta) \tag{16}$$

$$M(\theta)\overline{E}\frac{d}{dt}\left(N(\theta)\right) = 0 \tag{17}$$

$$\forall i \in \{1,2\}: \; M(\theta)\overline{B}_i(\theta) = \hat{B}_i(\theta) \tag{18}$$

$$\forall i \in \{1,2\}: \; \overline{C}_i(\theta)N(\theta) = \hat{C}_i(\theta) \tag{19}$$

We assume next that condition (9) hold for both systems $\left(\overline{\Sigma}\right)$ and $\left(\hat{\Sigma}\right)$. It's also assumed that $\overline{E} = \begin{pmatrix} I_{n_1} & 0 \\ 0 & 0 \end{pmatrix}$.

Theorem 1: Suppose that realizations $\left(\overline{\Sigma}\right)$ and $\left(\hat{\Sigma}\right)$ are strongly equivalent. If $\left(\overline{\Sigma}\right)$ is admissible and for a given $\gamma > 0$, $\left(\overline{\Sigma}\right)$ satisfies inequalities (12)-(14) for all $\theta(.)$, then $\left(\hat{\Sigma}\right)$ is admissible and achieves the same H_2 performance level $\gamma > 0$.

Proof: This result can be demonstrated following the same lines as done for the admissibility condition in [5]. The proof is omitted here for brevity reasons.

Note that the rational LPV system (Σ_r) and the descriptor system given by (Σ_d) are strongly equivalent. Indeed, when considering the following non singular and continuously differentiable functions (see **Remark 1**)

$$M(\theta)=\begin{pmatrix} I_{n_1} & -A_2(\theta)A_4(\theta)^{-1} \\ 0 & A_4(\theta)^{-1}\end{pmatrix},\ N(\theta)=\begin{pmatrix} I_{n_2} & 0 \\ -A_4(\theta)^{-1}A_3(\theta) & I_{n_1}\end{pmatrix}$$

it is easy to see that for all trajectories $\theta(.)$

$$M(\theta)E\frac{d}{dt}\{N(\theta)\}=0$$
$$M(\theta)EN(\theta)=E$$
$$M(\theta)\begin{pmatrix} A_1(\theta) & A_3(\theta) \\ A_2(\theta) & A_4(\theta)\end{pmatrix}N(\theta)=\begin{pmatrix} A_r(\theta) & 0 \\ 0 & I\end{pmatrix}$$
$$\forall i\in\{1,2\}:\ M(\theta)\begin{pmatrix} B_i \\ 0\end{pmatrix}=\begin{pmatrix} B_{r_i} \\ 0\end{pmatrix}$$
$$\forall i\in\{1,2\}:\ \begin{pmatrix} C_i & 0\end{pmatrix}N(\theta)=\begin{pmatrix} C_{r_i} & 0\end{pmatrix}$$

Consequently, if there exist matrices $W(\theta)$ and $S(\theta)$ such that inequalities (12), (13) and (14) hold for the descriptor system (Σ_d) then we can prove that there exist a matrix $Q(\theta)$ such that (6), (7) and (8) hold for the state-space rational LPV system (Σ_r) according to Theorem 1. Moreover, the H_2 performance index is the same for both systems.

Remark 3: The descriptor system (Σ_d) is affinely dependent on the time varying parameter. Hence it provides a suitable tool for analysis and gain scheduling control of the rational LPV system (Σ_r). Indeed, when considering quadratic stability, conditions (12), (13) and (14) lead to a finite set of LMIs since the descriptor matrices are polytopic. On the contrary, when considering the rational state-space realization even with constant matrices, conditions (6), (7) and (8) remain hardly tractable.

III. NEW LMI CONDITIONS FOR H_2 PERFORMANCE CHARACTERISATION

In this section we consider the polytopic descriptor system (Σ_d) given by (2).

