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Boundary Controllers for Euler-Bernoulli Beam with Arbitrary Decay Rate

Andrey Smyshlyaev, Bao-Zhu Guo, and Miroslav Krstic

***Abstract*— We consider a problem of stabilization of the Euler-Bernoulli beam. The beam is controlled at one end (using position and moment actuators) and has the "sliding" boundary condition at the opposite end. We design the controllers that achieve any prescribed decay rate of the closed loop system, improving upon the existing "boundary damper" controllers. The idea of the control design is to use the well-known representation of the Euler-Bernoulli beam model through the Schrödinger equation, and then adapt recently developed backstepping designs for the latter in order to stabilize the beam. We derive the explicit integral transformation (and its inverse) of the closed-loop system into an exponentially stable target system. The transformation is of a novel Volterra/Fredholm type. The design is illustrated with simulations.**

I. Introduction

In this paper we present a novel control design for the Euler-Bernoulli beam, which achieves arbitrary decay rate of the closed-loop system by boundary feedback. The beam is controlled at one boundary using position and moment actuators and has the "sliding" boundary condition at the uncontrolled boundary (Fig. 1). We assume that the full state (displacement and velocity) measurements are available.

The idea of our method is to use the well-known representation of the Euler-Bernoulli beam model through the Schrödinger equation [13]. For the Schrödinger equation, recently developed controllers [6] based on the backstepping method [16] improved on the common "passive damper" controllers by moving all open-loop eigenvalues arbitrarily to the left in the complex plane. In this paper we adapt the design from [6] to the Euler-Bernoulli beam. We should note that the design does not carry over trivially from one system to another because the boundary conditions of the beam do not directly correspond to the boundary conditions of the Schrödinger equation.

We derive the invertible integral transformation which, together with boundary feedbacks, converts the beam into an exponentially stable target system. The kernels of the transformation and control gains are given explicitly, expressed in terms of Kelvin functions. In contrast to other backstepping designs, that have been developed for more compicated models of the beams, such as the shear beam model [5] and the Timoshenko beam model [7], [8], here the transformation is not of a strict-feedback form. Instead, it contains both Volterra- and Fredholm-type integrals.

The existing literature on the control of the Euler-Bernoulli beam is extensive (see, e.g. [1], [2], [3], [10], [11], [9], [12] and references therein), however, unlike for the wave equation, the popular "passive damper" controllers provide only a limited damping for the beam. Two constructive approaches have been used to achieve an arbitrary decay rate for beams. In [4], and later in [17], by choosing a special cost function the authors design the controllers that do not require a solution of the Riccati equation. A pole assignment (or Riesz basis) approach was used in [15], with an application to Euler-Bernoulli beam with structural damping. In these three papers, the unbounded control operator is assumed to be admissible, which is not satisfied at least for the moment control for Euler-Bernoulli beam considered in this paper. An interesting extension of Riesz basis approach, which removes the admissibility assumption, was presented in [18], where a necessary and sufficient condition was given to assign the poles by bounded feedback. However, the resulting feedback is not as explicit as the controllers presented in this paper, as it is represented as an infinite sum of infinite products.

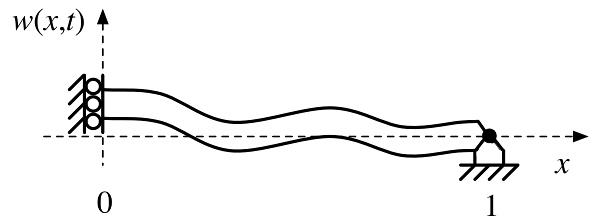

Fig. 1. Uncontrolled Euler-Bernoulli beam with the "sliding" boundary condition at $x = 0$ and the "hinged" boundary condition at $x = 1$.

