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Behavioral Controllability and Coprimeness for A Class of Infinite-Dimensional Systems

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***Abstract*— Behavioral system theory has become a successful framework in providing a viewpoint that does not depend on a priori notions of inputs/outputs. In particular, this theory provides notions as controllability, without an explicit reference to the state space formalism. One also obtains several interesting consequences of controllability, for example, direct sum decomposition of the signal space with a controllable behavior $\mathscr{B}$ as a direct summand. While there are some attempts to extend this theory to infinite-dimensional systems, for example, delay systems, the overall picture remains incomplete. This article extends this theory, particularly the notion of controllability, to a well-behaved class of infinite-dimensional systems, called pseudorational. A crucial notion in this context is the Bézout identity, and we relate a recent result to the context of behavioral controllability. We establish its relationships with notions as image representation and direct sum decompositions.**

I. Introduction

Behavioral system theory has become a successful framework in providing a viewpoint that does not depend on the a priori notions of inputs/outputs. An introductory and tutorial account is given in [9], [3]. In particular, this theory successfully provides such notions as controllability, without an explicit reference to state space formalism. One also obtains several interesting and illuminating consequences of controllability, for example, direct sum decomposition of the signal space with a controllable behavior $\mathscr{B}$ as a direct summand.

There are several attempts to extend this theory to infinite-dimensional systems, for example, delay systems, and some rank conditions for behavioral controllability have been obtained; see, e.g., [4], [2], [7], [8]. While these results give a nice generalization of their finite-dimensional counterparts, the overall picture still needs to be further studied in a more general and abstract setting. For example, one wants to see how the notion of zeros and poles can affect controllability in an abstract setting. This is to some extent accomplished in [4], [2], [7] to an extent that is amenable to delay-differential systems, and then generalized to a more general class in [8]. We here intend to give a theory in a general and unified setting of the class of pseudorational transfer functions that has been developed in [10], [11], etc.; this framework is close to the one used in [8], but has the advantage of closer relationships with state space realizations, and various spectral properties.

The paper is organized as follows: Section II introduces pseudorationality, and then generalizes this notion to the behavioral context. We briefly describe a state space formalism and realization procedures in Section III. Spectral properties and eigenfunction completeness are also reviewed, and they are crucial in characterizing coprimeness properties. Section IV introduces the notions of behavioral controllability in the present context, and gives various criteria for controllability. Of particular importance is the Bézout identity. Section V gives a proof for a condition for the Bézout identity, with generalization to the multivariable case.

II. Pseudorationality

We start by considering the following example of a behavior given by a delay-differential equation. We will use distributions in the subsequent developments, and the reader is referred to the Appendix for notation and conventions for distributions and spaces consisting of them.

Example 2.1:

$$\frac{d}{dt}w_1(t) - w_1(t-1) - \frac{d}{dt}w_2(t) = 0. \qquad (1)$$

Using distributions, we can express the solution space as

$$\begin{bmatrix} \delta' - \delta_1, & -\delta' \end{bmatrix} * \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = 0,$$

where $*$ denotes convolution (the signal space will be specified later in a more formal treatment). The delta distribution δ_1 and the derivative δ' may be regarded as elements of the polynomial ring $\mathbb{R}[\delta', \delta_1]$, and this yields a a common treatment in the literature. However, for a behavior defined by

$$\begin{bmatrix} \delta' - \delta_1 - \chi, & -\delta' \end{bmatrix} * \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = 0,$$

where χ is defined by

$$\chi(t) := \begin{cases} 1, & 0 \le t \le 1 \\ 0, & \text{otherwise,} \end{cases}$$

it is not possible to describe this equation with a convolution equation with elements in a polynomial ring like $\mathbb{R}[\delta', \delta_1]$.

On the other hand, all these elements have compact support. This leads to considering a convolution algebra consisting of elements (distributions) having compact support. This has an advantage that all system equations are essentially *finite-time determined*. This leads us to the following definition, which is an extension of the notion given in [10].

Definition 2.2: Let R be an $\mathtt{p} \times \mathtt{w}$ matrix ($\mathtt{w} \geq \mathtt{p}$) with entries in $\mathscr{E}'(\mathbb{R})$. It is said to be *pseudorational* if there exists a $\mathtt{p} \times \mathtt{p}$ submatrix P such that

1) $P^{-1} \in \mathscr{D}'_+(\mathbb{R})$ exists with respect to convolution where $\mathscr{D}'_+(\mathbb{R})$ denotes the subspace of $\mathscr{D}'$ having support bounded on the left;
2) $\mathrm{ord}(\det P^{-1}) = -\mathrm{ord}(\det P)$, where $\mathrm{ord}\,\psi$ denotes the order of a distribution ψ [5], [6] (for a definition, see the Appendix).