Theorem 2: For a given scalar $\gamma>0$, the descriptor LPV system (Σ_d) is admissible and $\|\Sigma_d\|_2<\gamma$ if there exist continuously differentiable functions $\tilde{W}:\theta\to\tilde{W}(\theta)$ $\tilde{S}_1:\theta\to\tilde{S}_1(\theta)$, $\tilde{S}_2:\theta\to\tilde{S}_2(\theta)$, $\tilde{S}_3:\theta\to\tilde{S}_3(\theta)$ and $\tilde{S}_4:\theta\to\tilde{S}_4(\theta)$ of appropriate dimensions such that for all $\theta(.)$:

$$\tilde{W}(\theta)=\tilde{W}(\theta)^T>0 \tag{20}$$

$$\begin{pmatrix} U^T\frac{d}{dt}\{\tilde{W}(\theta)\}U+B_1B_1^T & U\tilde{W}(\theta)U^T+\tilde{S}_1(\theta)U_0^T \\ \bullet & He\{\tilde{S}_2(\theta)U_0^T\}\end{pmatrix} +He\left\{\begin{pmatrix} A(\theta) \\ -I\end{pmatrix}\begin{pmatrix}\tilde{S}_3(\theta)^T & \tilde{S}_4(\theta)^T\end{pmatrix}\right\}<0 \tag{21}$$

$$tr\{C_1U\tilde{W}(\theta)U^TC_1^T\}<\gamma^2 \tag{22}$$

□ **Proof:** Is omitted for brevity reasons □

Conditions proposed in **Lemma 3** can be turned into a finite set of LMIs when choosing constant matrices for the solutions $W(\theta)$ and $S(\theta)$. The conservatism induced by this choice can be reduced thanks to **Theorem 2** since there is no multiplication between the state matrices and matrices $\tilde{W}(\theta)$, $\tilde{S}_1(\theta)$ and $\tilde{S}_2(\theta)$.

Corollary 1: For a given scalar $\gamma>0$, the descriptor LPV system (Σ_d) is admissible and $\|\Sigma_d\|_2<\gamma$ if there exist affine functions $\tilde{W}:\theta\to\tilde{W}(\theta)$ $\tilde{S}_1:\theta\to\tilde{S}_1(\theta)$, $\tilde{S}_2:\theta\to\tilde{S}_2(\theta)$, and constant matrices $\tilde{S}_3$, $\tilde{S}_4$ of appropriate dimensions such that for all admissible $\theta(.)$, $\forall(\omega_i,\tau_j)\in \mathrm{P}\times\Omega$:

$$\tilde{W}(\omega_i)=\tilde{W}(\omega_i)^T>0 \tag{23}$$

$$\begin{pmatrix} U\dot{\tilde{W}}(\tau_j)U^T+B_1B_1^T & U\tilde{W}(\omega_i)U^T+\tilde{S}_1(\omega_i)U_0^T \\ \bullet & He\{\tilde{S}_2(\omega_i)U_0^T\}\end{pmatrix} +He\left\{\begin{pmatrix} A(\omega_i) \\ -I\end{pmatrix}\begin{pmatrix}\tilde{S}_3^T & \tilde{S}_4^T\end{pmatrix}\right\}<0 \tag{24}$$

$$tr\{C_1U\tilde{W}(\omega_i)U^TC_1^T\}<\gamma^2 \tag{25}$$

It has been shown in section II that (Σ_r) and (Σ_d) are strongly equivalent. This means that the results presented in **Theorem 2** and **Corollary 1** are also sufficient conditions for the stability and the H_2 performance level of (Σ_r).

IV. H_2 GAIN SCHEDULED CONTROLLER DESIGN

In this section, we consider the output feedback H_2 control problem for the affine LPV descriptor system given by (2). We seek a gain scheduled controller given by:

$$(\Sigma_{d_K}):\begin{cases} E\dot{x}_K=A_K(\theta)x_K+B_K(\theta)y \\ u=C_K(\theta)x_K\end{cases} \tag{26}$$

with $x_K\in\mathbb{R}^n$, $A_K(\theta(t))\in\mathbb{R}^{n\times n}$, $B_K(\theta(t))\in\mathbb{R}^{n\times n_y}$ and $C_K(\theta(t))\in\mathbb{R}^{n_u\times n}$ are unknown matrices. The resulting closed-loop system is given by:

$$(\Sigma_{d_C}):\begin{cases} E_c\dot{x}_c=A_c(\theta)x_c+B_cw \\ y=C_cx_c\end{cases} \tag{27}$$

with: $x_c=\begin{bmatrix}x^T & x_K{}^T\end{bmatrix}^T$,

$$E_c=\begin{pmatrix} E & 0 \\ 0 & E\end{pmatrix},\ A_c(\theta)=\begin{pmatrix} A(\theta) & B_2C_K(\theta) \\ B_K(\theta)C_2 & A_K(\theta)\end{pmatrix},$$

$$B_c=\begin{pmatrix} B_1 \\ 0\end{pmatrix} \text{ and } C_c=\begin{pmatrix} C_1 & 0\end{pmatrix}.$$

Remark 4: Note that conditions $\mathrm{Img}\{E_c\}\supseteq\mathrm{Img}\{B_c\}$ and

$ker\{C_c\} \supseteq ker\{E_c\}$ still hold for the resulting closed-loop (27).