II. Problem Formulation

Consider the Euler-Bernoulli beam model

$$w_{tt}(x,t) + w_{xxxx}(x,t) = 0, \qquad 0 < x < 1\,. \tag{1}$$

We assume the "sliding end" boundary conditions at $x = 0$

$$w_x(0,t) = w_{xxx}(0,t) = 0\,. \tag{2}$$

and the following boundary conditions at $x = 1$:

$$w(1,t) = u_1(t), \quad w_{xx}(1,t) = u_2(t)\,. \tag{3}$$

Here w is a beam displacement and u_1, u_2 are the position and moment control inputs. The open-loop case $u_1 = u_2 \equiv 0$ corresponds to the "hinged end" (Fig. 1). The objective is to stabilize the zero equilibrium of the beam.

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Let us introduce a new complex variable

$$v = w_t - j w_{xx}\,, \tag{4}$$

where j is the imaginary unit. The direct substitution shows that v defined in this way satisfies the Schrödinger equation

$$v_t(x,t) = -j v_{xx}(x,t) \tag{5}$$

$$v_x(0,t) = 0, \quad v(1,t) = u(t)\,. \tag{6}$$

In a recent paper [6], the backstepping controllers were designed that achieve an arbitrary decay rate for the closed-loop system (5)–(6). This gives an idea to adapt the control design from [6] to the Euler-Bernoulli beam (1)–(3). However, the design is not going to be trivial because, as can be seen from (4), regulation of v to zero does not necessarily imply the regulation of w to zero.

Before we proceed, we summarize the backstepping control design for (5)–(6).

III. Summary of Backstepping Design for the Schrödinger Equation

As shown in [6], the controller

$$v(1,t) = \int_0^1 k(1,y) v(y,t)\,dy\,, \tag{7}$$

and the transformation

$$\psi(x,t) = v(x,t) - \int_0^x k(x,y) v(y,t)\,dy\,, \tag{8}$$

where $k(x,y)$ is a complex-valued function that satisfies

$$k_{xx}(x,y) = k_{yy}(x,y) + cjk(x,y) \tag{9}$$

$$k_y(x,0) = 0, \quad k(x,x) = -\frac{cj}{2}x \tag{10}$$

with $c > 0$, map (5), (6) into the following exponentially stable target system

$$\psi_t(x,t) = -j\psi_{xx}(x,t) - c\psi(x,t) \tag{11}$$

$$\psi_x(0,t) = \psi(1,t) = 0\,. \tag{12}$$

The eigenvalues of this system are $\sigma = -c + j\frac{\pi^2(2n+1)^2}{4}$, $n = 0,1,2,\ldots$, therefore the parameter c allows to move them arbitrarily to the left in the complex plane.

The solution to the PDE (9)–(10) is

$$\begin{aligned} k(x,y) &= -cjx\frac{I_1\left(\sqrt{cj(x^2-y^2)}\right)}{\sqrt{cj(x^2-y^2)}} \\ &= x\sqrt{\frac{c}{2\,(x^2-y^2)}}\left[(j-1)\mathrm{ber}_1\left(\sqrt{c\,(x^2-y^2)}\right)\right. \\ &\qquad \left. -(1+j)\mathrm{bei}_1\left(\sqrt{c\,(x^2-y^2)}\right)\right]. \end{aligned} \tag{13}$$

Here $I_1(\cdot)$ is the modified Bessel function and $\mathrm{ber}_1(\cdot)$ and $\mathrm{bei}_1(\cdot)$ are the Kelvin functions, which are defined in terms of I_1 as

$$\mathrm{ber}_1(x) = -\mathrm{Im}\left\{I_1\left(\frac{1+j}{\sqrt{2}}x\right)\right\} \tag{14}$$

$$\mathrm{bei}_1(x) = \mathrm{Re}\left\{I_1\left(\frac{1+j}{\sqrt{2}}x\right)\right\}. \tag{15}$$

The inverse transformation

$$v(x,t) = \psi(x,t) + \int_0^x l(x,y)\psi(y,t)\,dy \tag{16}$$

with

$$l(x,y) = -cjx\frac{J_1\left(\sqrt{cj(x^2-y^2)}\right)}{\sqrt{cj(x^2-y^2)}} \tag{17}$$

maps (11)–(12) back into (5), (6), (7).

