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Existence Conditions of Decoupling Controllers in the Generalized Plant Model

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***Abstract*—Necessary and sufficient conditions for the existence of diagonal, block-diagonal and triangular decoupling controllers in linear multivariable systems are presented for the most general setting. The plant model in this paper is general enough to accommodate non-square plant and non-unity feedback cases with 1DOF (one-degree-of-freedom) or 2DOF controller configuration. It is shown that the existence condition is finally described in terms of rank conditions on coefficient matrices in partial fraction expansion.**

I. Introduction

THE existence condition of decoupling controllers in linear multivariable systems has been studied in the past. Vardulakis [10] proposed a sufficient condition that a diagonal decoupling controller exists if there is no unstable pole-zero coincidence of the plant. Necessary and sufficient conditions for decoupling controllers were obtained in various ways. Lin [5,6] exploited the internal stability requirement as the constraints in constructing diagonal and block-diagonal input-output maps. Youla and Bongiorno [13] took similar approach with that of [5,6] for a diagonal decoupling problem but the class of all stabilizing decoupled transfer matrices were explicitly parameterized, which made it possible to derive the optimal H_2 decoupling controller. G □ mez and Goodwin [2] adopted an algebraic approach based on coprime factorizations to treat diagonal and triangular decoupling designs. In [12], a unifying approach was suggested to treat diagonal, block-diagonal and triangular decoupling problems. Above mentioned papers, however, considered the conventional model with unity feedback [2,5,6,10] or with state- feedback [12]. In [13], the unity feedback constraint was loosened but arbitrary non-unity feedback was still not assumed. Deseor and G □ ndes[15] derived all diagonal input-output maps achievable by stabilizing 2DOF(two-degree-of-freedom) controllers. Their derivation was based on a general setting which included delay or infinite dimensional systems.

In this paper, necessary and sufficient conditions for the existence of decoupling controllers are presented in the generalized plant model which accommodates non-square plants and non-unity feedback case with 1DOF or 2DOF configuration. The approach taken in this paper is so simple and direct that diagonal, block- diagonal and triangular decoupling problems are treated in a unified frame. It turns out that the existence condition is described in terms of rank conditions on the coefficient matrices in partial fraction expansion.

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Notations; Throughout the paper, only real rational matrices are considered. The notation T_{ba} stands for the transfer matrix from a to b. A rational matrix $G(s)$ is said to be stable if it is analytic in $\operatorname{Re} s \geq 0$ and iff stands for "if and only if". The Kronecker product of two matrices is denoted as $G \otimes R$. $vec(G)$ denotes the vector formed by stacking all the columns of the matrix G. The Khatri-Rao product of two matrices is denoted as $G \odot R$ and is the matrix whose i-column is given by $g_i \otimes r_i$ where g_i and r_i are the i-column of G and i- column of R, respectively. For a diagonal matrix, $vecd\,(G)$ denotes the vector formed by stacking all the diagonal elements of the matrix G. When V is a diagonal matrix, $vec\,(AVD) = (D' \otimes A) vec\,(V) = (D' \odot A) vecd(V)$ [1].

II. Internal Stability and Realizability Conditions

The generalized plant model under consideration is shown in Figure 1. The variable r is an exogenous input and the variable v is the target variable we are interested in. The variables u and y are the control input and the measured variable, respectively. The variables r and v are the ones such that the transfer matrix T_{vr} is to be decoupled. In most cases, r is the reference input and v is the plant output. The transfer matrix of the generalized plant is given by

$$\begin{bmatrix} v \\ y \end{bmatrix} = P \begin{bmatrix} r \\ u \end{bmatrix}, \quad P = \begin{bmatrix} P_{00} & P_{02} \\ P_{20} & P_{22} \end{bmatrix}. \tag{1}$$

The variables v and r have the same dimension of $m \times 1$. The variables u and y have the dimensions $m_1 \times 1$ and $m_2 \times 1$, respectively.

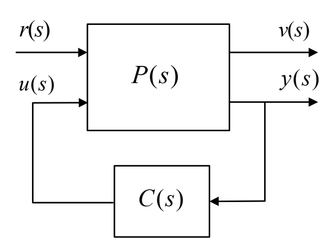

Figure 1. The generalized plant model

The following assumption is necessary and sufficient for the existence of a stabilizing controller [9].

Assumption 1: The general plant block $P(s)$ is free of hidden modes in $\operatorname{Re} s \geq 0$ and $\Psi_P{}^{+} = \Psi_{P_{22}}{}^{+}$.