Theorem 3: For a given scalar $\gamma > 0$, there exists a controller given by (26) such that the closed-loop (Σ_{d_C}) is admissible and $\|\Sigma_{d_C}\|_2 < \gamma$ if there exist symmetric matrices $W_1 > 0$, $W_2 > 0$, matrices S_1 and S_2, polytopic functions $F : \theta \to F(\theta)$ and $L : \theta \to L(\theta)$ such that the following inequalities hold for all $\theta(.)$:

$$He\{(A(\theta) + L(\theta)C_2)(UW_1U^T + S_1U_0^T)^T\} + B_1B_1^T < 0 \quad (28)$$

$$He\{(A(\theta) + B_2F(\theta))(UW_2U^T + S_2U_0^T)^T\} + B_1B_1^T < 0 \quad (29)$$

$$W_2 - W_1 > 0 \quad (30)$$

and

$$trace(C_1UW_2U^TC_1^T) < \gamma^2 \quad (31)$$

Moreover such a controller is given by

$$A_K(\theta) = \begin{pmatrix} PA(\theta)^T \\ +(A(\theta) + B_2F(\theta) + L(\theta)C_2)X^{-1} \\ +B_1B_1^T \end{pmatrix}(X^{-1} - P^T)^{-1} \quad (32)$$

$$B_K(\theta) = L(\theta) \quad (33)$$

$$C_K = -F(\theta)X^{-1}(X^{-1} - P^T)^{-1} \quad (34)$$

$$P = (UW_1U^T + S_1U_0^T) \quad (35)$$

$$X^{-1} = (UW_2U^T + S_2U_0^T)^T \quad (36)$$

□ **Proof:** Is omitted for brevity reasons □

Remark 5: Note that matrix inequalities (28) and (29) are not linear. However, these conditions can be turned into LMIs. Indeed, using the following change of variables $N(\theta) = F(\theta)X^{-1}$, inequality (29) becomes a LMI in N, W_2 and S_2. Therefore, using the fact that P is non singular and pre- and post-multiplying (28) by P^{-1} and P^{-T} leads to

$$\begin{pmatrix} He\{(A(\theta) + L(\theta)C_2)^T P^{-T}\} & P^{-1}B_1 \\ \bullet & -I \end{pmatrix} < 0$$

Then, considering the following change of variables $M(\theta) = L(\theta)^T P^{-T}$ we obtain a LMI condition in P^{-1} and M. Hence, according to **Remark 2**, there exist matrices $\tilde{W}_1, \tilde{S}_1$ of compatible dimensions satisfying

$$P^{-1} = (U\tilde{W}_1U^T + \tilde{S}_1U_0^T)$$

with $\bar{W}_1 = W_1^{-1}$, (28) becomes then a LMI in $\tilde{W}_1, \tilde{S}_1$ and M. Finally, (30) can be rewritten as $\begin{bmatrix} W_2 & I \\ I & \tilde{W}_1 \end{bmatrix} > 0$.

The choice of constant variables in **Theorem 3** induces some conservatism. The theorem proposed next reduces this conservatism by allowing parameter-dependent variables.

Theorem 4: For a given scalar $\gamma > 0$, there exist a controller such that the closed-loop (Σ_{d_C}) is admissible and $\|\Sigma_{d_C}\|_2 < \gamma$ if there exist affine functions $W_1(\theta) > 0$ $W_2(\theta) > 0$, $S_1^1(\theta)$, $S_2^1(\theta)$, $S_1^2(\theta)$, $S_2^2(\theta)$, $F(\theta)$, $L(\theta)$ and constant matrices S_3^1, $\tilde{S}_3^2$ of appropriate dimensions such that the following inequalities hold for all $\theta(.)$:

$$\begin{pmatrix} \frac{d}{dt}\{UW_1(\theta)U^T\} + B_1B_1^T & UW_1(\theta)U^T + S_1^1(\theta)U_0^T \\ \bullet & He\{S_2^1(\theta)U_0^T\} \end{pmatrix} + He\left\{\begin{pmatrix} A(\theta) + L(\theta)C_2 \\ -I \end{pmatrix}\begin{pmatrix} S_3^{1T} & S_3^{1T} \end{pmatrix}\right\} < 0 \quad (37)$$