IV. Target System

In this section we choose the target system which sets the desired behavior of the beam. Let us define

$$\alpha(x,t) = \int_x^1 \int_0^y \mathrm{Im}\left\{\psi(\xi,t)\right\} d\xi dy\,, \tag{18}$$

where ψ is the state of the target system (11)–(12) for the Schrödinger equation. It is straightforward to verify that α satisfies the following fourth-order PDE:

$$\alpha_{tt} + 2c\alpha_t + c^2\alpha + \alpha_{xxxx} = 0 \tag{19}$$

with boundary conditions

$$\alpha_x(0,t) = \alpha_{xxx}(0,t) = 0 \tag{20}$$

$$\alpha_{xx}(1,t) = \alpha(1,t) = 0\,. \tag{21}$$

That this target system is exponentially stable is easily seen from the definition (18) and the fact that (11)–(12) is exponentially stable. With a straightforward computation we obtain the eigenvalues of (19)–(21):

$$\sigma_n = -c \pm j\frac{\pi^2}{4}(2n+1)^2, \quad \text{for } n = 0,1,2,\ldots \tag{22}$$

To state the precise result, we define the energy state space $H_\alpha = H_L^2(0,1) \times L^2(0,1)$, with

$$H_L^2(0,1) = \{f \in H^2(0,1) | f'(0) = f(1) = 0\} \tag{23}$$

and with the inner product induced norm

$$\|(f,g)\|_{H_\alpha}^2 = \int_0^1 [f''(x)^2 + g(x)^2]\,dx \tag{24}$$

for all $(f,g) \in H_\alpha$. The system (19)–(21) can be written as

$$\frac{d}{dt}(\alpha,\alpha_t) = \mathcal{C}(\alpha,\alpha_t) + \mathcal{D}(\alpha,\alpha_t)\,, \tag{25}$$

where $\mathcal{C}$ is a skew-adjoint operator given by

$$\mathcal{C}(f,g) = (g, -f^{(4)}),\ \forall\,(f,g) \in D(\mathcal{C}), \tag{26}$$

$$\begin{aligned} D(\mathcal{C}) = \{(f,g) \in H_\alpha | f \in H^4(0,1), g \in H_L^2(0,1), \\ f'''(0) = f''(1) = 0\}, \end{aligned} \tag{27}$$

and $\mathcal{D}$ is a bounded operator given by

$$\mathcal{D}(f,g) = (0, -2cg - c^2 f),\ \forall\,(f,g) \in H_\alpha. \tag{28}$$

Theorem 1: Let $\mathcal{C}, \mathcal{D}$ be defined by (27)-(28). Then:

(i) There is a family of eigenfunctions of $\mathcal{C}+\mathcal{D}$, which form a Riesz basis for H_α. Hence $\mathcal{C}+\mathcal{D}$ generates a C_0-semigroup $e^{(\mathcal{C}+\mathcal{D})t}$ on H_α. For any $(\alpha(\cdot,0),\alpha_t(\cdot,0))\in H_\alpha$, there exists a unique (mild) solution to (25):

$$(\alpha(\cdot,t),\alpha_t(\cdot,t))=e^{(\mathcal{C}+\mathcal{D})t}(\alpha(\cdot,0),\alpha_t(\cdot,0))\in C(0,\infty;H_\alpha),$$

and if $(\alpha(\cdot,0),\alpha_t(\cdot,0))\in D(\mathcal{C})$, then

$$(\alpha(\cdot,t),\alpha_t(\cdot,t))\in C^1(0,\infty;D(\mathcal{C})).$$

(ii) The spectrum-determined growth condition holds for the semigroup $e^{(\mathcal{C}+\mathcal{D})t}$: $\omega(\mathcal{C}+\mathcal{D})=S(\mathcal{C}+\mathcal{D})=-c$, where $\omega(\mathcal{C}+\mathcal{D})$ is the growth bound of $e^{(\mathcal{C}+\mathcal{D})t}$ and $S(\mathcal{C}+\mathcal{D})$ is the spectral bound of $\mathcal{C}+\mathcal{D}$.

(iii). The system (25) is exponentially stable: for any given $\varepsilon>0$, there exists $C_\varepsilon>0$ such that

$$E_\alpha(t)\le C_\varepsilon e^{(-c+\varepsilon)t}E(0), \quad (29)$$

$$E_\alpha(t)=\frac{1}{2}\int_0^1[\alpha_{xx}^2(x,t)+\alpha_t^2(x,t)]dx. \quad (30)$$

Proof. Omitted.