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The notation Ψ_P denotes the characteristic denominator [14] of the rational matrix $P(s)$ and Ψ_P^+ absorbs all the zeros in $\mathrm{Re}\, s \geq 0$. Consider the polynomial coprime fractional expressions for P_{22};

$$P_{22} = A^{-1}(s)B(s) = B_1(s)A_1^{-1}(s). \tag{2}$$

There always exist polynomial matrices $X(s), Y(s), X_1(s)$ and $Y_1(s)$ such that

$$\begin{bmatrix} X_1 & Y_1 \\ -B & A \end{bmatrix}\begin{bmatrix} A_1 & -Y \\ B_1 & X \end{bmatrix} = \begin{bmatrix} A_1 & -Y \\ B_1 & X \end{bmatrix}\begin{bmatrix} X_1 & Y_1 \\ -B & A \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \tag{3}$$

with $\det X(s) \det X_1(s) \not\equiv 0$. (Adopting proper stable rational coprime fractions does not affect the remaining results of this paper). It is well known [7] that the condition $\Psi_P^+ = \Psi_{P_{22}}^+$ in Assumption 1 is equivalent to the one that

$$P_{00} - P_{02}A_1Y_1P_{20}, \quad P_{02}A_1 \text{ and } AP_{20} \tag{4}$$

are stable. $T_{vr}(s)$ is the transfer matrix to be decoupled and is given by

$$T := T_{vr}(s) = P_{00} + P_{02}(I - CP_{22})^{-1}CP_{20}. \tag{5}$$

Definition 1: A rational matrix $T(s)$ is said to be realizable for the given plant $P(s)$ if there exists a stabilizing controller $C(s)$ that realizes the transfer matrix $T_{vr}(s)$ of the system as the matrix $T(s)$.

From (1), it follows that $v = P_{00}\, r + P_{02}\, u$ and, usually, the variable v is the plant output and it does not contain a direct term of the reference input r. Hence, in almost all cases P_{00} becomes a null matrix. When $P_{00} = 0$, it follows that

$$T = P_{02}(I - CP_{22})^{-1}CP_{20} \tag{6}$$

In decoupling design, the transfer matrix T is required to be of full rank as well as the diagonal requirement. In view of (6) it is necessary that $m \leq m_1$ and $rank(P_{02}) = m$ for the full rank requirement of T. Similarly, it is required that $m \leq m_2$ and $rank(P_{20}) = m$. Although we presume that $P_{00} = 0$, we don't assume this to keep the plant model as general as possible and assume only the following rank conditions.

Assumption 2:
$m \leq m_1$, $m \leq m_2$ and $rank(P_{02}) = rank(P_{20}) = m$.

Next, consider the class of all stabilizing controllers characterized by the formula

$$C(s) = -(X_1 - KB)^{-1}(Y_1 + KA) \tag{7}$$

where $K(s)$ arbitrary real rational stable matrices such that $\det(X_1 - KB) \not\equiv 0$. Inserting this formula to (5), we obtain

$$T = T_0 - P_{02}A_1KAP_{20}, \tag{8}$$

where

$$T_0 := P_{00} - P_{02}A_1Y_1P_{20} \tag{9}$$

is a stable matrix by (4). Notice that T is also stable since $P_{02}A_1$, AP_{20} and K are stable. Since $rank(P_{02}) = rank(P_{20}) = m$, the ranks of $P_{02}A_1$ and AP_{20} are also m and in this case it is well known that there exist $m_1 \times m_1$ and $m_2 \times m_2$ unimodular matrices V_1 and V_2 [3,4] such that

$$P_{02}A_1V_1 = \begin{bmatrix} R_{10} & 0 \end{bmatrix} \text{ and } V_2AP_{20} = \begin{bmatrix} R_{20} \\ 0 \end{bmatrix} \tag{10}$$

with $rank(R_{10}) = rank(R_{20}) = m$. So it follows that

$$T = T_0 - P_{02}A_1KAP_{20} = T_0 - \begin{bmatrix} R_{10} & 0 \end{bmatrix}\hat{K}\begin{bmatrix} R_{20} \\ 0 \end{bmatrix}, \tag{11}$$

where $\hat{K} = V_1^{-1}KV_2^{-1}$. Now, consider the following partition

$$\hat{K} = V_1^{-1}KV_2^{-1} = \begin{bmatrix} \hat{K}_{11} & \hat{K}_{12} \\ \hat{K}_{21} & \hat{K}_{22} \end{bmatrix} \tag{12}$$