$$\begin{pmatrix} \frac{d}{dt}\{UW_2(\theta)U^T\} + B_1B_1^T & UW_2(\theta)U^T + S_1^2(\theta)U_0^T \\ \bullet & He\{S_2^2(\theta)U_0^T\} \end{pmatrix} + He\left\{\begin{pmatrix} A(\theta) + B_2F(\theta) \\ -I \end{pmatrix}\begin{pmatrix} S_3^{2T} & S_3^{2T} \end{pmatrix}\right\} < 0 \quad (38)$$

$$W_2(\theta) - W_1(\theta) > 0 \quad (39)$$

and

$$trace(C_1UW_2(\theta)U^TC_1^T) < \gamma^2 \quad (40)$$

Moreover such a controller is given by

$$A_K(\theta) = \begin{pmatrix} P(\theta)A(\theta)^T \\ +(A(\theta) + B_2F(\theta) + L(\theta)C_2)X(\theta)^{-1} \\ +B_1B_1^T \end{pmatrix}(X(\theta)^{-1} - P(\theta)^T)^{-1} \quad (41)$$

$$B_K(\theta) = L(\theta) \quad (42)$$

$$C_K = -F(\theta)X(\theta)^{-1}(X(\theta)^{-1} - P(\theta)^T)^{-1} \quad (43)$$

with

$$P = (UW_1(\theta)U^T + \hat{S}_1(\theta)U_0^T) \quad (44)$$

$$X(\theta)^{-1} = (UW_2(\theta)U^T + \hat{S}_2(\theta)U_0^T)^T \quad (45)$$

$$\hat{S}_1^1(\theta) = S_1^1(\theta) + S_2^1(\theta)(A(\theta) + L(\theta)C_2)^T \quad (46)$$

and

$$\hat{S}_1^2(\theta) = S_1^2(\theta) + S_2^2(\theta)(A(\theta) + B_2F(\theta))^T \quad (47)$$

□ **Proof:** Is omitted for brevity reasons □

Remark 6: Note that matrix inequalities (37) and (38) are not linear. However, they can be turned into LMIs by using the same technique as in **Remark 5**.

The descriptor controllers obtained in this section can be transformed into state-space rational controllers ensuring the stability and the H_2 performance for the rational LPV system (Σ_r).

V. Numerical Examples

A. Analysis of H_2 performance

We consider the following rational LPV system:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{pmatrix} -3 - \theta^2 - \dfrac{\theta^3}{1+\theta} & -\theta^2 - \dfrac{\theta^3}{1+\theta} \\ 1 + \theta - \theta^2 - \dfrac{\theta^3}{1+\theta} & -1 - \theta^2 - \dfrac{\theta^3}{1+\theta} \end{pmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} w$$

$$z = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

with $(\theta(t), \dot{\theta}(t)) \in [-0.9 \quad 0.9] \times [-1 \quad 1]$.

Setting $x_3 = -\theta(x_1 + x_2)$ and $x_4 = -\dfrac{\theta^2}{1+\theta}(x_1 + x_2)$ leads to the following equivalently affine descriptor realization:

$$\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}\begin{bmatrix}\dot{x}_1\\\dot{x}_2\\\dot{x}_3\\\dot{x}_4\end{bmatrix}=\begin{pmatrix}-3&0&\theta&\theta\\1+\theta&-1&\theta&\theta\\\theta&\theta&1&0\\0&0&\theta&1+\theta\end{pmatrix}\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}+\begin{pmatrix}1\\0\\0\\0\end{pmatrix}w$$

$$z=\begin{pmatrix}1&-1&0&0\\1&0&0&0\end{pmatrix}x$$

For this example, we apply the proposed analysis conditions of **Lemma 3** and **Theorem 2**. The minimum H_2 guaranteed cost obtained are respectively $\gamma = 1.209$ and $\gamma = 0.837$.

B. *Design of Gain scheduled H_2 controllers*

To compare the gain scheduling techniques based on **Theorem 3** and **Theorem 4**, a numerical example is given in this section.