V. Control Laws That Stabilize the Beam to a Constant Profile

From (18) and (11)–(12) it follows that the state ψ is expressed through α in the following way:

$$\psi=\alpha_t+c\alpha-j\alpha_{xx}\,. \quad (31)$$

Taking the real and imaginary parts of the transformation (8), we get

$$\alpha_t(x,t)+c\alpha(x,t)=w_t(x,t)-\int_0^x r(x,y)w_t(y,t)\,dy -\int_0^x s(x,y)w_{xx}(y,t)\,dy \quad (32)$$

$$\alpha_{xx}(x,t)=w_{xx}(x,t)+\int_0^x s(x,y)w_t(y,t)\,dy -\int_0^x r(x,y)w_{xx}(y,t)\,dy\,, \quad (33)$$

where the gains $r(x,y)$ and $s(x,y)$, defined correspondingly as the real and imaginary part of $k(x,y)=r(x,y)+js(x,y)$, satisfy two coupled PDEs

$$r_{xx}(x,y)=r_{yy}(x,y)-cs(x,y) \quad (34)$$

$$r_y(x,0)=r(x,x)=0 \quad (35)$$

and

$$s_{xx}(x,y)=s_{yy}(x,y)+cr(x,y) \quad (36)$$

$$s_y(x,0)=s(x,x)=-\frac{c}{2}x\,. \quad (37)$$

The solutions to these PDEs are obtained from (13):

$$r(x,y)=x\sqrt{\frac{c}{2\,(x^2-y^2)}}\left[-\mathrm{ber}_1\Big(\sqrt{c\,(x^2-y^2)}\Big) -\mathrm{bei}_1\Big(\sqrt{c\,(x^2-y^2)}\Big)\right] \quad (38)$$

$$s(x,y)=x\sqrt{\frac{c}{2\,(x^2-y^2)}}\left[\mathrm{ber}_1\Big(\sqrt{c\,(x^2-y^2)}\Big) -\mathrm{bei}_1\Big(\sqrt{c\,(x^2-y^2)}\Big)\right]. \quad (39)$$

The control laws are obtained by setting $x=1$ in (32), (33):

$$w_t(1,t)=\int_0^1 r(1,y)w_t(y,t)\,dy+\int_0^1 s(1,y)w_{xx}(y,t)\,dy \quad (40)$$

$$w_{xx}(1,t)=\int_0^1 r(1,y)w_{xx}(y,t)\,dy-\int_0^1 s(1,y)w_t(y,t)\,dy. \quad (41)$$

Note that the feedback (40) would be implemented as integral, not proportional, control.

The control laws (40), (41) stabilize the beam to a constant profile. To see this, we use the equations (4), (31), and the inverse transformation (16) to get

$$w_t(x,t)=\alpha_t(x,t)+c\alpha(x,t)+\int_0^x s(x,y)\alpha_{xx}(y,t)\,dy -\int_0^x r(x,y)(\alpha_t(y,t)+c\alpha(y,t))\,dy \quad (42)$$

$$w_{xx}(x,t)=\alpha_{xx}(x,t)-\int_0^x s(x,y)(\alpha_t(y,t)+c\alpha(y,t))\,dy -\int_0^x r(x,y)\alpha_{xx}(y,t)\,dy\,. \quad (43)$$

In deriving (42), (43) we used the fact that $l(x,y)=-r(x,y)+js(x,y)$, which can be shown from (17). We can see that when α converges to zero, w_t and w_{xx} converge to zero. Since $w_x(0)=0$, this implies that w converges to a constant. Therefore, the straightforward application of the control design for the Schrödinger equation to the Euler-Bernoulli beam equation results in controls that suppress oscillations without necessarily bringing the beam to the zero position.

VI. Control Laws that Guarantee Regulation to Zero

To achieve regulation to zero, we are going to modify the control law (40) to make it proportional, not integral, control. To this end, we want to express w_{xx} in (40) through the time derivatives w_t and w_{tt} and then integrate (40) with respect to time.