where the dimensions of $\hat{K}_{11}$ and $\hat{K}_{22}$ are $m \times m$ and $(m_1 - m) \times (m_2 - m)$, respectively. Then it follows that

$$T = T_0 - R_{10}\hat{K}_{11}R_{20} \tag{13}$$

and hence

$$\hat{K}_{11} = R_{10}^{-1}T_0R_{20}^{-1} - R_{10}^{-1}TR_{20}^{-1}. \tag{14}$$

In view of (7) and (14), a stable rational matrix T is realizable for $P(s)$ iff it makes $R_{10}^{-1}T_0R_{20}^{-1} - R_{10}^{-1}TR_{20}^{-1}$ stable. As we can see, a realizable T determines only $\hat{K}_{11}$, a part of $\hat{K}$, and the other parts of $\hat{K}$ can be obtained by other criterions of control system design.

The equation in (14) has the typical structure to which many realizability problems ultimately reduce. So, instead of solving the specific equation of (14) we will stick to the more general form in the following:

Standard problem for realizability(SPR): Given matrices $\Phi_i, i = \alpha, \beta, \gamma$, find a stable T that makes Φ_s stable where

$$\Phi_s = \Phi_\gamma - \Phi_\alpha T \Phi_\beta \tag{15}$$

and the dimensions of Φ_α, Φ_β and T are $m_3 \times n$, $n \times m_4$ and $n \times n$, respectively.

The realizability problem of T for the generalized plant model in Fig. 1 reduces to the SPR with

$$\Phi_\gamma = R_{10}^{-1}T_0R_{20}^{-1}, \quad \Phi_\alpha = R_{10}^{-1} \text{ and } \Phi_\beta = R_{20}^{-1} \tag{16}$$

For different realizability problems, we have different values of Φ_α, Φ_β and Φ_γ (see section V).

III. Decoupling problems

In the previous section, we have defined SPR in which we find a stable transfer matrix T that guarantees the existence of a stabilizing controller C. When an additional requirement is added to the transfer matrix T, we need to add this constraint on T in solving SPR. It will be shown shortly that when a decoupling constraint is added, the standard equation in SPR can be transformed to a new standard equation by vector operation and this invokes introduction of a new standard problem.

Standard Problem for Decoupling Design (SPDD): Given a vector ϕ and a matrix Ψ, find a stable vector $h(s)$ that makes ϕ_s stable where

$$\phi_s = \phi - \Psi h;\quad h : \tilde{n}\times 1,\ \Psi : m_5 \times \tilde{n}\,. \tag{17}$$

In the following, we will consider the three decoupling problems and explain the procedure of transforming SPR with a given decoupling constraint to SPDD.

A. Diagonal Decoupling

Suppose that we ask T to be diagonal in SPR. Taking vector operation on both sides of eq. (15) we get SPDD equation in (17) with

$$\phi = vec(\Phi_\gamma),\ \Psi = \Phi'_\beta \odot \Phi_\alpha \text{ and } h = vecd(T)\,. \tag{18}$$

B. Block Decoupling

Suppose that T in SPR is a block diagonal matrix of the form $T = diag\{T_i\}_{i=1}^{k}$ where T_i is an $n_i \times n_i$ block matrix and $n_1 + n_2 + \cdots + n_k = n$. Consider the partitions

$$\Phi_\alpha = [\,\Phi_1^\alpha \quad \Phi_2^\alpha \quad \cdots \quad \Phi_k^\alpha\,]\ , \tag{19}$$

$$\Phi_\beta = [(\Phi_1^\beta)' \quad (\Phi_2^\beta)' \quad \cdots \quad (\Phi_k^\beta)']\ ', \tag{20}$$

where the dimensions of Φ_i^α and Φ_i^β are $m_3 \times n_i$ and $n_i \times m_4$, respectively. Then it follows that

$$\Phi_\alpha T \Phi_\beta = \textstyle\sum_{i=1}^{k} \Phi_i^\alpha T_i \Phi_i^\beta\,. \tag{21}$$

Taking vector operation on both sides of (15) with (21), we get the equation (17) with

$$\Psi = [\ (\Phi_1^\beta)'\otimes\Phi_1^\alpha \quad (\Phi_2^\beta)'\otimes\Phi_2^\alpha \quad \cdots \quad (\Phi_k^\beta)'\otimes\Phi_k^\alpha\ ] \tag{22}$$

$$\phi = vec(\Phi_\gamma),\ h = [vec(T_1)' \quad vec(T_2)' \quad \cdots \quad vec(T_k)']'\,. \tag{23}$$