Let $L^2_{loc}(-\infty,\infty)$ be the space of locally square integrable functions. We give the following definition:

Definition 2.3: Let R be pseudorational as defined above. The *behavior* $\mathscr{B}$ defined by R is given by

$$\mathscr{B} := \{w \in (L^2_{loc}(-\infty,\infty))^{\mathtt{w}} | R * w = 0\} \tag{2}$$

The convolution $R * w$ is taken in the sense of distributions. Since R has compact support, this convolution is always well defined [5].

The behavior $\mathscr{B}$ is *time-invariant* in the sense that $\sigma_t \mathscr{B} \subset \mathscr{B}$ for every $t \in \mathbb{R}$, where σ_t is the left shift operator in $L^2_{loc}(-\infty,\infty)$ defined by

$$(\sigma_t w)(s) := w(s+t). \tag{3}$$

This clearly follows from the definition (2) since $R * (\sigma_t w) = R * \delta_{-t} * w = \delta_{-t} * R * w = 0$.

We introduce behaviors in a wider space of signals, namely in the space of distributions. Let $\mathscr{D}'$ be the space of distributions on $\mathbb{R}$, and let R be pseudorational. The *distributional behavior* $\mathscr{B}_{\mathscr{D}'}$ defined by R is given by

$$\mathscr{B}_{\mathscr{D}'} := \{w \in (\mathscr{D}')^{\mathtt{w}} | R * w = 0\}. \tag{4}$$

A. State Space Representation

Let $R \in \mathscr{E}'(\mathbb{R})^{\mathtt{p}\times\mathtt{w}}$ be pseudorational. Suppose, without loss of generality, that R is partitioned as $R = \begin{bmatrix} P & Q \end{bmatrix}$ such that P satisfies the invertibility condition of Definition 2.2, i.e., we consider the kernel representation

$$P * y + Q * u = 0 \tag{5}$$

where $w := \begin{bmatrix} y & u \end{bmatrix}^T$ is partitioned conformably with the sizes of P and Q.

When $G := P^{-1} * Q$ belongs to $L^2_{loc}(-\infty,\infty)^{\mathtt{p}\times\mathtt{m}}$, and $\mathrm{supp}\,G$ is contained in $[0,\infty)$, it is possible to give a state space model to (5).

To this end, it is possible to invoke realization theory developed in [10]; see also [13] for a comprehensive survey.

The idea is the following: Suppose that G belongs to $L^2_{loc}[0,\infty)$. Let 0 be the present time, and consider the set of all output functions $(G * u)|[0,\infty)$ generated by all input functions $u \in L^2(-\infty,0]$ having bounded support. Let X be the closure of all such output functions in $L^2_{loc}[0,\infty)$. X is easily seen to be invariant under left shift operators (3). Taking this as the state transition semigroup, one can obtain a canonical realization [10].

A nice consequence of pseudorationality is that this space X is always a closed subspace of the following more tractable space X^P:

$$X^P := \{x \in (L^2_{loc}[0,\infty))^{\mathtt{p}} \,|\, P * x|_{[0,\infty)} = 0\}, \tag{6}$$

and it is possible to give a realization using X^P as a state space. This realization turns out to be always observable ([10]), and whether $X = X^P$ depends on the coprimeness of the pair (P,Q) [11].

A remarkable feature of the realization given with X^P as a state space is that the spectrum of A is completely characterized in terms of the zeros of the Laplace transform of P.

Theorem 2.4: The spectrum $\sigma(A)$ is given by

$$\sigma(A) = \{\lambda \,|\, \det \hat{P}(\lambda) = 0\}, \tag{7}$$

where $\hat{P}$ denotes the Laplace transform of P (see the Appendix). Furthermore, every $\lambda \in \sigma(A)$ is an eigenvalue with finite multiplicity. The corresponding eigenfunction for $\lambda \in \sigma(A)$ is given by $e^{\lambda t} v$ where $\hat{P}(\lambda)v = 0$. Similarly for generalized eigenfunctions such as $te^{\lambda t}v'$. See [11] for details. Furthermore, for each $\lambda \ni \sigma(A)$, the resolvent operator $(\lambda I - A)^{-1}$ exists, and is compact.

Since $\hat{P}$ (and hence $\det \hat{P}$) is an entire function of exponential type by Theorem 6.1, the spectrum is discrete, and with finite multiplicities.

III. Controllability and Coprimeness

We now introduce the notion of controllability [3] in the present context.

Definition 3.1: Let R be pseudorational, and $\mathscr{B}$ the behavior associated to it. $\mathscr{B}$ is said to be *controllable* if for every pair $w_1, w_2 \in \mathscr{B}$, there exists $T \geq 0$ and $w \in \mathscr{B}$, such that $w(t) = w_1(t)$ for $t < 0$, and $w(t) = w_2(t-T)$ for $t \geq T$ (see Fig. III).

In other words, every pair of trajectories can be concatenated into one trajectory that agrees with them in the past and future.