A. Control Laws

First, let us calculate (integrating by parts twice)

$$\int_0^1 s(1,y)w_{xx}(y,t)\,dy = -\gamma_1 w_{xx}(1,t)-\int_0^1\left(\int_y^1\int_0^z s(1,\xi)\,d\xi\,dz\right)u_{xxxx}(y,t)\,dy, \quad (44)$$

where

$$\gamma_1=-\int_0^1 s(1,y)\,dy=\sinh\left(\sqrt{\frac{c}{2}}\right)\sin\left(\sqrt{\frac{c}{2}}\right). \quad (45)$$

Since

$$\int_y^1\int_0^z s(1,\xi)\,d\xi\,dz=(1-y)\int_0^1 s(1,\xi)\,d\xi -\int_y^1 s(1,\xi)(\xi-y)\,d\xi \quad (46)$$

and $w_{xxxx} = -w_{tt}$, from (44) and (40) we get

$$w_t(1,t) = \int_0^1 r(1,y)w_t(y,t)\,dy - \gamma_1 w_{xx}(1,t) - \int_0^1 w_{tt}(y,t)\int_y^1 s(1,\xi)(\xi-y)\,d\xi\,dy - \int_0^1 \gamma_1(1-y)w_{tt}(y,t)\,dy\,. \quad (47)$$

In a similar way, we get

$$\int_0^1 r(1,y)w_{xx}(y,t)\,dy = (1-\gamma_2)w_{xx}(1,t) - \int_0^1 (\gamma_2-1)(1-y)w_{tt}(y,t)\,dy - \int_0^1 w_{tt}(y,t)\int_y^1 r(1,\xi)(\xi-y)\,d\xi\,dy\,, \quad (48)$$

where

$$\gamma_2 = 1 - \int_0^1 r(1,y)\,dy = \cosh\left(\sqrt{\frac{c}{2}}\right)\cos\left(\sqrt{\frac{c}{2}}\right)\,. \quad (49)$$

Substituting (48) into (41) gives

$$w_{xx}(1,t) = -\frac{1}{\gamma_2}\int_0^1 s(1,y)w_t(y,t)\,dy - \frac{1}{\gamma_2}\int_0^1 (\gamma_2-1)(1-y)w_{tt}(y,t)\,dy - \frac{1}{\gamma_2}\int_0^1 w_{tt}(y,t)\int_y^1 r(1,\xi)(\xi-y)\,d\xi\,dy\,. \quad (50)$$

Substituting (50) into (47), after simplifications we get

$$w_t(1,t) = \int_0^1 (r(1,y)+\gamma s(1,y)w_t(y,t)\,dy + \int_0^1 w_{tt}(y,t)\int_y^1 (\gamma r(1,\xi)-s(1,\xi))(\xi-y)\,d\xi dy - \int_0^1 \gamma(1-y)w_{tt}(y,t)\,dy\,, \quad (51)$$

where

$$\gamma = \frac{\gamma_1}{\gamma_2} = \tanh\left(\sqrt{\frac{c}{2}}\right)\tan\left(\sqrt{\frac{c}{2}}\right)\,. \quad (52)$$

The control gains in (51) involve a division by γ_2, which may become zero for certain values of c. Therefore, c should satisfy the condition

$$c \neq \frac{\pi^2}{2}(2n+1)^2, \quad n = 0,1,2,\ldots\,, \quad (53)$$

which is easily achievable because c is the designer's choice.