C. Triangular Decoupling

We consider only the lower triangular case here. The formula for the upper triangular case can be obtained by minor modification of the results given below. Suppose that T in SPR is a lower triangular form of $T = [t_{ij}]$, $t_{ij} = 0$, for $i > j$. In this case, it can be shown that

$$\Phi_\alpha T \Phi_\beta = \textstyle\sum_{i=1}^{n} \Phi_{i,n}^\alpha T_{(i-1)d} \Phi_{1,(n+1-i)}^\beta \tag{24}$$

where Φ_{c_1,c_2}^α is the matrix consisting of the columns of Φ_α from c_1 - column to c_2 - column and Φ_{r_1,r_2}^β is the matrix consisting of the rows of Φ_β from $r_1 -$ row to $r_2 -$ row and T_{id} denotes the lower $i-th$ off-diagonal matrix of T with the dimension $(n-i)\times(n-i)$. That is,

$$T_{id} = diag\{t_{i+1,1},\quad t_{i+2,2},\quad t_{i+3,3},\quad \cdots,\quad t_{n,n-i}\} \tag{25}$$

for $i = 0, 1, \cdots, n-1$. Taking vector operation on both sides of (15) with (24), we get (17) with

$$\Psi = [(\Phi_{1,n}^\beta)'\odot\Phi_{1,n}^\alpha \vdots (\Phi_{1,n-1}^\beta)'\odot\Phi_{2,n}^\alpha \vdots\cdots\vdots (\Phi_{1,1}^\beta)'\odot\Phi_{n,n}^\alpha],$$
$$\phi = vec(\Phi_\gamma) \tag{26}$$

and

$$h = \lfloor vecd(T_{0d})' \quad vecd(T_{1d})' \quad \cdots \quad vecd(T_{(n-1)d})' \rfloor\ '\,. \tag{27}$$

IV. Solvability condition of SPDD

In the previous section, we have shown that the realizability problems associated with various decoupling constraints are reduced to SPDD. Now we will find the necessary and sufficient condition for the existence of a solution to SPDD.

A. Simple Pole Case

Suppose that s_i, $i = 1,2,\cdots,\nu$ are distinct unstable poles of ϕ or Ψ in (17) and they are simple. Then it is possible to express ϕ and Ψ as

$$\phi = \sum_{i=1}^{\nu} \frac{r_i}{s - s_i} + \phi_0(s) \quad \text{and} \quad \Psi = \sum_{i=1}^{\nu} \frac{R_i}{s - s_i} + \Psi_0(s), \tag{28}$$

where r_i and R_i are the residues of ϕ and Ψ at s_i, respectively, and $\phi_0(s)$ and $\Psi_0(s)$ are stable . Then it follows from (17) that

$$\phi_s = \sum_{i=1}^{\nu} \frac{r_i - R_i h(s)}{s - s_i} + \phi_0(s) - \Psi_0(s)h(s) \tag{29}$$

and the vector ϕ_s is stable iff the summation term is stable, which is equivalent to the condition that $R_i\, h(s_i) = r_i$, $i = 1,2,\cdots,\nu$. This linear equation has a solution $h(s_i)$ iff

$$rank(R_i) = rank([R_i \vdots r_i]) \tag{30}$$

(or, equivalently, r_i is included in the range space of R_i) for each $i = 1,2,\cdots,\nu$. Suppose that the above rank condition is satisfied and let a solution for $h(s_i)$ be μ_i. It is not difficult to show that there always exists a stable vector $h(s)$ satisfying the interpolation conditions $h(s_i) = \mu_i$, $i = 1 \to \nu$. Hence the rank condition in (30) is the necessary and sufficient condition for SPDD to have a solution.