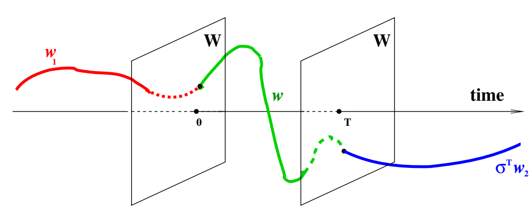

Fig. 1. Concatenation of trajectories

We also introduce an extended notion of controllability as follows:

Definition 3.2: Let R be pseudorational, and $\mathscr{B}_{\mathscr{D}'}$ be the distributional behavior (4). $\mathscr{B}_{\mathscr{D}'}$ is said to be *distributionally* *controllable* if for every pair $w_1, w_2 \in \mathscr{B}_{\mathscr{D}'}$, there exists $T \geq 0$ and $w \in \mathscr{B}_{\mathscr{D}'}$, such that $w|_{(-\infty,0)} = w_1$ on $(-\infty, 0)$, and $w|_{(T,\infty)} = \sigma_{-T} w_2$ on (T, ∞).

We now introduce various notions of coprimeness.

Definition 3.3: The pair (P,Q), $P,Q \in \mathscr{E}'(\mathbb{R})$ is said to be *spectrally coprime* if $\hat{P}(s)$ and $\hat{Q}(s)$ have no common zeros. It is *approximately coprime* if there exist sequences $\Phi_n, \Psi_n \in \mathscr{E}'(\mathbb{R})$ such that $P * \Phi_n + Q * \Psi_n \to \delta I$ in $\mathscr{E}'(\mathbb{R})$. The pair (P,Q) is said to satisfy the *Bézout identity* (or simply *Bézout*), if there exists $\Phi, \Psi \in \mathscr{E}'(\mathbb{R})$ such that

$$P * \Phi + Q * \Psi = \delta I, \tag{8}$$

Or equivalently,

$$\hat{P}(s)\hat{\Phi}(s) + \hat{Q}(s)\hat{\Psi}(s) = I \tag{9}$$

for some entire functions $\hat{\Phi}, \hat{\Psi}$ satisfying the Paley-Wiener estimate (31).

It is well known [3] that controllability admits various nice characterizations in terms of coprimeness, image representation, full rank conditions, etc. We here attempt to give a generalization of such results to the present context. To this end, we confine ourselves to the simplest scalar case, i.e., $p = m = 1$. We will also assume that q also satisfies the condition that the zeros of $\hat{q}(s)$ are contained in a half plane $\{s \,|\, \mathrm{Re}\, s < c\}$ for some $c \in \mathbb{R}$. (This is needed in the proof of Theorem 4.1.)

Theorem 3.4: Let R be pseudorational, and suppose without loss of generality that R is of form $R := \begin{bmatrix} p & q \end{bmatrix}$ where p satisfies the invertibility condition in Definition 2.2. Let $\mathscr{B}_{\mathscr{D}'}$ be the distributional behavior (4). Then the following statements are equivalent:

1) $\mathscr{B}_{\mathscr{D}'}$ is controllable.
2) There exist $\psi, \phi \in \mathscr{E}'(\mathbb{R})$ such that $p * \phi + q * \psi = \delta$.
3) $\mathscr{B}_{\mathscr{D}'}$ admits an image representation, i.e., there exists M over $\mathscr{E}'(\mathbb{R})$ such that for every $w \in \mathscr{B}_{\mathscr{D}'}$, there exists $\ell \in \mathscr{D}'$ such that $w = M * \ell$.
4) $\mathscr{B}_{\mathscr{D}'}$ is a direct summand of $(\mathscr{D}')^2$, i.e., there exists an distributional behavior $\mathscr{B}'$ such that $\mathscr{D}' = \mathscr{B}_{\mathscr{D}'} \oplus \mathscr{B}'$.
5) Let $\Lambda := \{\lambda \in \mathbb{C} | \hat{p}(\lambda) = 0\}$. Suppose that the algebraic multiplicities of the zeros $\lambda \in \Lambda$ are globally bounded. There exist $k \geq 0$ and $c > 0$ such that

$$|\lambda^k \hat{q}(\lambda)| \geq c, \quad \forall \lambda \in \Lambda. \tag{10}$$

Proof We will prove 2) ⇒ 3), 4), 5), and 3), 4) ⇒ 1), and 1) ⇒ 2), and then prove 5) → 2) separately in the next section.