We now integrate (51) with respect to time to get the controller

$$w(1,t) = \int_0^1 (r(1,y)+\gamma s(1,y)w(y,t)\,dy + \int_0^1 w_t(y,t)\int_y^1 (\gamma r(1,\xi)-s(1,\xi))(\xi-y)\,d\xi\,dy - \int_0^1 \gamma(1-y)w_t(y,t)\,dy\,, \quad (54)$$

where the constant of integration is chosen to be zero since this choice ensures the regulation of w to zero. To see this, note from the transformation (42), (43), and the boundary condition $w_x(0,t) = 0$ that w converges to a constant. Suppose $w(x,\infty) \equiv A$, then passing to the limit $t \to \infty$ in (54) we get

$$A = A\int_0^1 (r(1,y)+\gamma s(1,y))\,dy\,. \quad (55)$$

Computing the integral on the right hand side of (55), we obtain

$$0 = A\frac{\cosh(a)^2 - \sin(a)^2}{\cosh(a)\cos(a)}\,, \quad (56)$$

where $a = \sqrt{c/2}$. Note that $\cosh(a)^2 - \sin(a)^2 > 1$ for all $c > 0$, and $\cos(a) \neq 0$ due to the condition (53). Therefore, $A = 0$.

The other controller (33) can also be represented in terms of w and w_t as follows

$$w_{xx}(1,t) = -\int_0^1 s(1,y)w_t(y,t)\,dy + \frac{c^2}{8}w(1,t) + \int_0^1 r_{yy}(1,y)w(y,t)\,dy\,. \quad (57)$$

B. Transformation

To find out what the transformation from w to α is, we start with the definition (18) and note that

$$\mathrm{Im}\{\psi(x,t)\} = \mathrm{Im}\left\{v(x,t) - \int_0^x k(x,y)v(y,t)\,dy\right\} = -w_{xx}(x,t) + \int_0^x r(x,y)w_{xx}(y,t)\,dy - \int_0^x s(x,y)w_t(y,t)\,dy\,. \quad (58)$$

Substituting this into (18), we get

$$\alpha(x,t) = w(x,t) - w(1,t) + \int_x^1\int_0^y\int_0^z r(z,\xi)w_{xx}(\xi,t)\,d\xi dz dy - \int_x^1\int_0^y\int_0^z s(z,\xi)w_t(\xi,t)\,d\xi dz dy\,. \quad (59)$$

Integrating by parts the term with w_{xx}, changing the order of integration in both integral terms, and using (54), we obtain the final form of the transformation:

$$\alpha(x,t) = w(x,t) - \int_0^x (r(x,y)+cS(x,y))w(y,t)\,dy + \int_0^1 w_t(y,t)\left[-S(1,y)+\gamma(1-y) + \int_y^1 (s(1,\xi)-\gamma r(1,\xi))(\xi-y)\,d\xi\right]dy + \int_0^1 (cS(1,y)-\gamma s(1,y))\,w(y,t)\,dy + \int_0^x S(x,y)w_t(y,t)\,dy\,, \quad (60)$$

where

$$S(x,y) = \int_y^x (x-\xi)s(\xi,y)\,d\xi\,. \tag{61}$$

Note that this integral transformation is not strict-feedback, it is of a mixed Volterra/Fredholm type.

VII. Inverse Transformation

To prove stability of the closed-loop system through the stability of the target system, we need to derive the transformation which is inverse to (60).

It is natural to assume that the inverse transformation has the same structure as the direct one, consisting of two Volterra and two Fredholm integrals of the state of the target system and its time derivative. Therefore, we look for it in the form

$$\begin{aligned} w(x,t) = \alpha(x,t) &+ \int_0^x A(x,y)\alpha(y,t)\,dy \\ &+ \int_0^x B(x,y)\alpha_t(y,t)\,dy + \int_0^1 C(y)\alpha(y,t)\,dy \\ &+ \int_0^1 D(y)\alpha_t(y,t)\,dy\,, \end{aligned} \tag{62}$$

where A, B, C, D are the gains to be determined. Differentiating (62) w.r.t. time and space (twice) and matching the result to the equations (42) and (43) and to the boundary conditions $w_x(0,t) = w_{xxx}(0,t) = 0$, one can show that (we omit these straightforward calculations)

$$A(x,y) = -r(x,y) - 2cS(x,y)\,, \tag{63}$$

$$B(x,y) = -S(x,y)\,, \quad C(y) = 2cD(y)\,, \tag{64}$$

and $D(y)$ satisfies the ODE

$$D''''(y) = -c^2 D(y) \tag{65}$$

$$D'(0) = D(1) = D''(1) = 0, \quad D'''(0) = -c\,, \tag{66}$$

which has the solution

$$\begin{aligned} D(y) = &\frac{\sinh(ay)\cos(a(y-2)) + \cos(ay)\sinh(a(y-2))}{4a(\cosh(a)^2 - \sin(a)^2)} \\ &- \frac{\sin(ay)\cosh(a(y-2)) + \cosh(ay)\sin(a(y-2))}{4a(\cosh(a)^2 - \sin(a)^2)}. \end{aligned}$$

Note that the explicit form of the above transformation allows one to write the solution of the closed-loop system in closed form using the explicit solution of the target system.