B. General Case

Now we consider the general case. Let s_i, $i=1,2,\cdots,\nu$ be the distinct unstable poles of ϕ or Ψ in (17) and let $p_i=\max(p_\phi,p_\Psi)$ where p_ϕ and p_Ψ are the multiplicities of s_i as the pole of ϕ and Ψ, respectively. Then ϕ and Ψ are expressed as

$$\phi=\sum_{i=1}^{\nu}\sum_{k=1}^{p_i}\frac{r_i^k}{(s-s_i)^k}+\phi_0(s) \tag{31}$$

$$\Psi=\sum_{i=1}^{\nu}\sum_{k=1}^{p_i}\frac{R_i^k}{(s-s_i)^k}+\Psi_0(s) \tag{32}$$

where $\phi_0(s)$ and $\Psi_0(s)$ are stable. From (17), it follows that

$$\phi_s=\sum_{i=1}^{\nu}\phi_{si}+\phi_0(s)-\Psi_0(s)h(s) \tag{33}$$

where

$$\phi_{si}=\sum_{k=1}^{p_i}\frac{r_i^k-R_i^kh(s)}{(s-s_i)^k}\,. \tag{34}$$

Since $\phi_0(s)$, $\Psi_0(s)$ and $h(s)$ are stable, ϕ_s is stable iff ϕ_{si} is stable for each $i=1,2,\cdots,\nu$. Let's find the partial fraction expansion of ϕ_{si} at the pole s_i. For ease of presentation, we will consider the case $p_i=3$. After straightforward calculation, we get the results

$$\phi_{si}=\frac{\xi_i^3}{(s-s_i)^3}+\frac{\xi_i^2}{(s-s_i)^2}+\frac{\xi_i^1}{s-s_i}+\phi_{si0}(s) \tag{35}$$

where

$$\xi_i^3=r_i^3-R_i^3h(s_i) \tag{36.a}$$
$$\xi_i^2=-R_i^3h'(s_i)+r_i^2-R_i^2h(s_i) \tag{36.b}$$
$$\xi_i^1=-(1/2)R_i^3h''(s_i)-R_i^2h'(s_i)+r_i^1-R_i^1h(s_i) \tag{36.c}$$

and ϕ_{si0} a stable vector. Since ϕ_{si0} is stable, ϕ_{si} is stable iff

$$\xi_i^k=0,\ k=1,2,3\,. \tag{37}$$

Resolving (36) and (37), we get the following linear equation

$$\widetilde{R}_i\widetilde{h}_i=\widetilde{r}_i \tag{38}$$

with

$$\widetilde{R}_i=\begin{bmatrix}R_i^3 & 0 & 0\\ R_i^2 & R_i^3 & 0\\ R_i^1 & R_i^2 & R_i^3\end{bmatrix} \tag{39}$$

$$\widetilde{r}_i=\begin{bmatrix}r_i^3\\ r_i^2\\ r_i^1\end{bmatrix}\ \text{and}\ \widetilde{h}_i=\begin{bmatrix}h(s_i)\\ h'(s_i)\\ (1/2)h''(s_i)\end{bmatrix}. \tag{40}$$

Hence the condition in (37) is satisfied iff there exists a solution $\widetilde{h}_i$ for the equation (38) and this leads to the condition

$$rank(\widetilde{R}_i)=rank([\widetilde{R}_i\,\vdots\,\widetilde{r}_i])\,. \tag{41}$$

Suppose that the above rank condition is met and let $\widetilde{\mu}_i=[(\mu_i^0)'\ (\mu_i^1)'\ 1/2(\mu_i^2)']'$ be a solution for $\widetilde{h}_i$ so that

$$h(s_i)=\mu_i^0,\ h'(s_i)=\mu_i^1\ \ and\ \ h''(s_i)=\mu_i^2\,. \tag{42}$$

The remaining thing is to show that there exists a stable rational vector $h(s)$ satisfying the interpolation conditions in (42). Let's denote the elements of the vectors $h(s)$ and $\mu_i^q, q=1\rightarrow 3$ as following;

$$h(s)=[h_1(s)\ h_2(s)\cdots h_{\widetilde{n}}(s)]' \tag{43}$$
$$\mu_i^q=[\mu_{i1}^q\ \mu_{i2}^q\cdots\mu_{i\widetilde{n}}^q]'\ ,\ q=1,2,3\,. \tag{44}$$

Then finding a vector $h(s)$ satisfying the interpolation conditions in (42) becomes finding a scalar function $h_k(s)$ satisfying the following interpolation conditions for $k=1\rightarrow\widetilde{n}$;

$$h_k(s_i)=\mu_{ik}^0,\ h_k'(s_i)=\mu_{ik}^1\ \ and\ \ h_k''(s_i)=\mu_{ik}^2\,. \tag{45}$$