2) ⇒ 3) Consider the mapping

$$\pi_{\mathscr{B}_{\mathscr{D}'}} : \mathscr{D}' \ni \ell \mapsto \begin{bmatrix} q \\ -p \end{bmatrix} * \ell \in (\mathscr{D}')^2. \tag{11}$$

We claim that this gives an image representation. Since

$$\begin{bmatrix} p & q \end{bmatrix} \begin{bmatrix} q \\ -p \end{bmatrix} * \ell = 0,$$

the image of (11) clearly belongs to $\mathscr{B}_{\mathscr{D}'}$. We need only to prove that this mapping is surjective to $\mathscr{B}_{\mathscr{D}'}$. Take any $\begin{bmatrix} y \\ u \end{bmatrix}$ in $\mathscr{B}_{\mathscr{D}'}$, and set

$$\ell := \begin{bmatrix} \psi & \phi \end{bmatrix} * \begin{bmatrix} y \\ u \end{bmatrix}. \tag{12}$$

It follows that

$$\begin{aligned}
\begin{bmatrix} q \\ -p \end{bmatrix} * \begin{bmatrix} \psi & \phi \end{bmatrix} * \begin{bmatrix} y \\ u \end{bmatrix}
&= \begin{bmatrix} q*\psi & q*\phi \\ -p*\psi & -p*\phi \end{bmatrix} * \begin{bmatrix} y \\ u \end{bmatrix} \\
&= \begin{bmatrix} \delta - p*\phi & q*\phi \\ -p*\psi & q*\psi - \delta \end{bmatrix} * \begin{bmatrix} y \\ u \end{bmatrix} \\
&= \begin{bmatrix} y - \phi*(q*u - p*y) \\ u - \psi*(q*u - p*y) \end{bmatrix} = \begin{bmatrix} y \\ u \end{bmatrix}
\end{aligned}$$

Hence $\pi_{\mathscr{B}_{\mathscr{D}'}}$ is surjective and 3) follows.

2) ⇒ 4) To prove 4), first note that

$$\begin{bmatrix} p & q \\ -\psi & \phi \end{bmatrix}$$

is a unimodular matrix in $\mathscr{E}'(\mathbb{R})$. In fact, its determinant is $p*\phi + q*\psi = \delta$. Define $\tilde{\mathscr{B}}_{\mathscr{D}'}$ by

$$\tilde{\mathscr{B}}_{\mathscr{D}'} := \left\{ \begin{bmatrix} y \\ u \end{bmatrix} \,\middle|\, \begin{bmatrix} -\psi & \phi \end{bmatrix} * \begin{bmatrix} y \\ u \end{bmatrix} = 0 \right\}.$$

We first claim $\mathscr{B}_{\mathscr{D}'} \cap \tilde{\mathscr{B}}_{\mathscr{D}'} = \{0\}$. Indeed, If $\begin{bmatrix} y & u \end{bmatrix}^T$ belongs to both $\mathscr{B}_{\mathscr{D}'}$ and $\tilde{\mathscr{B}}_{\mathscr{D}'}$,

$$\begin{bmatrix} p & q \\ -\psi & \phi \end{bmatrix} * \begin{bmatrix} y \\ u \end{bmatrix} = 0$$

which readily yields $\begin{bmatrix} y & u \end{bmatrix}^T = 0$ because of the unimodularity of the matrix on the right.

Now take any $\begin{bmatrix} y & u \end{bmatrix}^T$ in $(\mathscr{D}')^{\mathtt{w}}$. Define

$$\begin{bmatrix} v \\ x \end{bmatrix} := \begin{bmatrix} p & q \\ -\psi & \phi \end{bmatrix} * \begin{bmatrix} y \\ u \end{bmatrix}. \tag{13}$$

Then

$$\begin{aligned}
\begin{bmatrix} y \\ u \end{bmatrix} &= \begin{bmatrix} p & q \\ -\psi & \phi \end{bmatrix}^{-1} * \begin{bmatrix} v \\ x \end{bmatrix} \\
&= \begin{bmatrix} \phi & -q \\ \psi & p \end{bmatrix} * \begin{bmatrix} v \\ x \end{bmatrix} \\
&= \begin{bmatrix} -q \\ p \end{bmatrix} * x + \begin{bmatrix} \phi \\ \psi \end{bmatrix} v.
\end{aligned}$$

The first term belongs to $\mathscr{B}_{\mathscr{D}'}$ while the second term to $\tilde{\mathscr{B}}_{\mathscr{D}'}$. Hence the correspondence (13) is surjective to $\mathscr{D}'^{w}$, and $\mathscr{D}'^{w} = \mathscr{B}_{\mathscr{D}'} \oplus \tilde{\mathscr{B}}_{\mathscr{D}'}$. Furthermore, since this correspondence is clearly continuous with respect to the topology of $\mathscr{D}'$, this direct sum decomposition is topological.

2) ⇒ 5) Suppose 2) holds. Substituting $\lambda \in \Lambda$, we obtain $\hat{q}(\lambda)\hat{\phi}(\lambda) = 1$. Since ϕ has compact support, $\hat{\phi}$ is at most of polynomial order [5]. Taking k to be such an order, 5) follows.