VIII. Main Result

The design procedure presented in previous sections makes it clear why the system (1)–(2) with the controllers (54), (57) is exponentially stable. In this section we give the precise statement of well-posedness and stability of the closed-loop system.

First, we define the state space

$$\begin{aligned} H = \Big\{&(f,g) \in H^2(0,1) \times L^2(0,1) | f'(0) = 0, \\ f(1) = &\int_0^1 (r(1,y) + \gamma s(1,y))f(y)dy \\ &+ \int_0^1 g(y) \int_y^1 (\gamma r(1,\xi) - s(1,\xi))(\xi - y)d\xi\,dy \\ &- \gamma \int_0^1 (1-y)g(y)\,dy \Big\}\,. \end{aligned} \tag{67}$$

It is easy to check that $\int_0^1 [r(1,y) + \gamma s(1,y)]dy \neq 1$. Therefore, we can define the inner product induced norm of H as the energy of the system:

$$\|(f,g)\|_H^2 = \int_0^1 [|f''(x)|^2 + |g(x)|^2]\,dx \tag{68}$$

for all $(f,g) \in H$. The system (1)–(2), (54), (57) can be written as

$$\frac{d}{dt}(w(\cdot,t), w_t(\cdot,t)) = \mathcal{A}(w(\cdot,t), w_t(\cdot,t))\,, \tag{69}$$

where

$$\begin{aligned} &\mathcal{A}(f,g) = (g, -f^{(4)}),\ \forall\,(f,g) \in D(\mathcal{A}), \\ &D(\mathcal{A}) = \{(f,g) \in H |\ \mathcal{A}(f,g) \in H,\ f'''(0) = 0 \\ &f''(1) = \frac{c^2 f(1)}{8} + \int_0^1 [r_{yy}(1,y)f(y) - s(1,y)g(y)]\,dy \Big\} \end{aligned} \tag{70}$$

Next two lemmas establish the existence and boundedness of $\mathcal{A}^{-1}$ and the existence and uniqueness of a classical solution. The proofs are straightforward and we omit them.

Lemma 2: Let $\mathcal{A}$ be defined by (70) and let the condition (53) hold. Then $\rho(\mathcal{A})$, the resolvent set of $\mathcal{A}$, is not empty. In fact, $0 \in \rho(\mathcal{A})$.

Lemma 3: Let $\mathcal{A}$ be defined by (70) and let the condition (53) hold. Then for any $(w(\cdot,0), w_t(\cdot,0)) \in D(A)$ there exists a unique classical solution to (69).

Now we are ready to state the main result of the paper.

Theorem 4: Let $\mathcal{A}$ be defined by (70) and let the condition (53) hold. Then:

(i) $\mathcal{A}$ generates a C_0-semigroup on H. For any initial value $(w(\cdot,0), w_t(\cdot,0)) \in H$, there exists a unique (mild) solution to (69):

$$(w(\cdot,t), w_t(\cdot,t)) = e^{\mathcal{A}t}(w(\cdot,0), w_t(\cdot,0)) \in C(0,\infty;H),$$

(ii) The system (69) is exponentially stable at the origin: for any given $\varepsilon > 0$, there exists $M_\varepsilon > 0$, which depends only on ε, such that for all initial conditions $(w(\cdot,0), w_t(\cdot,0)) \in H$,

$$\begin{aligned} E(t) &\le M_\varepsilon e^{(-c+\varepsilon)t} E(0) \\ E(t) &= \frac{1}{2}\int_0^1 [w_t^2(x,t) + w_{xx}^2(x,t)]\,dx\,. \end{aligned} \tag{71}$$

Proof. Statement (i) follows from Lemmas 2, 3 and Theorem 1.3 of [14] on p.102. Statement (ii) follows from the density of $D(\mathcal{A})$ in H and (42), (43) and Theorem 1.