Let $h_k(s)=m_k(s)/g_k(s)$ where $m_k(s)$ is an arbitrary polynomial and $g_k(s)$ is an arbitrary fixed strict-Hurwitz polynomial. Then,

$$g_k(s)h_k(s)=m_k(s) \tag{46}$$
$$g_k'(s)h_k(s)+g_k(s)h_k'(s)=m_k'(s) \tag{47}$$
$$g_k''(s)h_k(s)+2g_k'(s)h_k'(s)+g_k(s)h_k''(s)=m_k''(s) \tag{48}$$

Hence the values of $m_k(s_i)$, $m_k'(s_i)$ and $m_k''(s_i)$ are obtained from $h_k(s_i)$, $h_k'(s_i)$ and $h_k''(s_i)$ and let's denote them as $\widetilde{\mu}_{ik}^q, q=1,2,3$. Now the problem of finding $h_k(s)$ satisfying (45) becomes one of finding the polynomial $m_k(s)$ satisfying the interpolation conditions

$$m_k(s_i)=\widetilde{\mu}_{ik}^0,\ m_k'(s_i)=\widetilde{\mu}_{ik}^1\ \ and\ \ m_k''(s_i)=\widetilde{\mu}_{ik}^2 \tag{49}$$

and such a polynomial $m_k(s)$ always exists. Now let's extend these results to the general case. Since the interpolation conditions in (45) should be satisfied for each value of $i=1,2,\cdots,\nu$, the scalar function $h_k(s)$ should satisfy the interpolation conditions

$$h_k^{(q)}(s_i)=\mu_{ik}^q,\quad q=0,1,2,\cdots,p_i-1;\quad i=1,2,\cdots,\nu \tag{50}$$

As shown previously, the problem can be changed to one of finding a polynomial $m_k(s)$ satisfying the interpolation conditions

$$m_k^{(q)}(s_i)=\widetilde{\mu}_{ik}^q,\quad q=0,1,2,\cdots,p_i-1;\quad i=1,2,\cdots,\nu \tag{51}$$

It is well known that such a polynomial $m_k(s)$ always exists [page 49, 11]. Hence the rank equality in (41) for each i is the necessary and sufficient condition for SPDD to have a solution. Now we are ready to state the main theorem.

Theorem: Let s_i, $i=1,2,\cdots,\nu$, be the distinct unstable poles of ϕ or Ψ in (17) with multiplicity $p_i=\max(p_\phi,p_\Psi)$. The SPDD has a solution iff $\widetilde{R}_i\widetilde{h}_i=\widetilde{r}_i$ has a solution $\widetilde{h}_i$ for each i or, equivalently, the following rank conditions are satisfied;

$$rank(\widetilde{R}_i)=rank([\widetilde{R}_i \vdots \widetilde{r}_i]),\ \text{for } i=1,2,\cdots,\nu \tag{52}$$

where

$$\widetilde{R}_i=\begin{bmatrix} R_i^{p_i} & 0 & 0 & \cdots & 0 & 0\\ R_i^{p_i-1} & R_i^{p_i} & 0 & \ddots & \ddots & 0\\ \vdots & R_i^{p_i-1} & \ddots & \ddots & \ddots & \vdots\\ R_i^3 & \ddots & \ddots & \ddots & 0 & \vdots\\ R_i^2 & R_i^3 & \ddots & R_i^{p_i-1} & R_i^{p_i} & 0\\ R_i^1 & R_i^2 & R_i^3 & \cdots & R_i^{p_i-1} & R_i^{p_i}\end{bmatrix} \tag{53}$$

$$\widetilde{r}_i=\begin{bmatrix} r_i^{p_i}\\ r_i^{p_i-1}\\ \vdots\\ r_i^2\\ r_i^1\end{bmatrix},\quad \widetilde{h}_i=\begin{bmatrix} h(s_i)\\ h'(s_i)\\ (1/2)h''(s_i)\\ \vdots\\ (1/(p_i-2)!)h^{(p_i-2)}(s_i)\\ (1/(p_i-1)!)h^{(p_i-1)}(s_i)\end{bmatrix}. \tag{54}$$

Here, r_i^q and R_i^q are the coefficients of the term $1/(s-s_i)^q$ in partial fraction expansions of ϕ and Ψ, respectively.

To sum up, checking the existence of a decoupling controller for the generalized plant model is finally reduced to checking the rank conditions in (52). A realizable decoupling matrix T can be obtained by finding a stable $h(s)$ satisfying the interpolation constraints. Note also that $\widetilde{R}_i$ in (53) is a lower triangular block Toeplitz matrix.