3) ⇒ 1) Suppose 3) holds, and let $w_1, w_2 \in \mathscr{B}_{\mathscr{D}'}$. Then there exist $\ell_1, \ell_2 \in \mathscr{D}'$ such that $w_1 = M * \ell_1$ and $w_2 = M * \ell_2$. By suitably shifting M to the left, we may assume without loss of generality that supp$M$ is contained in $(-a,0)$ for some $a>0$. Now take any $T>a$. It is easy to see that

$$\pi_T(M*\sigma_T\ell_2)=\pi_T(M*\pi_T(\sigma_T\ell_2)) \tag{14}$$

where $\pi_T f = f|_{(T,\infty)}$. Now define $\ell\in\mathscr{D}'$ such that

$$\ell|_{(-\infty,0)} := \ell_1|_{(-\infty,0)}, \tag{15}$$
$$\ell|_{(T,\infty)} := \sigma_T\ell_2|_{(T,\infty)}. \tag{16}$$

It is possible to connect ℓ using the partition of unity [5]. It is also clear that $M*\ell$ agrees with $w_1=M*\ell_1$ and $\pi_T(\sigma_T w_2)$ on $(-\infty,0)$ and (T,∞), respectively. Hence $\mathscr{B}_{\mathscr{D}'}$ is controllable.
4) ⇒ 1) Suppose $(\mathscr{D}')^2=\mathscr{B}_{\mathscr{D}'}\oplus\tilde{\mathscr{B}}_{\mathscr{D}'}$ with

$$\tilde{\mathscr{B}}_{\mathscr{D}'}=\left\{\begin{bmatrix} y\\ u\end{bmatrix} \mid \begin{bmatrix} -\psi & \phi\end{bmatrix}\begin{bmatrix} y\\ u\end{bmatrix}=0\right\}. \tag{17}$$

We will show that

$$R:=\begin{bmatrix} p & q\\ -\psi & \phi\end{bmatrix}$$

is unimodular, i.e., det$R$ is invertible. Since $\mathscr{B}_{\mathscr{D}'}\cap\tilde{\mathscr{B}}_{\mathscr{D}'}=0$, $R*[y,u]^T=0$ clearly implies $[y,u]^T=0$. We claim that R is surjective. Let $r:=\det R=p*\phi+q*\psi$. It suffices to prove that r is unimodular. Since r belongs to $\mathscr{E}'(\mathbb{R})$, its Laplace transform $\hat{r}(s)$ admits the Hadamard factorization (33):

$$\hat{r}(s)=s^k e^{as}\prod_{n=1}^{\infty}\left(1-\frac{s}{\lambda_n}\right)\exp\left(\frac{s}{\lambda_n}\right).$$

It is easy to see that a is real. If $\hat{r}$ has a zero, say λ, then $x:=e^{\lambda t}$ satisfies $r*x=0$. Thus det$R$ has a nontrivial kernel, and hence R cannot be injective, which is a contradiction. Hence r is unimodular, i.e., $r^{-1}\in\mathscr{E}'(\mathbb{R})$. It now follows that $p*\phi*r^{-1}+q*\psi*r^{-1}=r*r^{-1}=\delta$.
1) ⇒ 2) We first consider the case for $\mathscr{B}$ with p^{-1} belonging to $L^2_{loc}(-\infty,\infty)$. By shifting p^{-1} suitably to the left, we may assume without loss of generality that p^{-1} belongs to $L^2_{loc}[0,\infty)$ and also that $p,q\in\mathscr{E}'(\mathbb{R}_-)$. Partition w conformably with p and q as $w=\begin{bmatrix} y & u\end{bmatrix}^T$. We can invoke realization theory for $p^{-1}*q$ as developed in [10] to obtain

$$x(t)=x_{\rm free}(t)+(p^{-1}*q*u)|_{(0,\infty)} \tag{18}$$

where $x_{\rm free}(t)$ is the solution to

$$p*x=0.$$

Hence every $x_{\rm free}(t)$ should take the form $p^{-1}*x_0$ for some x_0. Since $\mathscr{B}$ is controllable, there exist $T>0$ and $\begin{bmatrix} y & u\end{bmatrix}^T\in\mathscr{B}$ such that

$$(y,u)=\begin{cases}(0,0) & t<-T\\ (p^{-1},0) & t>0\end{cases}$$

This readily implies that there exists $\psi\in L^2[-T,0]$ such that $p^{-1}*q*\psi|_{(0,\infty)}=p^{-1}$. In other words,

$$p^{-1}*q*\psi=p^{-1}-\phi$$

for some $\phi\in L^2[-T,0]$. Convolving p from the left yields

$$p*\phi+q*\psi=\delta.$$

For the general case for $\mathscr{B}_{\mathscr{D}'}$ and $p^{-1}\in\mathscr{D}'_+(\mathbb{R})$, we need only to extend the above "state space formulas" to distributions. Formula (18) works equally well. We omit the details. □

Remark 3.5: Related results have been obtained in [8], but the relationships between various notions are not necessarily equivalences (not necessary and sufficient). The transparency of the results here is clearly due to the introduction of distributional behavior $\mathscr{B}_{\mathscr{D}'}$.