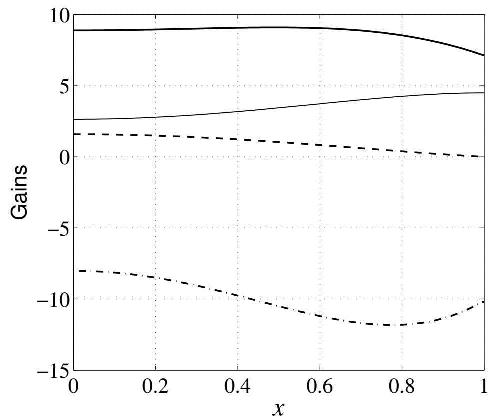

Fig. 2. Control gains for $c = 9$: from $w(x)$ to $w(1)$ (solid), from $w_t(x)$ to $w(1)$ (dashed), from $w(x)$ to $w_{xx}(1)$ (dash-dotted), from $w_t(x)$ to $w_{xx}(1)$ (thin solid).

IX. Simulation results

The results of simulation of the Euler-Bernoulli beam with the controllers (54), (57) are presented in Figs. 2–4. In Fig. 2 the control gains are shown for $c = 9$. In Fig. 3 we can see the oscillations of the uncontrolled beam. With control, the beam is quckly brought to the zero equilibrium (Fig. 4).

X. Future Work

In future work there are two extensions of the result of the paper to pursue. First, one would like to control beams with other types of boundary conditions. From the design procedure presented in the paper it is clear that an extension to a hinged type of the uncontrolled end should not pose any difficulties. One would just change the type of the uncontrolled boundary condition in the Schrödinger equation from Neumann to Dirichlet. However, it is not clear how to exploit the connection of the Euler-Bernoulli beam with the Schrödinger equation in case of the beam with a free uncontrolled end, the most important case from practical point of view.

One would also like to extend the results of the paper to the output-feedback case. For the Schrödinger equation, successful observer-based output-feedback design was developed in [6]. It seems that there are no conceptual obstacles in adapting this design to the Euler-Bernoulli beam.

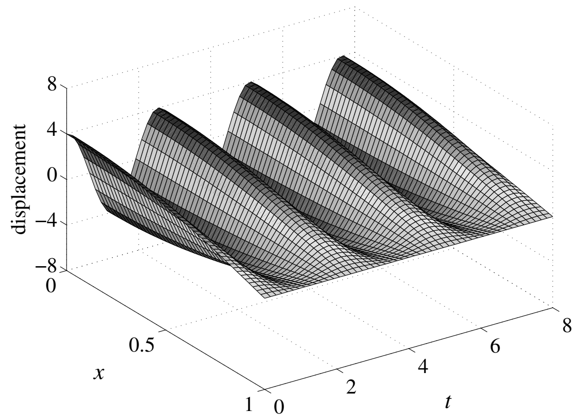

Fig. 3. Open-loop response of the Euler-Bernoulli beam.

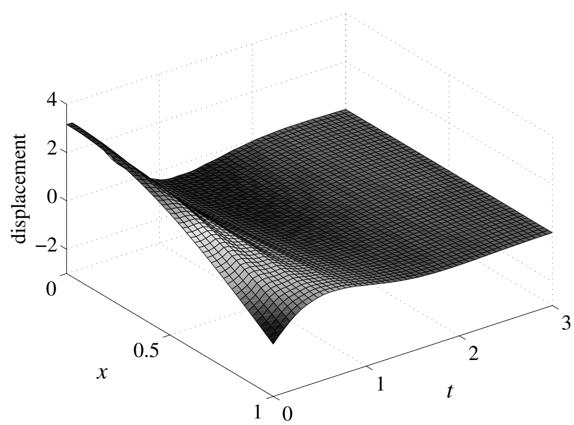

Fig. 4. Closed-loop response of the Euler-Bernoulli beam.

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