V. Special cases

The plant model in Fig. 1 is general enough to include the cases of non-square plants, non-unity feedback, 1DOF and 2DOF controller configurations. When we make some assumptions on the structure of the transfer matrices of $P(s)$, we get more specified results.

A. Square Plant with 1DOF Controller Case

Suppose that $P_{00}=0$, $m=m_1$ (square plant case) and $P_{20}=I$ (1DOF case). In this case, we can show from (8) that a stable rational matrix $T(s)$ is realizable iff $P_{02}{}^{-1}T$, $P_{02}{}^{-1}TP_{22}$, $P_{22}P_{02}{}^{-1}T$ and $(I+P_{22}P_{02}{}^{-1}T)P_{22}$ are stable and $\det(I+P_{22}P_{02}{}^{-1}T)\not\equiv 0$ (This condition can also be obtained by input-output stability requirement [8]). These five matrices can be compactly described by

$$[\ 0_{2m\times m}\ \ \hat{P}_{22}\ ]+\widetilde{P}_{22}P_{02}^{-1}T[\ I_m\ \ P_{22}\ ]=:\Phi_s \tag{55}$$

where

$$\hat{P}_{22}=[\,0_{m\times m}\ \ P_{22}{}'\,]'\ \text{ and }\ \widetilde{P}_{22}=[\,I_m\ \ P_{22}{}'\,]'. \tag{56}$$

Hence $T(s)$ is realizable iff Φ_s is stable, which leads to the SPR in (15) with

$$\Phi_\gamma=[\ 0_{2m\times m}\ \ \hat{P}_{22}\ ],\ \Phi_\alpha=\widetilde{P}_{22}P_{02}^{-1}\ \text{ and }\ \Phi_\beta=[\ I_m\ \ P_{22}\ ]. \tag{57}$$

The existence condition of various decoupling controllers for this special case can be checked by the procedures in sections III and IV and notice that, in this case, we don't need coprime factorizations.

When a diagonal decoupling T is sought, we can parameterize it further. From (55), T is realizable iff $\widetilde{P}_{22}P_{02}^{-1}T$ and $\hat{P}_{22}+\widetilde{P}_{22}P_{02}^{-1}TP_{22}$ are stable. For the term $\widetilde{P}_{22}P_{02}^{-1}T$ to be stable, T must be of the form $T=\Delta_\theta\Delta$ where Δ is an arbitrary diagonal stable matrix and $\Delta_\theta=diag\{\theta_i\}_{i=1}^m$ with θ_i being the monic polynomial of the minimal degree such that $\{\ i-$column of $\widetilde{P}_{22}P_{02}^{-1}\ \}\times\theta_i$ is stable. Hence T is realizable iff $\hat{P}_{22}+\widetilde{P}_{22}P_{02}^{-1}\Delta_\theta\Delta P_{22}$ is stable which leads to the SPDD with

$$\phi=vec(\hat{P}_{22}),\quad \Psi=(-P_{22}{}')\odot(\widetilde{P}_{22}P_{02}^{-1}\Delta_\theta) \tag{58}$$

$$h(s)=vecd(\Delta(s)). \tag{59}$$

B. Other Cases

By exploiting the results of Theorem we can easily prove that a diagonal decoupling controller exists for the following cases; 1) $P_{00}=0$ and the plant $P_{22}(s)$ is stable. 2) $P_{00}=0$, $P_{20}=[I\ \ 0\,]'$ and $P_{22}=[\,0\ \ P_a{}'\,]'$ (2DOF case). 3) $P_{00}=0$, $m=m_1$ (square plant), $P_{20}=I$ (1DOF case), $P_{02}=-P_{22}$ (unity feedback case) and there is no unstable pole-zero coincidence of the plant P_{22}.

Since the proofs of 1) and 2) are trivial, only the proof of 3) is given. Under the assumptions in 3), the equation in (14) becomes

$$\hat{K}_{11}=\hat{K}=K=-Y_1A^{-1}+B_1^{-1}TA^{-1}. \tag{60}$$

This leads to the standard equation of SPDD with

$$\phi=vec(Y_1A^{-1}),\quad \Psi=(A^{-1})'\odot B_1^{-1} \tag{61}$$

$$h(s)=vecd(T(s)). \tag{62}$$

Any unstable pole of ϕ or Ψ, if exists, comes from that of A^{-1} or B_1^{-1}. Let's consider an unstable pole s_i of B_1^{-1}. Since the poles of A^{-1} and B_1^{-1} are different by assumption, the coefficient vector $\widetilde{r}_i$ for s_i is zero and hence the condition $rank(\widetilde{R}_i)=$ $rank([\widetilde{R}_i \vdots \widetilde{r}_i])$ is trivially satisfied. Next, consider an unstable pole s_j of A^{-1}. From (3), we have $X_1A_1+Y_1B_1=I$ and it follows that $X_1A_1B_1^{-1}A^{-1}+Y_1A^{-1}=$ $B_1^{-1}A^{-1}$ and hence