Let us consider the following rational LPV system:

$$\begin{cases}\begin{bmatrix}\dot{x}_1\\\dot{x}_2\end{bmatrix}=\begin{pmatrix}\dfrac{\theta^2+\theta}{\theta+2}&\dfrac{3\theta+4}{\theta+2}\\1&-1\end{pmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}+\begin{pmatrix}2\\1\end{pmatrix}w+\begin{pmatrix}2\\1\end{pmatrix}u\\ z=x_1+x_2\end{cases}\quad (48)$$

with $\left(\theta(t),\dot{\theta}(t)\right)\in\begin{bmatrix}-1.5 & 1.5\end{bmatrix}\times\begin{bmatrix}-1 & 1\end{bmatrix}$.

Setting $x_3=\dfrac{\theta}{2+\theta}(x_1-x_2)$ leads to the following equivalent affine LPV descriptor representation:

$$\begin{cases}\begin{pmatrix}1&0&0\\0&1&0\\0&0&0\end{pmatrix}\begin{bmatrix}\dot{x}_1\\\dot{x}_2\\\dot{x}_3\end{bmatrix}=\begin{pmatrix}\theta&2&1\\1&-1&0\\\theta&-\theta&-(2+\theta)\end{pmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}+\begin{pmatrix}2\\1\\0\end{pmatrix}w+\begin{pmatrix}2\\1\\0\end{pmatrix}u\\ z=\begin{pmatrix}1&1&0\end{pmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}\end{cases}$$

When the design of the controller is based on **Theorem 3**, the minimum H_2 guaranteed level of $\gamma = 2.78$ can be achieved with controller $K_1(\theta)$.

With the method of **Theorem 4** the minimum H_2 guaranteed level of $\gamma = 2.19$ can be achieved with controller $K_2(\theta)$.

The matrices of the obtained controllers $K_1(\theta)$ and $K_2(\theta)$ are given by (49) and (50) with: $E_{K_1} = E_{K_2} = diag(1,1,0)$.

VI. Conclusion

In this paper, new LMI-based characterizations of the H_2 performance for rational LPV systems have been proposed using an equivalent descriptor affine LPV representation. Based on this, two methods for the design of H_2 gain scheduled output feedback controllers have been presented. The novelty of this approach is that it guarantees stability and H_2 performance of the rational LPV closed-loop system throughout the admissible parameter range using a finite set of LMIs. A comparison of the proposed methods has been shown through numerical examples. A practical application will be considered in a forthcoming paper.

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$$A_{K_1}(\theta)=10^3\begin{pmatrix}\text{-5.62-0.31}\theta & \text{-0.71-0.45}\theta & 6.53+2.33\theta\\ \text{-3.19-0.31}\theta & \text{-0.43-0.25}\theta & 3.56+1.92\theta\\ 0.18+0.9\theta & 0.03+0.01\theta & \text{-0.14-0.22}\theta\end{pmatrix},\ B_{K_1}(\theta)=10^3\begin{pmatrix}\text{-2.38-1.22}\theta\\ \text{-1.39-0.69}\theta\\ 0.09+0.04\theta\end{pmatrix},\ C_{K_1}(\theta)=\begin{pmatrix}527.51+10.76\theta & -33.96+2.38\theta & 0.07+0.10\theta\end{pmatrix}\quad (49)$$

$$A_{K_2}(\theta)=\begin{pmatrix}122.67+2.63\theta-141.7\theta^2 & -15.61-0.84\theta-1310.9\,\theta^2+0.63\theta^3 & -132.76+3.70\theta-139.6\,\theta^2+0.31\theta^3\\ 45.30+6.96\theta-293\,\theta^2+0.43\theta^3 & -3.49+0.92\theta-227\,\theta^2+0.25\theta^3 & -3.49+0.92\theta-227\,\theta^2+0.15\theta^3\\ 280.59+17.84\theta-4843\,\theta^2+0.22\theta^3 & -31.03+0.12\theta-4480\,\theta^2+2.22\theta^3 & 288.32+19.12\theta-552.4\,\theta^2+0.99\theta^3\end{pmatrix},\ B_{K_2}(\theta)=\begin{pmatrix}-68.01-0.57\theta\\ -19.44-0.22\theta\\ -132.11-1.23\theta\end{pmatrix},\ C_{K_2}(\theta)=\begin{pmatrix}2.56+112.17\theta-71.02\theta^2-0.2\theta^3\\ 11.35+104.61\theta-66.31\theta^2-0.01\theta^3\\ 17.68-5.45\theta-10.75\theta^2-0.15\theta^3\end{pmatrix}^T\quad (50)$$