IV. Bézout Identity

As we have seen in the previous section, the Bézout identity plays a crucial role in characterizing controllability.

This is first obtained in [12] for the case of $\mathscr{E}'(\mathbb{R}_-)$. We here extend this result to $\mathscr{E}'(\mathbb{R})$.

Let us first note that we may assume that both p and q belong to $\mathscr{E}'(\mathbb{R}_-)$ by suitably shifting them to the left. This would make a difference only by a factor δ_a, $a\le 0$, but since δ_a is unimodular in $\mathscr{E}'(\mathbb{R})$, this does not cause any difficulty. Likewise, we may also assume $\max\{r(p),r(q)\}=0$ by shifting them either to the left or right. So let us hereafter assume that one of p and q, say, p satisfies $r(p)=0$. This guarantees "eigenfunction completeness" [11], i.e., the eigenfunctions $\{t^{m_k k}e^{\lambda_k t}\}$ span a dense subspace of the space X^p [11].

The following theorem is obtained in [12]:

Theorem 4.1: Let $p^{-1}*q$ be pseudorational such that $r(p)=0$. Suppose that there exists a nonnegative integer m such that

$$|\lambda_n^m\hat{q}(\lambda_n)|\ge c, n=1,2,\dots \tag{19}$$

Then the pair (p,q) is Bézout.
The rest of this section is devoted to the proof of this theorem.

Note first that (8) means $[q]\cong[\delta]$ modulo p, namely $[q]$ is invertible over the quotient algebra $\mathscr{E}'(\mathbb{R}_-)/(p)$. This is characterized in [12]. We here briefly review the outline of the proof and indicate the basic idea, with indications for a generalization to the multivariable case.

We first observe that $\mathscr{E}'(\mathbb{R}_-)$ and $\mathscr{E}[0,\infty)$ are dual to each other with respect to the following duality:

$$\langle\alpha,f\rangle:=(\alpha*f)(0),\quad \alpha\in\mathscr{E}'(\mathbb{R}_-),f\in\mathscr{E}[0,\infty). \tag{20}$$

It is easy to see that (20) defines a separately continuous bilinear form on $\mathscr{E}'(\mathbb{R}_-)\times\mathscr{E}[0,\infty)$, and they are indeed dual to each other.

The outline of the proof is as follows:

1) To characterize the invertibility of $[q]$ in $\mathscr{E}'(\mathbb{R}_-)/(p)$, we view $\mathscr{E}'(\mathbb{R}_-)/(p)$ as the dual of a closed subspace (denoted $\mathscr{E}^{(p)}$) of $\mathscr{E}[0,\infty)$.
2) $\mathscr{E}^{(p)}$ admits a very simple representation. Due to the condition $r(p)=0$, $\mathscr{E}^{(p)}$ is eigenfunction complete [11], and every element admits an infinite series expansion: $x=\sum_n\alpha_n e^{\lambda_n t}$.
3) With respect to the duality (20), the action of q on $e^{\lambda_n t}$ is given by

$$\langle q,e^{\lambda_n t}\rangle=(q*e^{\lambda_n t})(0)=\hat{q}(\lambda_n). \tag{21}$$

4) Using (21), we see that the candidate for $\psi := [q]^{-1}$ should satisfy $\hat{\psi}(\lambda_n) = 1/\hat{q}(\lambda_n)$.
5) Whether this formula leads to a well defined element in $\mathscr{E}'(\mathbb{R}_-)/(p)$ is the crucial step.

Let us start with the following lemma:

Lemma 4.2: The dual space of $\mathscr{E}'(\mathbb{R}_-)/(p)$ is given by

$$\begin{aligned}(\mathscr{E}'(\mathbb{R}_-)/(p))' &= \{x \in \mathscr{E}[0,\infty) | p * x \in \mathscr{E}'(\mathbb{R}_-)\} \\ &=: \mathscr{E}^{(p)}. \end{aligned} \tag{22}$$

Proof Since $(\mathscr{E}'(\mathbb{R}_-))' = \mathscr{E}[0,\infty)$, we have

$$\begin{aligned}(\mathscr{E}'(\mathbb{R}_-)/(p))' &= \{x \in \mathscr{E}[0,\infty) \,|\, \langle \alpha, x\rangle = 0, \forall \alpha \in (p)\} \\ &= \{x \in \mathscr{E}[0,\infty) \,|\, (\delta_{-t} * p * x)(0) = 0, \forall t \geq 0\} \\ &= \{x \in \mathscr{E}[0,\infty) \,|\, p * x \in \mathscr{E}'(\mathbb{R}_-)\}.\end{aligned}$$

□

From here on suppose for simplicity that the zeros λ_n of $\hat{q}(s)$ are all simple zeros, and that m in (19) is 0 (although these are not at all necessary).