$$X_1 B^{-1} + Y_1 A^{-1} = B_1^{-1} A^{-1}. \tag{63}$$

Taking vector operation on both sides, we obtain

$$vec(X_1 B^{-1}) + vec(Y_1 A^{-1}) = ((A^{-1})' \odot B_1^{-1})\, vecd\,(I_m). \tag{64}$$

Inserting this equality to (61), we obtain another expression for $\phi(s)$ as

$$\phi(s) = ((A^{-1})' \odot B_1^{-1})\, vecd\,(I_m) - vec(X_1 B^{-1}). \tag{65}$$

Let's denote the partial fraction coefficient matrix of Ψ for s_j as $\widetilde{R}_j$. Since B^{-1} dose not have the pole s_j, the partial fraction coefficient vector $\widetilde{r}_j$ of ϕ for s_j becomes $\widetilde{R}_j vecd\,(I_m)$ and this implies that $\widetilde{r}_j$ is a linear combination of the columns of $\widetilde{R}_j$. Hence the condition $rank\,(\widetilde{R}_j) = rank\,([\widetilde{R}_j \vdots \widetilde{r}_j])$ is satisfied and this completes the proof.

VI. Example

Consider the case of 1DOF controller configuration with the square plant $P_a(s)$ and the non-unity feedback sensor F. In this case the transfer matrices in (1) are given by

$$P_{00} = 0,\ P_{02} = P_a(s),\ P_{20} = I,\ and\ \ P_{22} = -FP_a(s) \tag{66}$$

Consider the following plant [13] and the non-unity feedback

$$P_a = \begin{bmatrix} \frac{s-1}{s+2} & \frac{s-1}{s+2} \\ \frac{s+2}{s-1} & \frac{2(s+2)}{s-1} \end{bmatrix},\ \text{and}\quad F = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}. \tag{67}$$

Notice that we can use the formulas in (58) and (59). Since

$$P_a^{-1} = \begin{bmatrix} \frac{2(s+2)}{s-1} & \frac{1-s}{s+2} \\ \frac{s+2}{1-s} & \frac{s-1}{s+2} \end{bmatrix}, \tag{68}$$

we obtain $\Delta_\theta = diag\{s-1,\ 1\}$ after simple calculations. The vector $\phi(s)$ and the matrix $\Psi(s)$ have a simple pole at $s_1 = 1$. The residue values at $s_1 = 1$ are obtained as

$$r_1 = [0\ 0\ -3\ -3\ 0\ 0\ -6\ \ -6]' \tag{69}$$

$$R_1 = \begin{bmatrix} 0 & 0 & -3 & -3 & 0 & 0 & -6 & -6 \\ 18 & -9 & 0 & 0 & 36 & -18 & 0 & 0 \end{bmatrix}'. \tag{70}$$

Since $rank\,(R_1) = rank\,([R_1 \vdots r_1]) = 2$, a diagonal decoupling solution exists. Taking a solution for the equation $R_1\, h(s_1) = r_1$ as $h(1) = [0\ 1]'$, then a diagonal solution for $T(s)$ is parameterized as

$$T(s) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} s & 0 \\ 0 & s-1 \end{bmatrix} \begin{bmatrix} h_a & 0 \\ 0 & h_b \end{bmatrix} \tag{71}$$

where h_a and h_b are arbitrary stable rational functions. The controller C(s) in this case can be obtained from (5).

VII. Conclusion

The existence condition of diagonal, block-diagonal and triangular decoupling controllers are obtained for the generalized plant model. It is shown that these decoupling problems can be transformed to a solvable standard form SPDD and procedures to obtain solutions of SPDD by solving interpolation problems are explained. The existence condition of a solution for SPDD is described in terms of rank condition on a block Toeplitz matrix whose elements are the coefficient matrices in partial fraction expansion.

Possible future research works include the characterization of solution $h(s)$ for SPDD, analyzing the relationship with the previous related works on the existence conditions of decoupling controllers, and investigating the algebraic properties of lower-triangular block Toeplitz matrices.

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