Lemma 4.3: Under the hypothesis of $r(p) = 0$,

$$\operatorname{span}\{e^{\lambda_n t}\}_{n=1}^{\infty} \tag{23}$$

is dense in $\mathscr{E}^{(p)}$. Furthermore, every $x \in \mathscr{E}^{(p)}$ admits an expansion of type

$$x = \sum_{n=1}^{\infty} \alpha_n e^{\lambda_n t} \in \mathscr{E}^{(p)} \tag{24}$$

that converges with respect to the topology of $\mathscr{E}[0,\infty)$.

Proof The dense property of the subset (23) can be proven similarly as that given in [11]. (The proof given there is for $L^2_{loc}[0,\infty)$ instead of $\mathscr{E}[0,\infty)$ but the proof is similar).

We want to show (24).

Take any $x \in \mathscr{E}^{(p)}$. Then there exists a sequence x_i such that

$$x_i(t) = \sum_{n=1}^{n(i)} \alpha_n^{(i)} e^{\lambda_n t}$$

and $x_i \to x \in \mathscr{E}^{(p)}$ as $i \to \infty$. This means that every derivative of finite order $\sum_{n=1}^{n(i)} \alpha_n^{(i)} \lambda_n^m e^{\lambda_n t}$ converges to $(d/dt)^m x$. In particular, $\sum_{n=1}^{n(i)} \alpha_n^{(i)} \lambda_n^m$ is convergent for every $m \geq 0$. By the same argument as given for (27) below, $\sum_{n=1}^{n(i)} \alpha_n^{(i)} \lambda_n^m e^{\lambda_n t}$ is uniformly and absolutely convergent on every bounded interval $[0,T]$.

We first claim that for each fixed n, the sequence $\{\alpha_n^{(i)}\}$ is convergent as $i \to \infty$. By the Hahn-Banach theorem, take a continuous linear functional $f_n \in (\mathscr{E}^{(p)})'$ such that

$$\langle f_n, e^{\lambda_j t}\rangle = \delta_{jn}$$

where δ_{jn} denotes Kronecker's delta. Then $\langle f_n, \sum_{n=1}^{n(i)} \alpha_n^{(i)} e^{\lambda_n t}\rangle = \alpha_n^{(i)}$. By continuity, the left-hand side converges to $\langle f_n, x\rangle$, so that $\alpha_n^{(i)}$ is convergent, as $i \to \infty$.

Now define $\alpha_n := \lim_{i\to\infty} \alpha_n^{(i)}$. Then

$$x(t) = \lim_{i\to\infty} x_i(t) = \lim_{i\to\infty} \sum_{n=1}^{n(i)} \alpha_n^{(i)} e^{\lambda_n t}.$$

Since the last term converges locally uniformly and absolutely, we can exchange the order of lim and $\sum$, and see that the last term is equal to $\sum_{n=1}^{\infty} \alpha_n e^{\lambda_n t}$. The same can be said of every finite-order derivative, and this shows that the series

$$\sum_{n=1}^{\infty} \alpha_n e^{\lambda_n t}$$

actually converges in $\mathscr{E}^{(p)}$. This completes the proof. □

Note that the proof above works equally well for the multivariable case. All we need to do is to replace α_n by a corresponding eigenvector.

In view of the Lemma above, we are led to the definition

$$\langle \sum_{n=1}^{\infty} \alpha_n e^{\lambda_n t}, \psi\rangle = \sum_{n=1}^{\infty} \alpha_n / q(\lambda_n). \tag{25}$$

We need to show that this gives a continuous linear form on $\mathscr{E}^{(p)}$. Once this is shown to be convergent, it clearly gives the inverse of $[q]$ over $\mathscr{E}'(\mathbb{R}_-)/(p)$ in view of (21).

This is guaranteed by the following lemma:

Lemma 4.4: Let

$$x = \sum_{n=1}^{\infty} \alpha_n e^{\lambda_n t} \in \mathscr{E}^{(p)}. \tag{26}$$

Then for every r,

$$\sum_{n=1}^{\infty} \alpha_n n^r < \infty. \tag{27}$$

In particular,

$$\sum_{n=1}^{\infty} |\alpha_n| < \infty. \tag{28}$$

Sketch of Proof The idea of the proof is that if (26) is convergent (which is guaranteed by Lemma 4.3), then it means a very strong convergence since it should converge with respect to the topology of $\mathscr{E}[0,\infty)$. In particular, the derivative of an arbitrary order should converge. Since λ_n are the zeros of an entire function $\hat{p}(s)$ of exponential type, it grows with order as fast as n [1, Chapter 8]. This essentially yields (27). A complete proof may be found in [12]. □

V. Discussions

It is proven in [2], [4] that systems with commensurable delays are controllable if and only if the matrix R has constant rank for all $\lambda \in \mathbb{C}$. This is somewhat mysterious in the light of Theorem 3.4, since condition 5) requires that there be no "asymptotic cancellation at ∞," while the result by [2], [4] requires only "no cancellation in $\mathbb{C}$."

For the commensurate-delay case, p and q belong to $\mathbb{R}[\delta', \delta_{-a}]$, i.e., $\hat{p}, \hat{q} \in \mathbb{R}[s, e^{as}]$ for some $a > 0$. We can then write $\hat{p} = \hat{p}(s,z)$ and $\hat{q} = \hat{q}(s,z)$ with $z = e^{as}$. Regard $\hat{q}(s,z)$ as a polynomial of two variables in s, z. Then $\hat{q}(s,z)$ as $s \to \infty$ can go to zero only at most with polynomial order in s, z. Hence if there is an asymptotic cancellation as $s \to \infty$, this can be removed by multiplying a suitable factor s^m, because such a cancellation must be of polynomial order. Hence condition (10) is satisfied. This observation is in comformity with a result in [7, Proposition 4.2] which requires that the "cancellation" should be of polynomial type. It also agrees with the general observation why the above simple result for the commensurate delay does not carry over to the noncommensurable delay case.

Example 5.1: Consider the pair $(z, sz-1)$, $z = e^s$. This pair has an asymptotic cancellation for $z = 1/s$, as $s \to \infty$. But this cancellation can be removed by multiplying s to the first component z. Hence the pair $(e^s, se^s - 1)$ is Bézout over $\mathscr{E}'(\mathbb{R})$. Indeed, $e^s \times s + (se^s - 1) \times (-1) = 1$.

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Appendix: Notation and Nomenclature

Let $\mathscr{D}'$ be the space of distributions on $\mathbb{R}$, and $\mathscr{D}'_+(\mathbb{R})$ its subspace consisting of those having support bounded on the left. $\mathscr{E}'(\mathbb{R})$ denotes the subspace consisting of those with compact support. $\mathscr{E}'(\mathbb{R}_-)$ is the subspace of $\mathscr{E}'(\mathbb{R})$ consisting of those with support contained in the negative half line $(-\infty, 0]$. Each of these spaces constitutes a convolution algebra. Distributions such as Dirac's delta δ_a placed at $a \in \mathbb{R}$, its derivative δ'_a are examples of elements in $\mathscr{E}'(\mathbb{R}_-)$. A distribution α is said to be of *order* at most m if it can be extended as a continuous linear functional on the space of m-times continuously differentiable functions. Such a distribution is said to be of *finite order*. The largest number m, if one exists, is called the *order* of α ([5], [6]). The delta distribution δ_a, $a \in \mathbb{R}$ is of order zero, and its derivative δ'_a is of order one, etc. A distribution with compact support is known to be always of finite order ([5], [6]).

For a distribution $\alpha \in \mathscr{E}'(\mathbb{R})$, define real numbers $\ell(\alpha)$ and $r(\alpha)$ by

$$\ell(\alpha) \quad := \quad \inf\{t | t \in \operatorname{supp} \alpha\}, \tag{29}$$

$$r(\alpha) \quad := \quad \sup\{t | t \in \operatorname{supp} \alpha\}. \tag{30}$$

For a distribution $f \in \mathscr{D}'$, we denote its Laplace transform by $\hat{f}$, if it exists. Every $f \in \mathscr{E}'(\mathbb{R})$ has Laplace transform. The following Paley-Wiener theorem is fundamental for the Laplace transform of elements in $\mathscr{E}'(\mathbb{R})$.

Theorem 6.1 ([5]): A complex analytic function $f(s)$ is the Laplace transform of a distribution $\phi \in \mathscr{E}'(\mathbb{R})$ if and only if $f(s)$ is an entire function that satisfies the following growth estimate for some $C > 0, a > 0$ and integer $m \geq 0$:

$$|f(s)| \quad \leq \quad C(1+|s|)^m e^{a|\operatorname{Re} s|}. \tag{31}$$

In particular, $f(s) = \hat{\phi}(s)$ for some $\phi \in \mathscr{E}'(\mathbb{R}_-)$ if and only if it satisfies the estimate

$$\begin{aligned} |\hat{f}(s)| &\leq C(1+|s|)^m e^{a \operatorname{Re} s}, \operatorname{Re} s \geq 0, \\ &\leq C(1+|s|)^m, \operatorname{Re} s \leq 0 \end{aligned} \tag{32}$$

for some $C > 0, a > 0$ and integer $m \geq 0$. In this case, the support of ϕ is contained in $[-a, 0]$
We will refer to (31) as the Paley-Wiener estimate.

The zeros of $\hat{f}(s)$ are discrete, and each zero has a finite multiplicity. This in particular implies the following Hadamard factorization for $\hat{f}(s)$ [1]:

$$\hat{f}(s) = s^k e^{as} \prod_{n=1}^{\infty} \left(1 - \frac{s}{\lambda_n}\right) \exp\left(\frac{s}{\lambda_n}\right). \tag{33}$$

Since there are no finite accumulation point for $\{\lambda_n\}$, $\lambda_n \to \infty$ as $n \to \infty$.

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